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**Existence Analysis and Discretization of Nonlinear
Partial Differential Equations**

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Dedication

*To the loving memory of my late mother,
may Allah grant her eternal peace and Jannah.*

Her boundless love, patience, and sacrifices
will forever remain engraved in my heart,
and serve as a source of strength and inspiration.

To my dear father,
in gratitude for his continuous support, guidance, and encouragement.

To my beloved family,
for their unconditional love, understanding, and moral support.

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Abstract

This thesis investigates three representative wave equations, including one linear and two nonlinear waves equations, with viscoelastic effects under various boundary conditions such as dynamic, Neumann, and Dirichlet conditions. The global existence and uniqueness of weak solutions are established using the Faedo-Galerkin method. Energy techniques and the multiplier method are employed to analyze the long-time behavior, leading to exponential energy decay under suitable assumptions on the nonlinear and viscoelastic terms.

For the numerical approximation, the finite element method is used for spatial discretization, resulting in semi-discrete schemes, while implicit time-stepping methods construct fully discrete approximations. Optimal a priori error estimates are derived for both semi-discrete and fully discrete schemes, ensuring stability and convergence.

Numerical experiments validate the theoretical results and demonstrate the efficiency and accuracy of the proposed methods. Overall, this thesis provides a unified theoretical and numerical framework for nonlinear partial differential equations with strong damping and various boundary conditions.

Keywords: Existence, Uniqueness, Energy decay, Wave equation, Strong damping, Dynamic boundary conditions, Semi-discrete, Fully discrete.

Résumé

Cette thèse étudie trois équations des ondes représentatives, comprenant un exemple linéaire et deux exemples non linéaires, avec des effets viscoélastiques sous diverses conditions aux limites, telles que les conditions dynamiques, de Neumann et de Dirichlet. L'existence globale et l'unicité des solutions faibles sont établies à l'aide de la méthode de Faedo-Galerkin. Les techniques énergétiques et la méthode des multiplicateurs sont utilisées pour analyser le comportement à long terme, conduisant à une décroissance exponentielle de l'énergie sous des hypothèses appropriées sur les termes non linéaires et viscoélastiques.

Pour l'approximation numérique, la méthode des éléments finis est utilisée pour la discrétisation spatiale, produisant des schémas semi-discrets, tandis que des méthodes implicites de discrétisation temporelle construisent des schémas complètement discrets. Des estimations optimales d'erreur a priori sont obtenues pour les deux types de schémas, assurant stabilité et convergence.

Des expériences numériques confirment les résultats théoriques et démontrent l'efficacité et la précision des méthodes proposées. Dans l'ensemble, cette thèse fournit un cadre théorique et numérique unifié pour les équations aux dérivées partielles non linéaires avec amortissement fort et diverses conditions aux limites.

Mots-clés: Existence, Unicité, Décroissance de l'énergie, Équation des ondes, Amortissement fort, Conditions aux limites dynamiques, Semi-discrétisation, Schéma entièrement discret.

ملخص

تتناول هذه الأطروحة دراسة ثلاث نماذج لمعادلات الموجة، تشمل مثلاً خطياً ومثالين غير خطيين، مع تأثيرات لزجة مرنة وتحت شروط حدودية مختلفة مثل الشروط الحدودية الديناميكية وشروط نيومان وديريشليه. تم إثبات الوجود الشامل و وحدانية الحلول الضعيفة باستخدام طريقة فايدو-غاليركين. كما استخدمت تقنيات الطاقة وطريقة المضاعف لدراسة السلوك طويل الأمد، مما أدى إلى إثبات خاصية التناقص الأسّي للطاقة تحت فرضيات مناسبة على الحدود غير الخطية واللزجة. من الناحية العددية، تم اعتماد طريقة العناصر المنتهية في التقطيع المكاني، مما أدى إلى صيغ ذات تقطيع جزئي، بينما تم استخدام مخططات زمنية ضمنية لبناء صيغ ذات تقطيع كلي. وقد تم اشتقاق تقديرات خطأ قبلية مثلى لكل من الصيغتين، مما يضمن استقرار وتقارب الطرق المقترحة. تؤكد التجارب العددية النتائج النظرية وتظهر كفاءة ودقة الطرق المطورة. وبصفة عامة، توفر هذه الأطروحة إطاراً تحليلياً وعددياً موحدًا لدراسة المعادلات التفاضلية الجزئية غير الخطية مع التخميد القوي والشروط الحدودية المختلفة.

الكلمات المفتاحية: الوجود، الوحدانية، اضمحلال الطاقة، معادلة الموجة، التخميد القوي، الشروط الحدية الديناميكية، التقطيع الجزئي، التقطيع الكلي.

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Notations

Ω : Bounded domain in $\mathbb{R}^N$.

Γ : Topological boundary of Ω .

$x = (x_1, x_2, \dots, x_N)$: Generic point of $\mathbb{R}^N$.

$dx = dx_1 dx_2 \dots dx_N$: Lebesgue measure on Ω .

∇u : Gradient of u .

Δu : Laplacian of u .

a.e: Almost everywhere.

q : Conjugate of p , i.e $\frac{1}{p} + \frac{1}{q} = 1$.

$D(\Omega)$: Space of differentiable functions with compact support in Ω .

$D'(\Omega)$: Distribution space.

$C^k(\Omega)$: Space of functions k -times continuously differentiable in Ω .

$C_0(\Omega)$: Space of continuous functions null board in Ω .

$L^p(\Omega)$: Space of functions p -th power integrated on Ω with measure of dx .

$$\|f\|_p = \left(\int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega), \nabla u \in (L^p(\Omega))^N \right\}.$$

$W^{1,p}(\Omega)$: The closure of $D(\Omega)$ in $W^{1,p}(\Omega)$.

$$\|u\|_{1,p} = \left(\|u\|_p^p + \|\nabla u\|_p^p \right)^{\frac{1}{p}}.$$

$W_0^{1,p}(\Omega)$: The closure of $D(\Omega)$ in $W^{1,p}(\Omega)$.

$W_0^{-1,p'}(\Omega)$: The dual space of $W_0^{1,p}(\Omega)$.

H : Hilbert space.

$$H_0^1 = W_0^{1,2}(\Omega).$$

If X is a Banach space

$$L^p(0, T; X) = \left\{ f : (0, T) \rightarrow X \text{ is measurable; } \int_0^T \|f(t)\|_X^p dt < \infty \right\}.$$

$$L^\infty(0, T; X) = \left\{ f : (0, T) \rightarrow X \text{ is measurable; } \text{ess sup}_{t \in (0, T)} \|f(t)\|_X < \infty \right\}.$$

$C^k([0, T]; X)$: Space of functions k -times continuously differentiable for $[0, T] \rightarrow X$.

$D([0, T]; X)$: Space of functions continuously differentiable with compact support in $[0, T]$.

General Introduction

Partial differential equations of wave type play a fundamental role in describing many physical and engineering phenomena such as the vibration of elastic structures, sound propagation, and electromagnetic waves. Due to their complex nature, especially in two-dimensional domains $\Omega \subset \mathbb{R}^2$, analytical solutions are often difficult or even impossible to obtain. Therefore, numerical methods, and in particular the *Finite Element Method* (FEM), have become essential tools for approximating and analyzing such problems.

In recent years, wave equations with dynamic boundary conditions have attracted increasing attention because they provide a realistic representation of situations where the boundary itself exhibits dynamic behavior. Unlike classical boundary conditions (Dirichlet or Neumann), dynamic boundary conditions involve time derivatives on the boundary, reflecting the physical interaction between the domain and its boundary. They naturally arise in many practical applications such as vibrating membranes attached to elastic supports, thermoelastic materials, and fluid-structure interaction problems. The presence of dynamic boundary terms significantly affects the qualitative behavior of the solutions, influencing the energy exchange between the interior and the boundary. From a mathematical point of view, these conditions lead to additional analytical challenges related to well-posedness, stability, and long-time behavior.

The Finite Element Method (FEM) is one of the most powerful and flexible techniques for the numerical approximation of partial differential equations. It provides an efficient framework to handle complex geometries, heterogeneous materials, and various boundary conditions. In this work, the FEM is employed to construct both semi-discrete (space-discretized) and fully discrete (space and time-discretized) schemes for different types of wave equations. The accuracy, stability, and energy behavior of the discrete models are analyzed theoretically and confirmed through numerical results.

This thesis is devoted to the theoretical and numerical study of three classes of wave equations defined in a bounded domain $\Omega \subset \mathbb{R}^2$. For each problem, both analytical and numerical aspects are investigated as described below.

The originality of this thesis lies in the development and rigorous analysis of finite element approximations for several classes of wave equations involving strong damping and dynamic boundary conditions. For each problem under consideration, semi-discrete and fully discrete finite element schemes are constructed and analyzed. The theoretical study includes the establishment of stability and convergence results, while numerical experiments are carried out to validate the theoreti-

cal error estimates and convergence rates. This combination of theoretical analysis and numerical validation provides a comprehensive framework for the approximation of wave equations with complex dynamic effects.

1. We consider the nonlinear wave equation with strong damping given by

$$|u_t| u_{tt} - \Delta u - \gamma \Delta u_t - \Delta u_{tt} = 0, \quad \text{in } \Omega \times (0, T), \quad \gamma > 0$$

equipped with suitable initial and boundary conditions. In this part, we first construct a semi-discrete approximation using the Finite Element Method (FEM), and then extend the analysis to a fully discrete scheme by introducing a time discretization. Theoretical results on stability and convergence are provided, and numerical experiments are carried out to validate the predicted error estimates and order of convergence.

2. The second problem deals with a linear viscoelastic wave equation of the form

$$u_{tt}(x, t) - \Delta u(x, t) - \Delta u_t(x, t) - \Delta u_{tt}(x, t) = f(x, t), \quad \text{in } \Omega \times [0, T],$$

coupled with a dynamic boundary condition on $\partial\Omega$. The analysis begins with the proof of existence and uniqueness of weak solutions, followed by a study of the asymptotic behavior of the energy. Semi-discrete and fully discrete FEM schemes are then developed, and numerical tests are performed to confirm the theoretical findings regarding accuracy and convergence.

3. Finally, we consider the nonlinear wave equation with dynamic boundary conditions defined by

$$u_{tt}(x, t) - \Delta u(x, t) - \Delta u_t(x, t) + |u(x, t)|^{\rho-1} u(x, t) = f(x, t), \quad \text{in } \Omega \times [0, T],$$

coupled with a dynamic boundary condition on $\partial\Omega$, where $\rho > 1$. In this part, we establish the existence and uniqueness of solutions and analyze the asymptotic behavior of the energy. The corresponding semi-discrete and fully discrete FEM schemes are proposed, and their performance is assessed through numerical experiments validating the theoretical error estimates and convergence behavior.

Organization of the thesis. The main objective of this thesis is to develop and analyze efficient finite element methods for several classes of wave equations with damping and dynamic boundary conditions, and to validate the theoretical results through numerical simulations. The thesis is organized into four main chapters, in addition to this general introduction and a concluding section.

Chapter 1: This chapter introduces the fundamental mathematical tools and theoretical background needed for the analysis carried out in the subsequent chapters. It recalls essential concepts from functional analysis, basic algebraic and integral inequalities, the FaedoGalerkin approximation method, and the main ideas of the Finite Element Method, which serve as the foundation for both theoretical and numerical developments.

Chapter 2: This chapter deals with the analysis and numerical approximation of a nonlinear wave equation involving a strong damping term. After recalling the main

properties of the continuous problem, we construct semi-discrete and fully discrete FEM schemes, analyze their stability and convergence, and validate the theoretical results through numerical experiments.

Chapter 3: In this chapter, we study a linear viscoelastic wave equation coupled with dynamic boundary conditions. We establish the existence and uniqueness of weak solutions, analyze the asymptotic behavior of the associated energy, and develop both semi-discrete and fully discrete FEM schemes supported by numerical results.

Chapter 4: The last chapter is devoted to a nonlinear wave equation endowed with dynamic boundary conditions. We prove the existence and uniqueness of weak solutions, study the asymptotic energy behavior, and present semi-discrete and fully discrete FEM schemes. The chapter concludes with numerical experiments that confirm the theoretical predictions regarding error estimates and convergence rates.

Finally, the thesis concludes with a general summary of the main theoretical and numerical results, along with perspectives for future research, including extensions to higher-dimensional problems, more general damping mechanisms, and the development of adaptive or energy-preserving numerical schemes.

Chapter 1

PRELIMINARIES

In this chapter, we will introduce and state without proofs some important materials needed in the proof of our results.

1.1 Banach Spaces - Definitions and properties

We first review some basic concepts from functional analysis concerning the important class of linear spaces known as Banach spaces.

Definition 1.1.1. *A Banach space is a complete normed linear space X . Its dual space X' is the linear space of all continuous linear functional $f : X \rightarrow \mathbb{R}$.*

Proposition 1.1.1 ([39]). *X' equipped with the norm $\|\cdot\|_{X'}$ defined by*

$$\|f\|_{X'} = \sup \{|f(u)|; \|u\| \leq 1\}, \quad (1.1.1)$$

is also a Banach space. We shall denote the value of $f \in X'$ at $u \in X$ by either $f(u)$ or $\langle f, u \rangle_{X', X}$.

Remark 1.1.1. *From X' we construct the bidual or second dual $X'' = (X')'$. Furthermore, with each $u \in X$ we can define $\varphi(u) \in X''$ by $\varphi(u)(f) = f(u)$, $f \in X'$. This satisfies clearly $\|\varphi(u)\| \leq \|u\|$. Moreover, for each $u \in X$ there is an $f \in X'$ with $f(u) = \|u\|$ and $\|f\| = 1$. So it follows that $\|\varphi(u)\| = \|u\|$.*

Definition 1.1.2. *Since φ is linear, we see that*

$$\varphi : X \rightarrow X'',$$

is a linear isometry of X onto a closed subspace of X'' , we denote this by

$$X \hookrightarrow X''.$$

Definition 1.1.3. *If φ is onto X'' , then X is said to be reflexive, $X \cong X''$.*

Theorem 1.1.1 ([39]). *Let X be a Banach space. Then X is reflexive, if and only if,*

$$B_X = \{x \in X : \|x\| \leq 1\},$$

is compact with the weak topology $\sigma(X, X')$. "See the next subsection for the definition of $\sigma(X, X')$ ".

Definition 1.1.4. Let X be a Banach space, and let $(u_n)_{n \in \mathbb{N}}$ be a sequence in X . Then u_n converges strongly to u in X if and only if:

$$\lim_{n \rightarrow +\infty} \|u_n - u\|_X = 0,$$

and this is denoted by $u_n \rightarrow u$, or $\lim_{n \rightarrow +\infty} u_n = u$.

Definition 1.1.5. The Banach space E is said to be separable if there exists a countable subset D of E which is dense in E , i.e. $\overline{D} = E$.

Proposition 1.1.2 ([39]). If E is reflexive and if F is a closed vector subspace of E , then F is reflexive.

Corollary 1.1.1 ([39]). The following two assertions are equivalent:

- (i) E is reflexive.
- (ii) E' is reflexive.

1.1.1 The weak and weak star topologies

Let X be a Banach space and $f \in X'$. Denote by

$$\begin{aligned} \varphi_f : X &\rightarrow \mathbb{R} \\ x &\rightarrow \varphi_f(x), \end{aligned} \tag{1.1.2}$$

As f ranges over X' , we obtain a family $(\varphi_f)_{f \in X'}$ of applications to X in $\mathbb{R}$.

Definition 1.1.6. The weak topology on X , denoted by $\sigma(X, X')$, is the weakest topology on X , for which every $(\varphi_f)_{f \in X'}$ is continuous.

We will define the third topology on X' , weak star topology, denoted by $\sigma(X', X)$. For all $x \in X$. Denote by

$$\begin{aligned} \varphi_x : X' &\rightarrow \mathbb{R} \\ f &\rightarrow \varphi_x(f) = \langle f, x \rangle_{X', X}, \end{aligned} \tag{1.1.3}$$

As x ranges over X , we obtain a family $(\varphi_x)_{x \in X}$ of applications to X' in $\mathbb{R}$.

Definition 1.1.7. The weak star topology on X' is the weakest topology on X' for which every $(\varphi_x)_{x \in X}$ is continuous.

Remark 1.1.2. Since $X \subset X''$, it is clear that the weak-star topology $\sigma(X', X)$ is weaker than the topology $\sigma(X', X'')$, and the later is weaker than the strong topology.

Definition 1.1.8. A sequence (u_n) in X is weakly convergent to u if and only if

$$\lim_{n \rightarrow \infty} f(u_n) = f(u),$$

for every $f \in X'$, and this is denoted by $u_n \rightharpoonup u$

- Remark 1.1.3.** 1. If the weak limit exists, then it is unique.
 2. If $u_n \rightarrow u \in X$ (strongly) then $u_n \rightharpoonup u$ (weakly).
 3. If $\dim X < +\infty$, then weak convergence is equivalent to strong convergence.

Proposition 1.1.3 ([39]). On the compactness in the three topologies in the Banach space X :

1. First, the unit ball

$$B \equiv \{x \in X : \|x\| \leq 1\}, \quad (1.1.4)$$

in X is compact if and only if $\dim(x) < \infty$.

2. Second, the unit ball B' in X' (The closed subspace of a product of compact spaces) is weakly compact in X' if and only if X is reflexive.
 3. Third, B' is always weakly star compact in the weak star topology of X' .

Proposition 1.1.4 ([39]). Let (f_n) be a sequence in X' . We have:

1. $[f_n \rightharpoonup^* f \text{ in } \sigma(X', X)] \Leftrightarrow [f_n(x) \rightharpoonup^* f(x), \forall x \in X]$.
2. If $f_n \rightarrow f$ (strongly) then $f_n \rightharpoonup f$, in $\sigma(X', X'')$.
3. If $f_n \rightharpoonup f$ in $\sigma(X', X'')$, then $f_n \rightharpoonup^* f$, in $\sigma(X', X)$.
4. If $f_n \rightharpoonup^* f$ in $\sigma(X', X)$ then $\|f_n\|$ is bounded and $\|f\| \leq \liminf \|f_n\|$.
5. If $f_n \rightharpoonup^* f$ in $\sigma(X', X)$ and $x_n \rightarrow x$ (strongly) in X , then $f_n(x_n) \rightarrow f(x)$.

1.1.2 Hilbert spaces

Definition 1.1.9. A Hilbert space H , is a vector space supplied with inner product $\langle u, v \rangle$, such that $\|u\| = \sqrt{\langle u, u \rangle}$ is the norm which let H complete.

Theorem 1.1.2 (Riesz [39]). If $(H; \langle \cdot, \cdot \rangle)$ is a Hilbert space, $\langle \cdot, \cdot \rangle$ being a scalar product on H , then $H' = H$ in the following sense: to each $f \in H'$ there corresponds a unique $u \in H$ such that $f = \langle u, \cdot \rangle$ and $\|f\|_H' = \|u\|_H$.

Remark 1.1.4. From this theorem we deduce that $H'' = H$. This means that a Hilbert space is reflexive.

Theorem 1.1.3 ([39]). Let $(u_n)_{n \in \mathbb{N}}$ is a bounded sequence in the Hilbert space H , it possess a subsequence which converges in the weak topology of H .

Theorem 1.1.4 ([39]). In the Hilbert space, all sequence which converges in the weak topology is bounded.

Theorem 1.1.5 ([39]). Let $(u_n)_{n \in \mathbb{N}}$ be a sequence which converges to u , in the weak topology and $(v_n)_{n \in \mathbb{N}}$ is an other sequence which converge weakly to v , then

$$\lim_{n \rightarrow \infty} \langle v_n, u_n \rangle = \langle v, u \rangle \quad (1.1.5)$$

Theorem 1.1.6 (Banach-Alaoglu-Bourbaki [39]). Let X be a normed space, then the unit ball

$$B' \equiv \{x \in X' : \|x\| \leq 1\}, \quad (1.1.6)$$

of X' is compact in $\sigma(X', X)$.

1.2 Functional Spaces

1.2.1 The $L^p(\Omega)$ spaces

Definition 1.2.1. Let $1 \leq p < \infty$ and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$ by

$$L^p(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ is measurable and } \int_{\Omega} |f(x)|^p dx < \infty \right\}. \quad (1.2.1)$$

If $p = \infty$, we have

$$L^\infty(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ is measurable and there exists a constant } C \text{ such that } |f(x)| \leq C \text{ a.e in } \Omega. \right\}.$$

Notation 1.2.1. We denote:

$$\|f\|_p = \left[\int_{\Omega} |f(x)|^p dx \right]^{\frac{1}{p}}. \quad (1.2.2)$$

$$\|f\|_\infty = \inf \{ C, |f(x)| \leq C \text{ a.e in } \Omega \}. \quad (1.2.3)$$

Notation 1.2.2. For $p \in \mathbb{R}$ and $1 \leq p \leq \infty$, we denote by q the conjugate of p i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.2.1 ([16]). L^p is a vector space, and $\|\cdot\|_p$ is a norm for all $1 \leq p \leq \infty$,

Theorem 1.2.2 (Fischer-Riesz [16]). L^p is a Banach space for all $1 \leq p \leq \infty$, and

Remark 1.2.1. In particular, when $p = 2$, $L^2(\Omega)$ equipped with the inner product

$$\langle f, g \rangle_{L^2(\Omega)} = \int_{\Omega} f(x) g(x) dx, \quad (1.2.4)$$

is a Hilbert space.

Theorem 1.2.3 ([16]). For $1 < p < \infty$, $L^p(\Omega)$ is a reflexive space.

1.2.2 Some integral inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapters.

Theorem 1.2.4 (Hölder's inequality [16]). Let $1 \leq p \leq \infty$. Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and

$$\int_{\Omega} |fg| dx \leq \|f\|_p \|g\|_q.$$

Lemma 1.2.1 ([16]). Let $f \in L^p(\Omega) \cap L^q(\Omega)$, where $1 \leq p \leq r \leq q$, $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$, and $0 \leq \alpha \leq 1$. Then

$$\|u\|_{L^r} \leq \|u\|_{L^p}^\alpha \|u\|_{L^q}^{1-\alpha}.$$

Proposition 1.2.1 ([16]). If $\mu(\Omega) < \infty$, $1 < p < q < \infty$, then $L^q \hookrightarrow L^p$ and, if $u \in L^q(\Omega)$

$$\|u\|_{L^p} \leq \mu(\Omega)^{\frac{1}{p} - \frac{1}{q}} \|u\|_{L^q}.$$

if $u \in L^\infty(\Omega)$ then $u \in L^p(\Omega)$ and

$$\|u\|_{L^p} \leq \mu(\Omega)^{\frac{1}{p}} \|u\|_{L^\infty}.$$

Lemma 1.2.2 (Young's inequality [16]). Let $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$ with $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$. Then $f * g \in L^r(\mathbb{R}^N)$ and

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

1.2.3 The $W^{m,p}(\Omega)$ spaces

The theory of Sobolev spaces has been developed by generalizing the notion of classical derivatives and introducing the idea of weak or generalized derivatives.

Definition and basic properties. Let Ω be an open subset of $\mathbb{R}^d$ and $d \in \mathbb{N}$. Let α a multi-index where $\alpha = (\alpha_1, \dots, \alpha_d)$ with $\alpha_i \geq 0$ for any $i \in 1, \dots, d$, $|\alpha| = \sum_{i=1}^d \alpha_i$ and

$$D^\alpha = D_1^{\alpha_1} \dots D_d^{\alpha_d} \text{ with } D_i = \frac{\partial}{\partial x_i}.$$

Proposition 1.2.2 ([16]). Let Ω be an open domain in $\mathbb{R}^N$. Then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that

$$\langle T, \varphi \rangle = \int_{\Omega} f(x) \varphi(x) dx, \text{ for all } \varphi \in D(\Omega),$$

where $1 \leq p \leq \infty$ and it's well-known that f is unique.

Now, we will introduce the Sobolev spaces. Let $1 \leq p \leq \infty$, and $k \in \mathbb{N}$. The Sobolev space $W^{k,p}(\Omega)$ is the space of functions $f \in L^p(\Omega)$ which have generalized up to order k such that $D^\alpha f \in L^p(\Omega)$ for all $|\alpha| \leq k$. For $k = 0$, we set $W^{0,p}(\Omega) = L^p(\Omega)$.

$$W^{k,p}(\Omega) = \left\{ f \in L^p(\Omega); D^\alpha f \in L^p(\Omega), \forall \alpha \in \mathbb{N}^d; |\alpha| \leq k \right\}.$$

The space $W^{k,p}(\Omega)$ becomes a Banach space with the norm:

$$\|f\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\Omega)}^p \right)^{1/p}, & \text{for } 1 \leq p < +\infty. \\ \max_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)}, & \text{for } p = +\infty \end{cases}$$

$W^{k,p}(\Omega)$ is a reflexive space for $1 < p < \infty$, and a separable space for $1 \leq p < \infty$.

Definition 1.2.2. Let $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. Then, the Sobolev space $W_0^{k,p}(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$ in the norm of the space $W^{k,p}(\Omega)$. It follows from the definition above that the space $W_0^{k,p}(\Omega)$ is a Banach space with the norm $\|\cdot\|_{W^{k,p}(\Omega)}$.

We write $H_0^k(\Omega) = W_0^{k,2}(\Omega)$.

For $1 \leq p \leq \infty$, the dual space of $W^{k,p}(\Omega)$ is denoted by $W^{-k,q}(\Omega)$ where q is the conjugate exponent of p . We usually use the notation $W^{-1,2}(\Omega) = H^{-1}(\Omega)$. Moreover, for $k, l \in \mathbb{N}, k \leq l$ we have the inclusions $C_0^\infty(\Omega) \subset H_0^1(\Omega) \subset H_0^k(\Omega) \subset L^2(\Omega) \subset H^{-k}(\Omega) \subset H^{-l}(\Omega) \subset (C_0^\infty(\Omega))'$, each of these spaces being dense in the following one.

Embedding results. We now turn to embedding and compact embedding results concerning the Sobolev spaces.

Theorem 1.2.5 ([16]). Let Ω be an open bounded set of $\mathbb{R}^d$ with a Lipschitz boundary. For nonnegative integers k, l such that $0 \leq l \leq k$, we have the continuous embeddings $W^{k,p}(\Omega) \subset W^{l,p}(\Omega)$ for all $1 \leq p \leq \infty$. Moreover, for $k \geq 0$, we have $W^{k,r}(\Omega) \subset W^{k,p}(\Omega)$ for all $1 \leq p \leq r \leq \infty$.

Definition 1.2.3. If $1 \leq p < n$, the Sobolev conjugate of p is defined as

$$p^* = \frac{np}{n-p}$$

Equivalently, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Also, $p^* > p$.

Theorem 1.2.6 (Gagliardo-Nirenberg-Sobolev Inequality [16]). Let $1 \leq p < n$. Then there exists a constant $C > 0$ (depending on p and n) such that

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^N)}, \quad \forall u \in W^{1,p}(\mathbb{R}^N)$$

In particular, we have the continuous embedding

$$W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$$

Corollary 1.2.1 ([16]). For any $1 \leq p < n$, $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is continuously embedded for all $r \in [p, p^*]$.

We now extend the results to proper subsets of $\mathbb{R}^n$.

Corollary 1.2.2 ([16]). Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary and $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have

$$\begin{aligned} \text{if } 1 \leq p < \infty, \text{ then } W^{1,p}(\Omega) &\subset L^{p^*}(\Omega), \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \\ \text{if } p = n, \text{ then } W^{1,p}(\Omega) &\subset L^r(\Omega), \quad \forall r \in [p, +\infty[. \\ \text{if } p > n, \text{ then } W^{1,p}(\Omega) &\subset L^\infty(\Omega) \end{aligned}$$

with continuous embedding.

Moreover, if $p > n$ we have:

$$\forall u \in W^{1,p}(\Omega), \quad |u(x) - u(y)| \leq C |x - y|^\delta \|u\|_{W^{1,p}(\Omega)} \text{ a.e. with } x, y \in \Omega$$

where $\delta = 1 - \frac{n}{p} > 0$, and C is a constant depending on p, n , and Ω . In particular $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Theorem 1.2.7 (Rellich-Kondrachov [16]). *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary and $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

$$\begin{aligned} \text{if } p < n, \text{ then } W^{1,p}(\Omega) &\subset L^r(\Omega) \forall r \in [1, p^*] \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \\ \text{if } p = n, \text{ then } W^{1,p}(\Omega) &\subset L^r(\Omega), \forall r \in [p, +\infty[. \\ \text{if } p > n, \text{ then } W^{1,p}(\Omega) &\subset C(\overline{\Omega}) \end{aligned}$$

with compact embedding.

Remark 1.2.2. We remark in particular that

$$W^{1,p}(\Omega) \subset L^r(\Omega)$$

with compact embedding for $1 \leq p \leq \infty$ and for $p \leq r < p^*$.

Generalised Sobolev Embedding: We now generalize the results of previous section to all derivative orders of $k \geq 2$.

Theorem 1.2.8 ([16]). *Let $k \geq 1$ be an integer and $1 \leq p < \infty$ and. Then*

1. *If $p < n/k$, then $W^{k,p}(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$ for all $r \in [p, np/(n - pk)]$*
2. *If $p = n/k$, then $W^{k,n/k}(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$ for all $r \in [n/k, \infty)$*
3. *If $p > n/k$, then $W^{k,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$*

with continuous embedding.

Lemma 1.2.3 (Sobolev-Poincaré's inequality [16]). *Let Ω be a bounded open subset of $\mathbb{R}^n$, then there is a constant $C_{p,\Omega}$ (depending on p and Ω) such that*

$$\text{If } 2 \leq p \leq \frac{2n}{n-2}, n \geq 3 \quad \text{and} \quad p \geq 2, n = 1, 2,$$

then

$$\|u\|_{L^p(\Omega)} \leq C_{p,\Omega} \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

Remark 1.2.3. For all $\varphi \in H^2(\Omega)$, $\Delta\varphi \in L^2(\Omega)$ and for Γ sufficiently smooth, we have

$$\|\varphi(t)\|_{H^2(\Omega)} \leq C \|\Delta\varphi(t)\|_{L^2(\Omega)}.$$

Proposition 1.2.3 (Green's formula [40]). *For all $u \in H^2(\Omega)$, $v \in H^1(\Omega)$ we have*

$$-\int_{\Omega} \Delta u v dx = \int_{\Omega} \nabla u \nabla v dx - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v d\sigma,$$

where $\frac{\partial u}{\partial \eta}$ is a normal derivation of u at Γ .

1.2.4 The $L^p(0, T, X)$ spaces

Let X be a Banach space, denote by $L^p(0, T, X)$ the space of measurable functions

Definition 1.2.4. *Let*

$$\begin{aligned} f :]0, T[&\rightarrow X \\ t &\rightarrow f(t). \end{aligned} \quad (1.2.5)$$

such that

$$\left(\int_0^T \|f(t)\|_X^p dt \right)^{\frac{1}{p}} = \|f\|_{L^p(0, T, X)} < \infty, \text{ for } 1 \leq p < \infty. \quad (1.2.6)$$

If $p = \infty$,

$$\|f\|_{L^p(0, T, X)} = \sup_{t \in]0, T[} \text{ess} \|f(t)\|_X. \quad (1.2.7)$$

Theorem 1.2.9 ([39]). *The space $L^p(0, T, X)$ is complete.*

We denote by $D'(0, T, X)$ the space of distributions in $]0, T[$ which take its values in X and let us define

$$D'(0, T, X) = \mathcal{L}(D]0, T[, X),$$

where $\mathcal{L}(\phi, \varphi)$ is the space of the linear continuous applications of ϕ to φ . Since $u \in D'(0, T, X)$, we define the distribution derivation as

$$\frac{\partial u}{\partial t}(\varphi) = -u\left(\frac{d\varphi}{dt}\right), \quad \forall \varphi \in D(]0, T[),$$

and since $u \in L^p(0, T, X)$, we have

$$u(\varphi) = \int_0^T u(t) \varphi(t) dt, \quad \forall \varphi \in D(]0, T[),$$

We will introduce several basic results on the $L^p(0, T, X)$ space. These results will be very useful in subsequent chapters of this thesis.

Lemma 1.2.4 ([39]). *Let $f \in L^p(0, T, X)$ and $\frac{\partial f}{\partial t} \in L^p(0, T, X)$, ($1 \leq p \leq \infty$), then the function f is continuous from $[0, T] \rightarrow X$. i.e. $f \in C^1(0, T, X)$.*

Lemma 1.2.5 ([39]). *Let $\varphi =]0, T[\times \Omega$ an open bounded domain in $\mathbb{R} \times \mathbb{R}^n$, and g_μ, g are two functions in $L^q(]0, T[, L^q(\Omega))$, $1 < q < \infty$ such that*

$$\|g_\mu\|_{L^q(]0, T[, L^q(\Omega))} \leq C, \quad \forall \mu \in N \quad (1.2.8)$$

and $g_\mu \rightarrow g$ in φ , then $g_\mu \rightarrow g$ in $L^q(\varphi)$.

Theorem 1.2.10 ([39]). *$L^p(0, T, X)$ equipped with the norm $\|\cdot\|_{L^p(0, T, X)}$, $1 \leq p \leq \infty$ is a Banach space.*

Proposition 1.2.4 ([39]). *Let X be a reflexive Banach space, X' its dual, and $1 \leq p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of $L^p(0, T, X)$ is identify algebraically and topologically with $L^q(0, T, X')$.*

Proposition 1.2.5 ([39]). *Let X, Y be Banach space, $X \subset Y$ with continuous embedding, then we have*

$$L^p(0, T, X) \subset L^p(0, T, Y),$$

with continuous embedding.

The following compactness criterion will be useful for nonlinear evolution problem, especially in the limit of the nonlinear terms.

Proposition 1.2.6 ([39]). *Let B_0, B, B_1 be Banach spaces with $B_0 \subset B \subset B_1$. Assume that the embedding $B_0 \hookrightarrow B$ is compact and $B \hookrightarrow B_1$ are continuous. Let $1 < p, q < \infty$. Assume further that B_0 and B_1 are reflexive. Define*

$$W \equiv \{u \in L^p(0, T, B_0) : u' \in L^q(0, T, B_1)\}. \quad (1.2.9)$$

Then, the embedding $W \hookrightarrow L^p(0, T, B)$ is compact.

Now, we introduce a space of functions with values in a space X (a separable Hilbert space).

The space $L^2(a, b, X)$ is a Hilbert space for the inner product

$$(f, g)_{L^2(a, b, X)} = \int_a^b (f(t), g(t))_X dt$$

We note that $L^\infty(a, b, X) = (L^1(a, b, X))'$. Now, we define the Sobolev spaces with values in a Hilbert space X . For $k \in \mathbb{N}$, $p \in [1, \infty]$, we set:

$$W^{k,p}(a, b, X) = \left\{ v \in L^p(a, b, X) ; \frac{\partial v}{\partial x_i} \in L^p(a, b, X), \forall i \leq k \right\}$$

The Sobolev space $W^{k,p}(a, b, X)$ is a Banach space with the norm

$$\begin{aligned} \|f\|_{W^{k,p}(a, b, X)} &= \left(\sum_{i=0}^k \left\| \frac{\partial f}{\partial x_i} \right\|_{L^p(a, b, X)}^p \right)^{1/p}, \text{ for } p < +\infty \\ \|f\|_{W^{k,\infty}(a, b, X)} &= \sum_{i=0}^k \left\| \frac{\partial f}{\partial x_i} \right\|_{L^\infty(a, b, X)}, \text{ for } p = +\infty \end{aligned}$$

The spaces $W^{k,2}(a, b, X)$ form a Hilbert space and it is noted $H^k(0, T, X)$. The $H^k(0, T, X)$ inner product is defined by:

$$(u, v)_{H^k(a, b, X)} = \sum_{i=0}^k \int_a^b \left(\frac{\partial u}{\partial x^i}, \frac{\partial v}{\partial x^i} \right)_X dt.$$

1.2.5 Some Algebraic inequalities

Since our study based on some known algebraic inequalities, we recall several of them here.

Lemma 1.2.6 (The Cauchy–Schwarz inequality [40]). *Every inner product satisfies*

the Cauchy–Schwarz inequality

$$\langle x_1, x_2 \rangle \leq \|x_1\| \|x_2\|. \quad (1.2.10)$$

The equality sign holds if and only if x_1 and x_2 are linearly dependent.

Lemma 1.2.7 (Young’s inequality). *For all $a, b \in \mathbb{R}^+$, we have*

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad (1.2.11)$$

where δ is any positive constant.

Lemma 1.2.8 ([40]). *For $a, b \geq 0$, the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad (1.2.12)$$

where, $\frac{1}{p} + \frac{1}{q} = 1$.

1.3 Integral Inequalities

We will recall some fundamental integral inequalities to estimate the decay rate of the energy.

1.3.1 A result of exponential decay

The estimation of the energy decay for some dissipative problems is based on the following lemma:

Lemma 1.3.1 ([20]). *Let $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-increasing function and assume that there is a constant $A > 0$ such that*

$$\forall t \geq 0, \quad \int_t^{+\infty} E(\tau) d\tau \leq \frac{1}{A} E(t). \quad (1.3.1)$$

Then we have

$$\forall t \geq 0, \quad E(t) \leq E(0) e^{1-At}. \quad (1.3.2)$$

Proof of Lemma 1.3.1. The inequality (1.3.2) is verified for $t \leq \frac{1}{A}$, this follows from the fact that E is a decreasing function. We prove that (1.3.2) is verified for $t \geq \frac{1}{A}$. Introduce the function

$$h : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad h(t) = \int_t^{+\infty} E(\tau) d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (1.3.1) we find that

$$\forall t \geq 0, \quad h'(t) + Ah(t) \leq 0.$$

Let

$$T_0 = \sup \{t, h(t) > 0\}. \quad (1.3.3)$$

For every $t < T_0$, we have

$$\frac{h'(t)}{h(t)} \leq -A,$$

thus

$$h(t) \leq e^{-At} \leq \frac{1}{A} E(0) e^{-At}, \quad \text{for } 0 \leq t < T_0. \quad (1.3.4)$$

Since $h(t) = 0$ if $t \geq T_0$, this inequality holds in fact for every $t \in \mathbb{R}_+$. Let $\varepsilon > 0$. As E is positive and decreasing, we deduce that

$$\forall t \geq \varepsilon, \quad E(t) \leq \frac{1}{\varepsilon} \int_{t-\varepsilon}^t E(\tau) d\tau \leq \frac{1}{\varepsilon} h(t-\varepsilon) \leq \frac{1}{A\varepsilon} E(0) e^{\varepsilon t} e^{-At}.$$

Choosing $\varepsilon = \frac{1}{A}$, we obtain

$$\forall t \geq 0, \quad E(t) \leq E(0) e^{1-At}.$$

The proof of Lemma 1.3.1 is now completed. $\square$

1.3.2 A result of polynomial decay

Lemma 1.3.2 ([20]). *Let $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-increasing function and assume that there are two constants $q > 0$ and $A > 0$ such that*

$$\forall t \geq 0, \quad \int_t^{+\infty} E^{q+1}(\tau) d\tau \leq \frac{1}{A} E^q(0) E(t). \quad (1.3.5)$$

Then we have

$$\forall t \geq 0, \quad E(t) \leq E(0) \left(\frac{1+q}{1+AQ t} \right)^{1/q}. \quad (1.3.6)$$

Remark 1.3.1. It is clear that Lemma 1.3.1 is similar to Lemma 1.3.2 in the case of $q = 0$.

Proof of Lemma 1.3.2. If $E(0) = 0$, then $E \equiv 0$ and there is nothing to prove. Otherwise, replacing the function E by the function $\frac{E}{E(0)}$ we may assume that $E(0) = 1$. Introduce the function

$$h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \quad h(t) = \int_t^{+\infty} E(\tau) d\tau.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (1.3.5) we find that

$$\forall t \geq 0, \quad -h' \geq (Ah)^{1+q}.$$

where

$$T_0 = \sup \{t, h(t) > 0\}.$$

Integrating in $[0, t]$ we obtain that

$$\forall 0 \leq t < T_0, h(t)^{-q} - h(0)^{-q} \geq \sigma \omega^{1+q} t,$$

hence

$$0 \leq t < T_0, \quad h(t) \leq (h^{-q}(0) + qA^{1+q}t)^{-1/q}. \quad (1.3.7)$$

Since $h(t) = 0$ if $t \geq T_0$, this inequality holds in fact for every $t \in \mathbb{R}_+$. Since

$$h(0) \leq \frac{1}{A} E(0)^{1+q} = \frac{1}{A},$$

by (1.3.5), the right-hand side of (1.3.7) is less than or equal to:

$$(h^{-q}(0) + qA^{1+q}t)^{-1/q} \leq \frac{1}{A} (1 + Aq t)^{-1/q}. \quad (1.3.8)$$

From other hand, E being nonnegative and non-increasing, we deduce from the definition of h and the above estimate that:

$$\begin{aligned} \forall s \geq 0, E\left(\frac{1}{A} + (q+1)s\right)^{q+1} &\leq \frac{1}{\frac{1}{A} + q+1} \int_s^{\frac{1}{A} + (q+1)s} E(\tau)^{q+1} d\tau \\ &\leq \frac{A}{1 + Aqs} h(s) \leq \frac{A}{1 + Aqs} \frac{1}{A} (1 + Aqs)^{-\frac{1}{q}}, \end{aligned}$$

hence

$$\forall S \geq 0, \quad E\left(\frac{1}{A} + (q+1)S\right) \leq \frac{1}{(1 + AqS)^{1/q}}.$$

Choosing $t = \frac{1}{A} + (1+q)s$ then the inequality (1.3.6) follows. Note that letting $q \rightarrow 0$ in this theorem we obtain (1.3.6). $\square$

1.3.3 New integral inequalities of P.Martinez

The above inequalities are verified only if the energy function is integrable. We will try to resolve this problem by introducing some weighted integral inequalities, so we can estimate the decay rate of the energy when it is slow.

Lemma 1.3.3 ([20]). *Let $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-increasing function and $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ an increasing C^1 function such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \rightarrow +\infty \quad \text{when} \quad t \rightarrow +\infty. \quad (1.3.9)$$

Assume that there exist $q \geq 0$ and $A > 0$ such that

$$\int_S^{+\infty} E(t)^{q+1} \phi'(t) dt \leq \frac{1}{A} E(0)^q E(S), \quad 0 \leq S < +\infty. \quad (1.3.10)$$

then we have

$$\text{if } q > 0, \text{ then } E(t) \leq E(0) \left(\frac{1+q}{1+qA\phi(t)} \right)^{\frac{1}{q}}, \quad \forall t \geq 0,$$

$$\text{if } q = 0, \text{ then } E(t) \leq E(0) e^{1-A\phi(t)}, \quad \forall t \geq 0.$$

Proof of Lemma 1.3.3. This Lemma is a generalization of Lemma 1.3.1. Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$, (we notice that ϕ^{-1} has a meaning by the hypotheses assumed on ϕ). f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain

$$\begin{aligned} \int_{\phi(S)}^{\phi(T)} f(x)^{q+1} dx &= \int_{\phi(S)}^{\phi(T)} E(\phi^{-1}(x))^{q+1} dx = \int_S^T E(t)^{q+1} \phi'(t) dt \\ &\leq \frac{1}{A} E(0)^q E(S) \\ &= \frac{1}{A} E(0)^q f(\phi(S)), \quad 0 \leq S < T < +\infty. \end{aligned}$$

Setting $s := \phi(S)$ and letting $T \rightarrow +\infty$, we deduce that

$$\forall s \geq 0, \quad \int_s^{+\infty} f(x)^{q+1} dx \leq \frac{1}{A} E(0)^q f(s).$$

By Lemma 1.3.1, we obtain the desired results. $\square$

1.4 Existence Methods

1.4.1 Faedo-Galerkin's approximations

We consider the Cauchy problem abstract's for a second order evolution equation in the separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$.

$$(P) \quad \begin{cases} u''(t) + A(t)u(t) = f(t), & t \in [0, T] \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases}$$

where u and f denote the unknown and the given functions, respectively. mapping the closed interval $[0, T] \subset \mathbb{R}$ into a real separable Hilbert space H . $A(t)$ ($0 \leq t \leq T$) are linear bounded operators in H acting in the energy space $V \subset H$.

Assume that $\langle A(t)u(t), v(t) \rangle = a(t; u(t), v(t))$, for all $u, v \in V$; where $a(t; \cdot, \cdot)$ is a bilinear continuous on V . The problem (P) can be formulated as: Find a solution $u(t)$ such that

$$(\tilde{P}) \quad \begin{cases} u \in C([0, T]; V), \quad u' \in C([0, T]; H) \\ \langle u''(t), v \rangle + a(t; u(t), v) = \langle f, v \rangle \text{ in } D'([0, T]) \\ u_0 \in V, \quad u_1 \in H. \end{cases}$$

This problem can be solved using the Faedo–Galerkin approximation method.

Let V_m be a finite-dimensional subspace of V , and let $\{w_{jm}\}$ a basis of V_m such that:

1. $V_m \subset V$ ($\dim V_m < \infty$), $\forall m \in \mathbb{N}$.
2. $V_m \rightarrow V$ such that, there exists a dense subspace ϑ in V and for all $v \in \vartheta$ we can get sequence $\{u_m\}_{m \in \mathbb{N}} \in V_m$ and $u_m \rightarrow u$ in V .
3. $V_m \subset V_{m+1}$ and $\overline{\cup_{m \in \mathbb{N}} V_m} = V$.

We define the solution u_m of the approximate problem

$$(P_m) \quad \begin{cases} u_m(t) = \sum_{j=1}^{d_m} g_j(t) w_{jm} \\ u_m \in C([0, T]; V_m), \quad u'_m \in C([0, T]; V_m), \quad u_m \in L^2(0, T; V_m) \\ \langle u''_m(t), w_{jm} \rangle + a(t; u_m(t), w_{jm}) = \langle f, w_{jm} \rangle, \quad 1 \leq j \leq d_m \\ u_m(0) = \sum_{j=1}^{d_m} \xi_j(t) w_{jm}, \quad u'_m(0) = \sum_{j=1}^{d_m} \eta_j(t) w_{jm} \end{cases}$$

where

$$\begin{aligned} \sum_{j=1}^{d_m} \xi_j(t) w_{jm} &\longrightarrow u_0 \text{ in } V \text{ as } m \longrightarrow \infty \\ \sum_{j=1}^{d_m} \eta_j(t) w_{jm} &\longrightarrow u_1 \text{ in } V \text{ as } m \longrightarrow \infty \end{aligned}$$

By virtue of the theory of ordinary differential equations, the system (P_m) admits a unique local solution which can be extended to a maximal interval $[0, t_m[$ by Zorn's lemma since the non-linear terms have the suitable regularity. In the next step, we obtain a priori estimates for the solution, so that can be extended outside $[0, t_m[$ to obtain one solution defined for all $t > 0$.

1.4.2 Gronwall's lemma

To establish a priori estimates for the Faedo-Galerkin approximations, we recall a fundamental tool known as Gronwall's inequality:

Lemma 1.4.1 ([42]). *Let $T > 0$, $g \in L^1(0, T)$, $g \geq 0$ a.e and c_1, c_2 are positive constants. Let $\varphi \in L^1(0, T)$ $\varphi \geq 0$ a.e such that $g\varphi \in L^1(0, T)$ and*

$$\varphi(t) \leq c_1 + c_2 \int_0^t g(s) \varphi(s) ds \quad \text{a.e in } (0, T).$$

then, we have

$$\varphi(t) \leq c_1 \exp \left(c_2 \int_0^t g(s) ds \right) \quad \text{a.e in } (0, T).$$

1.4.3 A priori estimation and convergence

Using the following estimation

$$\|u_m\|^2 + \|u'_m\|^2 \leq C \left\{ \|u_m(0)\|^2 + \|u'_m(0)\|^2 + \int_0^T \|f(s)\|^2 ds \right\}; \quad 0 \leq t \leq T$$

and the Gronwall lemma we deduce that the solution u_m of the approximate problem (P_m) converges to the solution u of the initial problem (P) . The uniqueness proves that u is the solution.

1.4.4 The Discrete Gronwall Inequality

We now present a discrete version of Gronwall's inequality, which is widely used in the analysis of sequences arising from numerical approximations. In particular, B. G. Pachpatte proved the following discrete Gronwall-type inequality in [43]:

Lemma 1.4.2. *Let $x, f, g, h : \mathbb{N} \rightarrow \mathbb{R}_+$ be sequences such that*

$$x(n) \leq f(n) + g(n) \sum_{s=0}^{n-1} h(s) x(s), \quad n \geq 1.$$

Then

$$x(n) \leq f(n) + g(n) \sum_{s=0}^{n-1} h(s) f(s) \prod_{\tau=s+1}^{n-1} (h(\tau) g(\tau) + 1), \quad \forall n \in \mathbb{N}^*.$$

1.5 Finite Element Approximation of the Wave Equation

To illustrate the finite element method, let us consider the classical linear wave equation

$$\begin{cases} u_{tt}(x, t) - \Delta u(x, t) = f(x, t), & x \in \Omega, t \in (0, T), \\ u(x, t) = 0, & x \in \partial\Omega, t \in (0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary.

Weak formulation. Let $V := H_0^1(\Omega)$ with inner product

$$a(w, v) = (\nabla w, \nabla v), \quad (w, v) = (w, v)_{L^2(\Omega)}.$$

The weak form reads: find $u : [0, T] \rightarrow V$ such that for all $v \in V$,

$$(u_{tt}(t), v) + a(u(t), v) = (f(t), v).$$

Finite element space. Let $\mathcal{T}_h$ be a regular triangulation of Ω (in the sense of [13]), and define the classical finite element space of continuous, piecewise linear functions:

$$V_h = \left\{ v_h \in C^0(\bar{\Omega}) : v_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h, v_h|_{\partial\Omega} = 0 \right\}.$$

Denote by $P^h : V \rightarrow V_h$ the classical interpolation operator associated with the nodes of the mesh.

Semi-discrete problem. Find $u_h : [0, T] \rightarrow V_h$ such that

$$(u_{h,tt}(t), v_h) + a(u_h(t), v_h) = (f(t), v_h), \quad \forall v_h \in V_h, \quad t \in (0, T),$$

with initial conditions $u_h(0) = P^h u_0$ and $u_{h,t}(0) = P^h u_1$.

Matrix form. Let $\{\phi_i\}_{i=1}^N$ be the nodal basis of V_h . Writing $u_h(t) = \sum_{i=1}^N U_i(t) \phi_i$ gives the second-order ODE system

$$M\ddot{U}(t) + AU(t) = F(t),$$

where

$$M_{ij} = (\phi_i, \phi_j), \quad A_{ij} = a(\phi_i, \phi_j), \quad F_j(t) = (f(t), \phi_j).$$

Approximation and error estimate. If the exact solution satisfies $u(\cdot, t) \in H^2(\Omega)$ for each t , the interpolation error of P^h gives

$$\|u(\cdot, t) - P^h u(\cdot, t)\|_{H^1(\Omega)} \leq Ch \|u(\cdot, t)\|_{H^2(\Omega)}. \quad (\text{see [13]})$$

Consequently, under standard regularity assumptions and with $u_h(0) = P^h u_0$, $u_{h,t}(0) = P^h u_1$, one obtains

$$\|u(t) - u_h(t)\|_{H^1(\Omega)} + \|u_t(t) - u_{h,t}(t)\|_{L^2(\Omega)} \leq Ch, \quad t \in [0, T],$$

where C is independent of h .

Remark. This formulation provides the *semi-discrete* model, in which the spatial variable is discretized by finite elements while time remains continuous. The corresponding fully discrete schemes, obtained by introducing a time-stepping method, will be presented in the subsequent chapters according to the considered problems.

Chapter 2

Numerical analysis and simulation for a nonlinear viscoelastic equation with strong damping

2.1 Introduction

The viscoelastic wave equation describes how waves travel through materials that have both elastic and viscous characteristics. Understanding the evolution of waves in materials that can simultaneously dissipate energy and deform is made possible by the use of viscoelastic wave equations, for the physical aspects, see Love [1].

Let Ω be a bounded open set of $\mathbb{R}^n$, $n = 1, 2$, with boundary Γ of class C^1 and $Q = \Omega \times (0, T)$ for $T > 0$, with lateral boundary defined by $\Sigma = \Gamma \times (0, T)$. We shall consider the following the nonlinear viscoelastic problem

$$\begin{cases} |u_t| u_{tt} - \Delta u - \gamma \Delta u_t - \Delta u_{tt} = 0, & \text{in } Q, \quad \gamma > 0 \\ u = 0, & \text{on } \Sigma, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \end{cases} \quad (2.1.1)$$

where $u = u(x, t)$ represents the displacement at time t and the space coordinate x .

In [2], M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. Ferreira studied the global existence of weak solutions and uniform decay for a nonlinear viscoelastic equation with strong damping

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau) \Delta u(\tau) d\tau - \gamma \Delta u_t = 0, & \text{in } \Omega \times (0, \infty), \\ u = 0, & \text{on } \Gamma \times (0, \infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \end{cases} \quad (2.1.2)$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary Γ and ρ is a real number such that $0 < \rho \leq \frac{2}{(n-2)}$ if $n \geq 3$ or $\rho > 0$ if $n = 1, 2$. The authors considered $\gamma \geq 0$ to prove the existence of solutions of the problem (2.1.2) and in order to obtain the uniform decay rates of the energy they restricted themselves to the case $\gamma > 0$.

It is important to note that if $\rho = 1$ and $g = 0$ in (2.1.2), then we have obtain problem (2.1.1), which is a special case of the problem (2.1.2). By following the same argumants and hypotheses presented by the authors in Reference [2], we can establish the global existence of weak solutions for problem (2.1.1). Furthermore, we can derive the uniform decay rates of the energy. The objective of this work is to study the convergence of finite element solution to the exact solution of the problem (2.1.1) in the $L^\infty(L^2)$ norm. Furthermore, a fully discrete scheme based on a symmetric difference approximation is considered and optimal order convergence in $L^\infty(L^2)$ norm is established.

A finite difference approximation with artificial viscosity, semi-discrete in space, for the damped wave equation was considered in [3] and [4]. In [5] existence results and numerical simulation were presented for a hyperbolic equation with variable coefficients and acoustic boundary conditions. For the numerical analysis of nonlinear hyperbolic problems, references such as [6] and [7] are notable. Additionally, in [8], researchers studied error estimates for the finite element method applied to a 2D wave equation with dynamic boundary control. They examined methods for the wave model, structure-preserving properties, and a fully discrete implicit approach, emphasizing the role of artificial viscosity in defining the numerical method. In addition to the works mentioned in the previous paragraphs, there are recent studies focusing on the application of the finite element method and full discretization to relatively complex problems, which have shown promising results (see references [9], [10], [11], and [12]). These studies indicate that the methods are not only applicable but also effective in addressing intricate problems, leading to improved accuracy and efficiency.

This chapter is organized as follows. Section 2.2 presented a detailed analysis of the semi-discrete Galerkin approximation to the solution of (2.1.1). Section 2.3 is devoted to the formulation and examination of a fully discrete scheme, which employs piecewise linear basis functions in space together with the backward Euler method for temporal discretization. The corresponding error estimates are rigorously derived. Section 2.4 reports a series of numerical experiments conducted to demonstrate the accuracy and effectiveness of the proposed numerical method. The obtained numerical errors and the associated convergence rates for various mesh sizes are summarized in tables, and graphical illustrations of the computed solutions are provided. The numerical results are reported for different values of the parameter γ .

2.2 Semi-Discrete Approximation

Notations. We introduce the following space:

$$V = H_0^1(\Omega) = \{u \in H^1(\Omega); u|_\Gamma = 0\},$$

and we denote by $\|\cdot\|_p$, $\|\cdot\|_{m,\Omega}$ and $\|\cdot\|_{H_0^1}$ the norms in $L^p(\Omega)$, $H^m(\Omega)$ and $H_0^1(\Omega)$, respectively, and by $(\cdot, \cdot)$ the scalar product in $L^2(\Omega)$.

We also introduce the constant $C_{p,\Omega} > 0$, depending only on p and the domain

Ω , such that the **Sobolev-Poincaré inequality** holds:

$$\|u\|_{L^p(\Omega)} \leq C_{p,\Omega} \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega), \quad \forall p \geq 1. \quad (2.2.1)$$

In the particular case $p = 2$, the inequality (2.2.1) reduces to the classical Poincaré inequality, and for simplicity we denote the constant $C_{2,\Omega}$ by C_Ω .

Weak Formulation. Multiplying the equation (2.1.1)₁ by a test function $v \in V$. Next, we apply Green's formulation with the boundary condition to derive the weak formulation of the problem (2.1.1). Find $u(x, t) \in V$ such that

$$\begin{cases} (|u_t| u_{tt}, v) + (\nabla u, \nabla v) + \gamma (\nabla u_t, \nabla v) + (\nabla u_{tt}, \nabla v) = 0, & \forall v \in V, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega. \end{cases} \quad (2.2.2)$$

2.2.1 Semi-Discrete Approximation

Assuming that $\bar{\Omega}$ is a polyhedral domain. Let V^h be a finite-dimensional space, approximating the variational space V , given by

$$V^h = \left\{ v^h \in C(\bar{\Omega}); v_K^h \in \mathbb{P}_1(K), K \in \mathcal{T}_h, v^h = 0 \text{ on } \Gamma \right\},$$

where $\mathbb{P}_1(K)$ represents the space of polynomials of global degree less than or equal to one in K and we denote by $(\mathcal{T}_h)_{h>0}$ a regular family of polygonalization of $\bar{\Omega}$ (see for instance [13]), compatible with the partition of the boundary $\Gamma = \partial\Omega$, i.e. the finite element space V^h is composed of continuous and piecewise affine functions. Let h_K be the diameter of an element $K \in \mathcal{T}_h$ and let $h = \max_{K \in \mathcal{T}_h} h_K$ denote the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $u_0^h = P^h u_0$ and $u_1^h = P^h u_1$, where P^h is the classical finite element interpolation operator defined by

$$(\nabla(P^h \psi), \nabla \chi) = (\nabla \psi, \nabla \chi), \quad \forall \chi \in V^h,$$

satisfying the following estimations (see [13])

$$\begin{cases} \|\psi - P^h \psi\|_2 \leq Ch^2 \|\psi\|_{2,\Omega}, & \forall \psi \in H^2(\Omega), \\ \|\psi - P^h \psi\|_{1,\Omega} \leq Ch \|\psi\|_{2,\Omega}, & \forall \psi \in H^2(\Omega). \end{cases} \quad (2.2.3)$$

The semi-discrete finite element solution $u^h(t)$ of problem (2.1.1) is defined as follows: for all $t \in [0, T]$, find $u^h(t) \in V^h$ such that for all $v^h \in V^h$

$$\begin{cases} (|u_t^h| u_{tt}^h, v^h) + (\nabla u^h, \nabla v^h) + \gamma (\nabla u_t^h, \nabla v^h) + (\nabla u_{tt}^h, \nabla v^h) = 0, \\ u^h(0) = P^h u_0, \quad u_t^h(0) = P^h u_1. \end{cases} \quad (2.2.4)$$

In the following theorem we provide an error estimate for the discrete problem (2.2.4)

Theorem 2.2.1. Assume that $u_0, u_1 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u \in H^3(0, T, H^2(\Omega))$, then,

for each $T > 0$, we have

$$\left\| \nabla u - \nabla u^h \right\|_{L^\infty(0, T, L^2(\Omega))}^2 + \left\| \nabla u_t - \nabla u_t^h \right\|_{L^\infty(0, T, L^2(\Omega))}^2 \leq Ch^2,$$

where C is a positive constant independent of h .

Before proving Theorem 2.2.1, we need the following lemma:

Lemma 2.2.1. *Let $u^h(t)$ be the solution defined in (2.2.4). If $u_0, u_1 \in H_0^1(\Omega) \cap H^2(\Omega)$, then, for each $T > 0$, we have the following properties*

$$u^h \in L^\infty(0, T, H_0^1(\Omega)), \quad u_t^h \in L^\infty(0, T, H_0^1(\Omega)), \quad u_{tt}^h \in L^2(0, T, H_0^1(\Omega)).$$

Proof. The proof is divided into two parts: the first estimate and the second estimates.

The first estimate: Setting $v^h = u_t^h$ in (2.2.4), we obtain

$$\frac{d}{dt} \left(\frac{1}{3} \|u_t^h\|_3^3 + \frac{1}{2} \|\nabla u^h\|_2^2 + \frac{1}{2} \|\nabla u_t^h\|_2^2 \right) + \gamma \|\nabla u_t^h\|_2^2 = 0.$$

Integrating the last equality from 0 to t , it follows that

$$\begin{aligned} \frac{1}{3} \|u_t^h\|_3^3 + \frac{1}{2} \|\nabla u^h\|_2^2 + \frac{1}{2} \|\nabla u_t^h\|_2^2 + \gamma \int_0^t \|\nabla u_t^h(s)\|_2^2 ds \\ = \frac{1}{3} \|u_t^h(0)\|_3^3 + \frac{1}{2} \|\nabla u^h(0)\|_2^2 + \frac{1}{2} \|\nabla u_t^h(0)\|_2^2. \end{aligned}$$

By applying the Sobolev-Poincaré inequality, we find that

$$\begin{aligned} \frac{1}{3} \|u_t^h\|_3^3 + \frac{1}{2} \|\nabla u^h\|_2^2 + \frac{1}{2} \|\nabla u_t^h\|_2^2 + \gamma \int_0^t \|\nabla u_t^h(s)\|_2^2 ds \\ \leq \frac{1}{3} C_{3,\Omega} \|\nabla u_t^h(0)\|_2^3 + \frac{1}{2} \|\nabla u^h(0)\|_2^2 + \frac{1}{2} \|\nabla u_t^h(0)\|_2^2 \\ \leq C \left(\|\nabla u^h(0)\|_2^2 + \|\nabla u_t^h(0)\|_2^2 + \|\nabla u_t^h(0)\|_2^3 \right). \end{aligned} \quad (2.2.5)$$

Note that

$$\begin{aligned} \|\nabla u^h(0)\|_2^2 &= \|\nabla(P^h u_0)\|_2^2 = (\nabla(P^h u_0), \nabla(P^h u_0)) \\ &= (\nabla u_0, \nabla(P^h u_0)) = (\nabla u_0, \nabla u_0) \\ &= \|\nabla u_0\|_2^2, \end{aligned} \quad (2.2.6)$$

where we used the definition of P^h .

Similarly we have

$$\|\nabla u_t^h(0)\|_2^2 = \|\nabla(P^h u_1)\|_2^2 = \|\nabla u_1\|_2^2. \quad (2.2.7)$$

Substituting (2.2.6) and (2.2.7) in (2.2.5), we infer

$$\begin{aligned} & \frac{1}{3} \|u_t^h\|_3^3 + \frac{1}{2} \|\nabla u^h\|_2^2 + \frac{1}{2} \|\nabla u_t^h\|_2^2 + \gamma \int_0^t \|\nabla u_t^h(s)\|_2^2 ds \\ & \leq C (\|\nabla u_0\|_2^2 + \|\nabla u_1\|_2^2 + \|\nabla u_1\|_2^3) \\ & \leq C, \end{aligned} \quad (2.2.8)$$

where we used the regularity assumptions on u_0 and u_1 .

The second estimate: Taking $v^h = u_{tt}^h$ in (2.2.4), we obtain

$$\frac{\gamma}{2} \frac{d}{dt} \left(\|\nabla u_t^h\|_2^2 \right) + \|\nabla u_{tt}^h\|_2^2 + \int_{\Omega} |u_t^h| |u_{tt}^h|^2 dx = -(\nabla u^h, \nabla u_{tt}^h).$$

Applying Young's inequality yields

$$\frac{\gamma}{2} \frac{d}{dt} \left(\|\nabla u_t^h\|_2^2 \right) + \|\nabla u_{tt}^h\|_2^2 + \int_{\Omega} |u_t^h| |u_{tt}^h|^2 dx \leq \frac{1}{2} \|\nabla u^h\|_2^2 + \frac{1}{2} \|\nabla u_{tt}^h\|_2^2.$$

Therefore,

$$\gamma \frac{d}{dt} \left(\|\nabla u_t^h\|_2^2 \right) + \|\nabla u_{tt}^h\|_2^2 \leq \|\nabla u^h\|_2^2.$$

Integrating both sides over $(0, t)$, we find

$$\begin{aligned} \gamma \|\nabla u_t^h\|_2^2 + \int_0^t \|\nabla u_{tt}^h(s)\|_2^2 ds & \leq \|\nabla u_t^h(0)\|_2^2 + \int_0^t \|\nabla u^h(s)\|_2^2 ds \\ & \leq \|\nabla u_1\|_2^2 + \int_0^t \|\nabla u^h(s)\|_2^2 ds. \end{aligned}$$

Employing the boundedness of $\|u^h\|_{L^\infty(0, T, H_0^1(\Omega))}$ provided by (2.2.8) together with the regularity assumptions on u_1 , we conclude that

$$\int_0^t \|\nabla u_{tt}^h(s)\|_2^2 ds \leq C. \quad (2.2.9)$$

From estimates (2.2.8) and (2.2.9), it follows that Lemma 2.2.1 holds. $\square$

Proof of Theorem 2.2.1. In particular, (2.2.2) is true for all $v^h \in V^h$. Take $v = v^h \in V^h$ and subtract to get

$$\begin{aligned} & \left(|u_t| u_{tt} - |u_t^h| u_{tt}^h, v^h \right) + \left(\nabla u - \nabla u^h, \nabla v^h \right) + \gamma \left(\nabla u_t - \nabla u_t^h, \nabla v^h \right) \\ & + \left(\nabla u_{tt} - \nabla u_{tt}^h, \nabla v^h \right) = 0. \end{aligned} \quad (2.2.10)$$

Consider the decomposition of the error: $e^h = u^h - P^h u$. Therefore

$$\begin{aligned} & \left(\nabla e^h, \nabla v^h \right) + \gamma \left(\nabla e_t^h, \nabla v^h \right) + \left(\nabla e_{tt}^h, \nabla v^h \right) \\ & = \left(|u_t| u_{tt} - |u_t^h| u_{tt}^h, v^h \right) + \left(\nabla u - \nabla(P^h u), \nabla v^h \right) \\ & + \gamma \left(\nabla u_t - \nabla(P^h u_t), \nabla v^h \right) + \left(\nabla u_{tt} - \nabla(P^h u_{tt}), \nabla v^h \right). \end{aligned} \quad (2.2.11)$$

Taking $v^h = e_t^h$, we find

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 \right) + \gamma \|\nabla e_t^h\|_2^2 \\ &= \left(|u_t| u_{tt} - |u_t^h| u_{tt}^h, e_t^h \right) + \left(\nabla u - \nabla(P^h u), \nabla e_t^h \right) \\ & \quad + \gamma \left(\nabla u_t - \nabla(P^h u_t), \nabla e_t^h \right) + \left(\nabla u_{tt} - \nabla(P^h u_{tt}), \nabla e_t^h \right). \end{aligned} \quad (2.2.12)$$

By the definition of P^h we have

$$\begin{aligned} \left(\nabla u - \nabla(P^h u), \nabla e_t^h \right) &= \left(\nabla u, \nabla e_t^h \right) - \left(\nabla(P^h u), \nabla e_t^h \right) \\ &= \left(\nabla u, \nabla e_t^h \right) - \left(\nabla u, \nabla e_t^h \right) \\ &= 0. \end{aligned}$$

Similarly, we obtain

$$\left(\nabla u_t - \nabla(P^h u_t), \nabla e_t^h \right) = 0 \quad \text{and} \quad \left(\nabla u_{tt} - \nabla(P^h u_{tt}), \nabla e_t^h \right) = 0.$$

Combine these results with (2.2.12) to obtain

$$\frac{1}{2} \frac{d}{dt} \left(\|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 \right) + \gamma \|\nabla e_t^h\|_2^2 = \left(|u_t| u_{tt} - |u_t^h| u_{tt}^h, e_t^h \right).$$

Applying the identity

$$|a_1| b_1 - |a_2| b_2 = (|a_1| - |a_2|) b_1 + (b_1 - b_2) |a_2|, \quad (2.2.13)$$

we obtain that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 \right) + \gamma \|\nabla e_t^h\|_2^2 \\ &= \left((|u_t| - |u_t^h|) u_{tt}, e_t^h \right) + \left((u_{tt} - u_{tt}^h) |u_t^h|, e_t^h \right) \\ &\leq \int_{\Omega} |u_t - u_t^h| |e_t^h| |u_{tt}| dx + \left((u_{tt} - u_{tt}^h) |u_t^h|, e_t^h \right) \\ &\leq \int_{\Omega} |u_t - P^h u_t| |e_t^h| |u_{tt}| dx + \int_{\Omega} |e_t^h|^2 |u_{tt}| dx \\ & \quad + \left((u_{tt} - P^h u_{tt}) |u_t^h|, e_t^h \right) - \left(e_{tt}^h |u_t^h|, e_t^h \right). \end{aligned} \quad (2.2.14)$$

On the other hand, we have

$$\left(e_{tt}^h |u_t^h|, e_t^h \right) \geq \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} |u_t^h| |e_t^h|^2 dx \right) - \frac{1}{2} \int_{\Omega} |u_{tt}^h| |e_t^h|^2 dx. \quad (2.2.15)$$

Therefore, from (2.2.15) and (2.2.14), we get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \int_{\Omega} |u_t^h| |e_t^h|^2 dx \right) + \gamma \|\nabla e_t^h\|_2^2 \\
& \leq \int_{\Omega} |u_t - P^h u_t| |e_t^h| |u_{tt}| dx + \int_{\Omega} |e_t^h|^2 |u_{tt}| dx + \frac{1}{2} \int_{\Omega} |u_{tt}^h| |e_t^h|^2 dx \\
& \quad + \left((u_{tt} - P^h u_{tt}) |u_t^h|, e_t^h \right) \\
& \leq \int_{\Omega} |u_t - P^h u_t| |e_t^h| |u_{tt}| dx + \int_{\Omega} |e_t^h|^2 |u_{tt}| dx + \frac{1}{2} \int_{\Omega} |u_{tt}^h| |e_t^h|^2 dx \\
& \quad + \int_{\Omega} |u_{tt} - P^h u_{tt}| |u_t^h| |e_t^h| dx.
\end{aligned} \tag{2.2.16}$$

Applying Hölder inequality and considering $p > 0$ and $q > 0$, such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, it follows that

$$\begin{aligned}
& \int_{\Omega} |u_{tt}| |e_t^h|^2 dx \leq \|u_{tt}\|_p \|e_t^h\|_q \|e_t^h\|_2, \\
& \int_{\Omega} |u_t - P^h u_t| |e_t^h| |u_{tt}| dx \leq \|u_{tt}\|_p \|u_t - P^h u_t\|_q \|e_t^h\|_2, \\
& \int_{\Omega} |u_{tt} - P^h u_{tt}| |u_t^h| |e_t^h| dx \leq \|u_t^h\|_p \|u_{tt} - P^h u_{tt}\|_q \|e_t^h\|_2, \\
& \int_{\Omega} |u_{tt}^h| |e_t^h|^2 dx \leq \|u_{tt}^h\|_p \|e_t^h\|_q \|e_t^h\|_2.
\end{aligned}$$

Combining the above estimates with (2.2.16), we find that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \int_{\Omega} |u_t^h| |e_t^h|^2 dx \right) + \gamma \|\nabla e_t^h\|_2^2 \\
& \leq \left(\|u_{tt}\|_p \|u_t - P^h u_t\|_q + \|u_t^h\|_p \|u_{tt} - P^h u_{tt}\|_q \right) \|e_t^h\|_2 \\
& \quad + \left(\|u_{tt}\|_p + \frac{1}{2} \|u_{tt}^h\|_p \right) \|e_t^h\|_q \|e_t^h\|_2.
\end{aligned} \tag{2.2.17}$$

By [14] we can always choose p and q to make the continuous embeddings of $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$. Thus there are constants $C_1, C_2 > 0$ independent of t , such that

$$\begin{aligned}
& \|u_t^h\|_p \leq C_1 \|u_t^h\|_{H_0^1}, \quad \|u_t - P^h u_t\|_q \leq C_2 \|u_t - P^h u_t\|_{H_0^1}, \\
& \|u_{tt}\|_p \leq C_1 \|u_{tt}\|_{H_0^1}, \quad \|u_{tt} - P^h u_{tt}\|_q \leq C_2 \|u_{tt} - P^h u_{tt}\|_{H_0^1}, \\
& \|u_{tt}^h\|_p \leq C_1 \|u_{tt}^h\|_{H_0^1}.
\end{aligned} \tag{2.2.18}$$

Combining (2.2.17) with (2.2.18), we find

$$\frac{1}{2} \frac{d}{dt} \left(\|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \int_{\Omega} |u_t^h| |e_t^h|^2 dx \right) + \gamma \|\nabla e_t^h\|_2^2$$

$$\leq \left(\beta_1 \|u_t - P^h u_t\|_{H_0^1} + \beta_2 \|u_{tt} - P^h u_{tt}\|_{H_0^1} \right) \|e_t^h\|_2 + \beta_3 \|e_t^h\|_q \|e_t^h\|_2.$$

where

$$\begin{aligned} \beta_1 &= C_1 C_2 \|u_{tt}\|_{H_0^1}, & \beta_2 &= C_1 C_2 \|u_t^h\|_{H_0^1}, \\ \beta_3 &= C_1 \left(\|u_{tt}\|_{H_0^1} + \frac{1}{2} \|u_{tt}^h\|_{H_0^1} \right). \end{aligned}$$

Using the Sobolev-Poincaré inequality, we find

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \int_{\Omega} |u_t^h| |e_t^h|^2 dx \right) + \gamma \|\nabla e_t^h\|_2^2 \\ & \leq C_{\Omega} \left(\beta_1 \|u_t - P^h u_t\|_{H_0^1} + \beta_2 \|u_{tt} - P^h u_{tt}\|_{H_0^1} \right) \|\nabla e_t^h\|_2 \\ & \quad + \beta_3 C_{\Omega} C_{q,\Omega} \|\nabla e_t^h\|_2^2. \end{aligned}$$

Using Young's inequality, there exists $\delta_1 > 0$ such that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \int_{\Omega} |u_t^h| |e_t^h|^2 dx \right) + \gamma \|\nabla e_t^h\|_2^2 \\ & \leq C \left(\beta_1^2 \|u_t - P^h u_t\|_{H_0^1}^2 + \beta_2^2 \|u_{tt} - P^h u_{tt}\|_{H_0^1}^2 \right) \\ & \quad + \delta_1 \|\nabla e_t^h\|_2^2 + \beta_3 C_{\Omega} C_{q,\Omega} \|\nabla e_t^h\|_2^2. \end{aligned}$$

Now, choosing $\delta_1 = \gamma$, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \int_{\Omega} |u_t^h| |e_t^h|^2 dx \right) \\ & \leq C \left(\beta_1^2 \|u_t - P^h u_t\|_{H_0^1}^2 + \beta_2^2 \|u_{tt} - P^h u_{tt}\|_{H_0^1}^2 \right) + \beta_3 C_{\Omega} C_{q,\Omega} \|\nabla e_t^h\|_2^2. \end{aligned}$$

Integrating the last inequality from 0 to t and multiplying by 2 yields

$$\begin{aligned} & \|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \int_{\Omega} |u_t^h| |e_t^h|^2 dx \\ & \leq \|\nabla e^h(0)\|_2^2 + \|\nabla e_t^h(0)\|_2^2 + \int_{\Omega} |P^h u_1| |e_t^h(0)|^2 dx + C \int_0^t \beta_3(s) \|\nabla e_t^h(s)\|_2^2 ds \\ & \quad + C \int_0^t \left(\beta_1^2(s) \|u_t(s) - P^h u_t(s)\|_{H_0^1}^2 + \beta_2^2(s) \|u_{tt}(s) - P^h u_{tt}(s)\|_{H_0^1}^2 \right) ds. \end{aligned}$$

Applying Hölder inequality and considering $p > 0$ and $q > 0$, such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, it follows that

$$\int_{\Omega} |P^h u_1| |e_t^h(0)|^2 dx \leq \|P^h u_1\|_p \|e_t^h(0)\|_q \|e_t^h(0)\|_2.$$

Using the embeddings $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, we

obtain

$$\int_{\Omega} \left| P^h u_1 \right| \left| e_t^h(0) \right|^2 dx \leq C \left\| P^h u_1 \right\|_{H_0^1} \left\| e_t^h(0) \right\|_{H_0^1}^2. \quad (2.2.19)$$

Note that

$$\begin{aligned} \left\| e_t^h(0) \right\|_{H_0^1} &= \left\| u_t^h(0) - P^h u_t(0) \right\|_{H_0^1} \\ &= \left\| u_1 - P^h u_1 \right\|_{H_0^1} \\ &\leq Ch \left\| u_1 \right\|_{2,\Omega}, \end{aligned} \quad (2.2.20)$$

where we used the second inequality of (2.2.3).

Similarly we have

$$\left\| \nabla e^h(0) \right\|_2 \leq \left\| e^h(0) \right\|_{1,\Omega} \leq Ch \left\| u_0 \right\|_{2,\Omega}, \quad (2.2.21)$$

$$\left\| \nabla e_t^h(0) \right\|_2 \leq \left\| e_t^h(0) \right\|_{1,\Omega} \leq Ch \left\| u_1 \right\|_{2,\Omega}, \quad (2.2.22)$$

and

$$\left\| P^h u_1 \right\|_{H_0^1} \leq Ch \left\| u_1 \right\|_{2,\Omega} + C \left\| u_1 \right\|_{H_0^1}. \quad (2.2.23)$$

Combining estimates (2.2.19), (2.2.20), and (2.2.23), we infer that

$$\int_{\Omega} \left| P^h u_1 \right| \left| e_t^h(0) \right|^2 dx \leq Ch^2 \left(h \left\| u_1 \right\|_{2,\Omega} + \left\| u_1 \right\|_{H_0^1} \right) \left\| u_1 \right\|_{2,\Omega}^2. \quad (2.2.24)$$

Using the second inequality of (2.2.3), we get

$$\left\| u_t(s) - P^h u_t(s) \right\|_{H_0^1} \leq Ch \left\| u_t(s) \right\|_{2,\Omega}, \quad (2.2.25)$$

$$\left\| u_{tt}(s) - P^h u_{tt}(s) \right\|_{H_0^1} \leq Ch \left\| u_{tt}(s) \right\|_{2,\Omega}. \quad (2.2.26)$$

Putting (2.2.21), (2.2.22), (2.2.24), (2.2.25), and (2.2.26) together, we find that

$$\begin{aligned} \left\| \nabla e^h \right\|_2^2 + \left\| \nabla e_t^h \right\|_2^2 &\leq Ch^2 \left\{ \left\| u_0 \right\|_{2,\Omega}^2 + \left\| u_1 \right\|_{2,\Omega}^2 + h \left\| u_1 \right\|_{2,\Omega}^3 + \left\| u_1 \right\|_{H_0^1} \left\| u_1 \right\|_{2,\Omega}^2 \right. \\ &\quad \left. + \int_0^t \left(\beta_1^2(s) \left\| u_t(s) \right\|_{2,\Omega}^2 + \beta_2^2(s) \left\| u_{tt}(s) \right\|_{2,\Omega}^2 \right) ds \right. \\ &\quad \left. + C \int_0^t \beta_3(s) \left\| \nabla e_t^h(s) \right\|_2^2 ds \right\} \end{aligned}$$

Employing Hölder's inequality, it follows that

$$\begin{aligned} \left\| \nabla e^h \right\|_2^2 + \left\| \nabla e_t^h \right\|_2^2 &\leq Ch^2 \left\{ \left\| u_0 \right\|_{2,\Omega}^2 + \left\| u_1 \right\|_{2,\Omega}^2 + \left\| u_1 \right\|_{H_0^1} \left\| u_1 \right\|_{2,\Omega}^2 \right. \\ &\quad \left. + h \left\| u_1 \right\|_{2,\Omega}^3 + \left\| \beta_1 \right\|_{L^\infty(0,T)}^2 \left\| u_t \right\|_{L^2(0,T,H^2(\Omega))}^2 \right. \\ &\quad \left. + \left\| \beta_2 \right\|_{L^\infty(0,T)}^2 \left\| u_{tt} \right\|_{L^2(0,T,H^2(\Omega))}^2 \right. \\ &\quad \left. + C \left\| \beta_3 \right\|_{L^2(0,T)} \left(\int_0^t \left\| \nabla e_t^h(s) \right\|_2^4 ds \right)^{\frac{1}{2}} \right\}. \end{aligned}$$

Using Lemma 2.2.1 together with the regularity assumptions on u_0, u_1 , and u , we have

$$\|\nabla e^h\|_2^4 + \|\nabla e_t^h\|_2^4 \leq Ch^4 + C \int_0^t \|\nabla e_t^h(s)\|_2^4 ds.$$

Applying Gronwall's inequality and taking the essential supremum $t \in [0, T]$, we obtain

$$\|\nabla e^h\|_{L^\infty((0,T),L^2(\Omega))}^2 + \|\nabla e_t^h\|_{L^\infty((0,T),L^2(\Omega))}^2 \leq Ch^2.$$

Finally, we get

$$\|\nabla u - \nabla u^h\|_{L^\infty((0,T),L^2(\Omega))}^2 + \|\nabla u_t - \nabla u_t^h\|_{L^\infty((0,T),L^2(\Omega))}^2 \leq Ch^2.$$

This completes the proof of Theorem 2.2.1. $\square$

2.3 Fully-Discrete Approximation

In this section, we first present a fully discrete numerical scheme of the problem (2.1.1) and then perform the error analysis. Let $(n, N) \in \mathbb{N} \times \mathbb{N}^*$. Concerning the discretization in time, we divide the interval $[0, T]$ into N subintervals.

Let $\tau = \frac{T}{N}$ denote the time step. Define $t_n = n\tau$, for $n \in \{0, 1, 2, \dots, N\}$.

Given an integer $N > 0$, our numerical scheme can be stated as find u^n , $n = 2, \dots, N$ such that

$$\begin{cases} \frac{1}{\tau} \left(|\hat{u}^{n+1}| (\hat{u}^{n+1} - \hat{u}^n), v^h \right) + \left(\nabla u^{n+1}, \nabla v^h \right) + \gamma \left(\nabla \hat{u}^{n+1}, \nabla v^h \right) \\ \quad + \frac{1}{\tau} \left(\nabla \hat{u}^{n+1} - \nabla \hat{u}^n, \nabla v^h \right) = 0, \quad \forall v^h \in V^h, \\ u^0 = P^h u_0, \quad u^1 = u^0 + \tau P^h u_1, \end{cases} \quad (2.3.1)$$

where

$$\hat{u}^{n+1} = \frac{1}{\tau} (u^{n+1} - u^n).$$

We first establish two auxiliary lemmas that will be used in the proof of the main result of this section.

Lemma 2.3.1. *Let u^n , $0 \leq n \leq N$, be the fully discrete finite element solution defined in (2.3.1). If $u_0, u_1 \in H_0^1(\Omega)$, then we have*

$$\frac{2}{3} \|\hat{u}^{n+1}\|_3^3 + \|\nabla u^{n+1}\|_2^2 + \|\nabla \hat{u}^{n+1}\|_2^2 \leq C, \quad \forall 0 \leq n \leq N-1. \quad (2.3.2)$$

Proof. Taking $v^h = \hat{u}^{n+1}$ in (2.3.1), we get

$$\begin{aligned} & \frac{1}{\tau} \left(|\hat{u}^{n+1}| (\hat{u}^{n+1} - \hat{u}^n), \hat{u}^{n+1} \right) + \frac{1}{\tau} \left(\nabla u^{n+1}, \nabla \hat{u}^{n+1} - \nabla u^n \right) \\ & + \frac{1}{\tau} \left(\nabla \hat{u}^{n+1} - \nabla \hat{u}^n, \nabla \hat{u}^{n+1} \right) + \gamma \|\nabla \hat{u}^{n+1}\|_2^2 = 0. \end{aligned}$$

Using the elementary equality

$$(a, b) = \frac{1}{2} (\|a\|^2 + \|b\|^2 - \|a - b\|^2), \quad (2.3.3)$$

we have

$$\begin{aligned} & \frac{1}{\tau} (|\hat{u}^{n+1}| (\hat{u}^{n+1} - \hat{u}^n), \hat{u}^{n+1}) + \frac{1}{2\tau} (\|\nabla u^{n+1}\|_2^2 - \|\nabla u^n\|_2^2 + \|\nabla u^{n+1} - \nabla u^n\|_2^2) \\ & + \frac{1}{2\tau} (\|\nabla \hat{u}^{n+1}\|_2^2 - \|\nabla \hat{u}^n\|_2^2 + \|\nabla \hat{u}^{n+1} - \nabla \hat{u}^n\|_2^2) + \gamma \|\nabla \hat{u}^{n+1}\|_2^2 = 0. \end{aligned}$$

It follows that

$$\begin{aligned} & 2(|\hat{u}^{n+1}| (\hat{u}^{n+1} - \hat{u}^n), \hat{u}^{n+1}) + \|\nabla u^{n+1}\|_2^2 - \|\nabla u^n\|_2^2 \\ & + \|\nabla \hat{u}^{n+1}\|_2^2 - \|\nabla \hat{u}^n\|_2^2 \leq 0. \end{aligned} \quad (2.3.4)$$

For the nonlinear term, we apply the inequality

$$|a| a(a - b) \geq \frac{1}{3} (|a|^3 - |b|^3), \quad \forall (a, b) \in \mathbb{R}^2 \quad (2.3.5)$$

to find that

$$(|\hat{u}^{n+1}| (\hat{u}^{n+1} - \hat{u}^n), \hat{u}^{n+1}) \geq \frac{1}{3} (\|\hat{u}^{n+1}\|_3^3 - \|\hat{u}^n\|_3^3). \quad (2.3.6)$$

From (2.3.4) and (2.3.6), we have

$$\frac{2}{3} (\|\hat{u}^{n+1}\|_3^3 - \|\hat{u}^n\|_3^3) + \|\nabla u^{n+1}\|_2^2 - \|\nabla u^n\|_2^2 + \|\nabla \hat{u}^{n+1}\|_2^2 - \|\nabla \hat{u}^n\|_2^2 \leq 0.$$

Summing up the last inequality for n , we can get

$$\frac{2}{3} \|\hat{u}^{n+1}\|_3^3 + \|\nabla u^{n+1}\|_2^2 + \|\nabla \hat{u}^{n+1}\|_2^2 \leq \frac{2}{3} \|\hat{u}^1\|_3^3 + \|\nabla u^1\|_2^2 + \|\nabla \hat{u}^1\|_2^2.$$

Applying the Sobolev-Poincaré inequality, we obtain

$$\frac{2}{3} \|\hat{u}^{n+1}\|_3^3 + \|\nabla u^{n+1}\|_2^2 + \|\nabla \hat{u}^{n+1}\|_2^2 \leq \frac{2}{3} C_{3,\Omega} \|\nabla \hat{u}^1\|_2^3 + \|\nabla u^1\|_2^2 + \|\nabla \hat{u}^1\|_2^2. \quad (2.3.7)$$

Next, we estimate $\|\nabla u^1\|_2$ and $\|\nabla \hat{u}^1\|_2$. Using (2.2.6) and (2.2.7), together with the definitions of u^1 and $\hat{u}^1$, we obtain

$$\|\nabla u^1\|_2 = \left\| \nabla (P^h u_0) + \tau \nabla (P^h u_1) \right\|_2 \leq C (\|\nabla u_0\|_2 + \|\nabla u_1\|_2),$$

and

$$\|\nabla \hat{u}^1\|_2 = \left\| \nabla (P^h u_1) \right\|_2 = \|\nabla u_1\|_2.$$

Since $u_0, u_1 \in H_0^1(\Omega)$, we conclude that

$$\|\nabla u^1\|_2 \leq C, \quad \text{and} \quad \|\nabla \hat{u}^1\|_2 \leq C. \quad (2.3.8)$$

Therefore, (2.3.2) follows from (2.3.7) and (2.3.8). $\square$

Remark 2.3.1. Inequality (2.3.5) follows from the convexity of $F(x) = \frac{1}{3}|x|^3$, since for all $(a, b) \in \mathbb{R}^2$ we have

$$F'(a)(a - b) \geq F(a) - F(b).$$

Lemma 2.3.2. Let u^n , $0 \leq n \leq N$, be the fully discrete finite element solution defined in (2.3.1). If $u_0, u_1 \in H_0^1(\Omega)$, then we have

$$\|\nabla \hat{u}^{n+1} - \nabla \hat{u}^n\|_2 \leq C\tau, \quad \forall \quad 0 \leq n \leq N-1. \quad (2.3.9)$$

Proof. Choosing $v^h = \hat{u}^{n+1} - \hat{u}^n$ in (2.3.1), we have

$$\begin{aligned} \frac{1}{\tau} \|\nabla \hat{u}^{n+1} - \nabla \hat{u}^n\|_2^2 &= -(\nabla u^{n+1}, \nabla \hat{u}^{n+1} - \nabla \hat{u}^n) - \gamma(\nabla \hat{u}^{n+1}, \nabla \hat{u}^{n+1} - \nabla \hat{u}^n) \\ &\quad - \frac{1}{\tau} \int_{\Omega} |\hat{u}^{n+1}| (\hat{u}^{n+1} - \hat{u}^n)^2 dx \\ &\leq -(\nabla u^{n+1}, \nabla \hat{u}^{n+1} - \nabla \hat{u}^n) - \gamma(\nabla \hat{u}^{n+1}, \nabla \hat{u}^{n+1} - \nabla \hat{u}^n). \end{aligned}$$

By a simple application of the Cauchy–Schwarz inequality, one obtains

$$\frac{1}{\tau} \|\nabla \hat{u}^{n+1} - \nabla \hat{u}^n\|_2^2 \leq (\|\nabla u^{n+1}\|_2 + \gamma \|\nabla \hat{u}^{n+1}\|_2) \|\nabla \hat{u}^{n+1} - \nabla \hat{u}^n\|_2,$$

which implies that

$$\|\nabla \hat{u}^{n+1} - \nabla \hat{u}^n\|_2 \leq \tau (\|\nabla u^{n+1}\|_2 + \gamma \|\nabla \hat{u}^{n+1}\|_2).$$

Using (2.3.2) yields

$$\|\nabla \hat{u}^{n+1} - \nabla \hat{u}^n\|_2 \leq C\tau.$$

□

Now, we are in a position to prove the error estimate for the convergence of the approximate solution of (2.3.1) to the weak solution of (2.2.2). The following theorem is the main result of this section.

Theorem 2.3.1. Suppose that the assumptions of Theorem 2.2.1 hold, with the additional condition that $u \in W^{2,\infty}(0, T, H^2(\Omega))$. If $\tau < \left(2(C_{3,\Omega})^3 M_1\right)^{-1}$, then we have the following estimate

$$\|\nabla u(t_n) - \nabla u^n\|_{L^\infty((0,T),L^2(\Omega))}^2 + \|\nabla u_t(t_n) - \nabla \hat{u}^n\|_{L^\infty((0,T),L^2(\Omega))}^2 \leq C(h^2 + \tau^2),$$

where

$$M_1 = \max_{1 \leq n \leq N-1} \left\{ \frac{\|\nabla \hat{u}^{n+1} - \nabla \hat{u}^n\|_2}{\tau} \right\}.$$

Proof. We denote $\hat{u}(t) = u_t(t)$, and then we use the standard decompositions

$$\begin{cases} u^n - u(t_n) = e^n + \rho^n, \\ \hat{u}^n - \hat{u}(t_n) = \hat{e}^n + \hat{\rho}^n, \end{cases} \quad \text{where} \quad \begin{cases} e^n = u^n - P^h u(t_n), & \rho^n = P^h u(t_n) - u(t_n), \\ \hat{e}^n = \hat{u}^n - P^h \hat{u}(t_n), & \hat{\rho}^n = P^h \hat{u}(t_n) - \hat{u}(t_n). \end{cases}$$

Using (2.2.2) with $t = t_{n+1}$ and $v = v^h$, we get

$$\begin{aligned} & \left(|\hat{u}(t_{n+1})| \hat{u}_t(t_{n+1}), v^h \right) + \left(\nabla u(t_{n+1}), \nabla v^h \right) + \gamma \left(\nabla \hat{u}(t_{n+1}), \nabla v^h \right) \\ & + \left(\nabla \hat{u}_t(t_{n+1}), \nabla v^h \right) = 0, \quad \forall v^h \in V^h, \end{aligned} \quad (2.3.10)$$

Subtracting (2.3.10) from (2.3.1)₁, we find

$$\begin{aligned} & \left(|\hat{u}^{n+1}| \frac{\hat{u}^{n+1} - \hat{u}^n}{\tau} - |\hat{u}(t_{n+1})| \hat{u}_t(t_{n+1}), v^h \right) + \left(\nabla u^{n+1} - \nabla u(t_{n+1}), \nabla v^h \right) \\ & + \gamma \left(\nabla \hat{u}^{n+1} - \nabla \hat{u}(t_{n+1}), \nabla v^h \right) + \left(\frac{\nabla \hat{u}^{n+1} - \nabla \hat{u}^n}{\tau} - \nabla \hat{u}_t(t_{n+1}), \nabla v^h \right) = 0. \end{aligned}$$

Applying the identity (2.2.13), we obtain

$$\begin{aligned} & \left((|\hat{u}^{n+1}| - |\hat{u}(t_{n+1})|) \frac{\hat{u}^{n+1} - \hat{u}^n}{\tau} + \left(\frac{\hat{u}^{n+1} - \hat{u}^n}{\tau} - \hat{u}_t(t_{n+1}) \right) |\hat{u}(t_{n+1})|, v^h \right) \\ & + \left(\nabla u^{n+1} - \nabla u(t_{n+1}), \nabla v^h \right) + \gamma \left(\nabla \hat{u}^{n+1} - \nabla \hat{u}(t_{n+1}), \nabla v^h \right) \\ & + \left(\frac{\nabla \hat{u}^{n+1} - \nabla \hat{u}^n}{\tau} - \nabla \hat{u}_t(t_{n+1}), \nabla v^h \right) = 0, \quad \forall v^h \in V^h. \end{aligned}$$

Hence

$$\begin{aligned} & \frac{1}{\tau} \left((\hat{e}^{n+1} - \hat{e}^n) |\hat{u}(t_{n+1})|, v^h \right) + \left(\nabla e^{n+1} + \gamma \nabla \hat{e}^{n+1}, \nabla v^h \right) + \frac{1}{\tau} \left(\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla v^h \right) \\ & = \left(\nabla \hat{u}_t(t_{n+1}) - \frac{\nabla(P^h \hat{u}(t_{n+1})) - \nabla(P^h \hat{u}(t_n))}{\tau}, \nabla v^h \right) - \left(\nabla \rho^{n+1} + \gamma \nabla \hat{\rho}^{n+1}, \nabla v^h \right) \\ & + \left(\left(\hat{u}_t(t_{n+1}) - \frac{P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)}{\tau} \right) |\hat{u}(t_{n+1})|, v^h \right) \\ & + \left((|\hat{u}(t_{n+1})| - |\hat{u}^{n+1}|) \frac{\hat{u}^{n+1} - \hat{u}^n}{\tau}, v^h \right). \end{aligned}$$

By the definition of P^h we have

$$\begin{aligned} & \frac{1}{\tau} \left((\hat{e}^{n+1} - \hat{e}^n) |\hat{u}(t_{n+1})|, v^h \right) + \left(\nabla e^{n+1} + \gamma \nabla \hat{e}^{n+1}, \nabla v^h \right) + \frac{1}{\tau} \left(\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla v^h \right) \\ & = \left(\nabla \hat{u}_t(t_{n+1}) - \frac{\nabla \hat{u}(t_{n+1}) - \nabla \hat{u}(t_n)}{\tau}, \nabla v^h \right) + \left((|\hat{u}(t_{n+1})| - |\hat{u}^{n+1}|) \frac{\hat{u}^{n+1} - \hat{u}^n}{\tau}, v^h \right) \\ & + \left(\left(\hat{u}_t(t_{n+1}) - \frac{P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)}{\tau} \right) |\hat{u}(t_{n+1})|, v^h \right) - \left(\nabla \rho^{n+1} + \gamma \nabla \hat{\rho}^{n+1}, \nabla v^h \right). \end{aligned}$$

Now, taking $v^h = \hat{e}^{n+1}$, we get

$$\begin{aligned} & \frac{1}{\tau} \left((\hat{e}^{n+1} - \hat{e}^n) |\hat{u}(t_{n+1})|, \hat{e}^{n+1} \right) + \frac{1}{\tau} \left(\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla \hat{e}^{n+1} \right) + \left(\nabla e^{n+1}, \nabla \hat{e}^{n+1} \right) \\ & + \gamma \left\| \nabla \hat{e}^{n+1} \right\|_2^2 + \left(\nabla \rho^{n+1} + \gamma \nabla \hat{\rho}^{n+1}, \nabla \hat{e}^{n+1} \right) \end{aligned}$$

$$\begin{aligned}
&= \left(\nabla \hat{u}_t(t_{n+1}) - \frac{\nabla \hat{u}(t_{n+1}) - \nabla \hat{u}(t_n)}{\tau}, \nabla \hat{e}^{n+1} \right) + \left((|\hat{u}(t_{n+1})| - |\hat{u}^{n+1}|) \frac{\hat{u}^{n+1} - \hat{u}^n}{\tau}, \hat{e}^{n+1} \right) \\
&\quad + \left(\left(\hat{u}_t(t_{n+1}) - \frac{P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)}{\tau} \right) |\hat{u}(t_{n+1})|, \hat{e}^{n+1} \right).
\end{aligned}$$

Employing the inequality $a(a-b) \geq \frac{1}{2}(a^2 - b^2)$, for all $(a, b) \in \mathbb{R}^2$, we find that

$$\begin{aligned}
&\frac{1}{2\tau} \left(|\hat{e}^{n+1}|^2 - |\hat{e}^n|^2, |\hat{u}(t_{n+1})| \right) + \frac{1}{\tau} (\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla \hat{e}^{n+1}) + (\nabla \hat{e}^{n+1}, \nabla \hat{e}^{n+1}) \\
&\quad + \gamma \|\nabla \hat{e}^{n+1}\|_2^2 + (\nabla \rho^{n+1} + \gamma \nabla \hat{\rho}^{n+1}, \nabla \hat{e}^{n+1}) \\
&\leq \left(\nabla \hat{u}_t(t_{n+1}) - \frac{\nabla \hat{u}(t_{n+1}) - \nabla \hat{u}(t_n)}{\tau}, \nabla \hat{e}^{n+1} \right) + \left((|\hat{u}(t_{n+1})| - |\hat{u}^{n+1}|) \frac{\hat{u}^{n+1} - \hat{u}^n}{\tau}, \hat{e}^{n+1} \right) \\
&\quad + \left(\left(\hat{u}_t(t_{n+1}) - \frac{P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)}{\tau} \right) |\hat{u}(t_{n+1})|, \hat{e}^{n+1} \right).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\frac{1}{2\tau} \left(|\hat{e}^{n+1}|^2, |\hat{u}(t_{n+1})| \right) - \frac{1}{2\tau} \left(|\hat{e}^n|^2, |\hat{u}(t_n)| \right) + \frac{1}{\tau} (\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla \hat{e}^{n+1}) \\
&\quad + (\nabla \hat{e}^{n+1}, \nabla \hat{e}^{n+1}) + \gamma \|\nabla \hat{e}^{n+1}\|_2^2 + (\nabla \rho^{n+1} + \gamma \nabla \hat{\rho}^{n+1}, \nabla \hat{e}^{n+1}) \\
&\leq \left((|\hat{u}(t_{n+1})| - |\hat{u}^{n+1}|) \frac{\hat{u}^{n+1} - \hat{u}^n}{\tau}, \hat{e}^{n+1} \right) + \frac{1}{2\tau} \left(|\hat{e}^n|^2, |\hat{u}(t_{n+1})| - |\hat{u}(t_n)| \right) \\
&\quad + \left(\left(\hat{u}_t(t_{n+1}) - \frac{P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)}{\tau} \right) |\hat{u}(t_{n+1})|, \hat{e}^{n+1} \right) \tag{2.3.11} \\
&\quad + \left(\nabla \hat{u}_t(t_{n+1}) - \frac{\nabla \hat{u}(t_{n+1}) - \nabla \hat{u}(t_n)}{\tau}, \nabla \hat{e}^{n+1} \right) \\
&\leq \frac{1}{\tau} (|\hat{e}^{n+1} + \hat{\rho}^{n+1}| |\hat{u}^{n+1} - \hat{u}^n|, |\hat{e}^{n+1}|) + \frac{1}{2\tau} \left(|\hat{e}^n|^2, \left| \int_{t_n}^{t_{n+1}} \hat{u}_t(s) ds \right| \right) \\
&\quad + \left(\left(\hat{u}_t(t_{n+1}) - \frac{P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)}{\tau} \right) |\hat{u}(t_{n+1})|, \hat{e}^{n+1} \right) \\
&\quad + \left(\nabla \hat{u}_t(t_{n+1}) - \frac{\nabla \hat{u}(t_{n+1}) - \nabla \hat{u}(t_n)}{\tau}, \nabla \hat{e}^{n+1} \right).
\end{aligned}$$

Applying Hölder, Sobolev-Poincaré, and Minkowski inequalities, it follows that

$$\begin{aligned}
&(|\hat{e}^{n+1} + \hat{\rho}^{n+1}| |\hat{u}^{n+1} - \hat{u}^n|, |\hat{e}^{n+1}|) + \frac{1}{2} \left(|\hat{e}^n|^2, \left| \int_{t_n}^{t_{n+1}} \hat{u}_t(s) ds \right| \right) \\
&\leq \|\hat{e}^{n+1} + \hat{\rho}^{n+1}\|_3 \|\hat{u}^{n+1} - \hat{u}^n\|_3 \|\hat{e}^{n+1}\|_3 + \frac{1}{2} \|\hat{e}^n\|_4^2 \left\| \int_{t_n}^{t_{n+1}} \hat{u}_t(s) ds \right\|_2 \\
&\leq (C_{3,\Omega})^3 \|\nabla \hat{u}^{n+1} - \nabla \hat{u}^n\|_2 \left(\|\nabla \hat{e}^{n+1}\|_2^2 + \|\nabla \hat{\rho}^{n+1}\|_2 \|\nabla \hat{e}^{n+1}\|_2 \right) \tag{2.3.12} \\
&\quad + \frac{(C_{4,\Omega})^2}{2} \left\| \hat{u}_t(s) \right\|_{L^\infty(t_n, t_{n+1})} \int_{t_n}^{t_{n+1}} ds \left\| \nabla \hat{e}^n \right\|_2^2 \\
&\leq (C_{3,\Omega})^3 M_1 \tau \left(\|\nabla \hat{e}^{n+1}\|_2^2 + \|\nabla \hat{\rho}^{n+1}\|_2 \|\nabla \hat{e}^{n+1}\|_2 \right)
\end{aligned}$$

$$+ \frac{(C_{4,\Omega})^2}{2} M_2 \tau \|\nabla \hat{e}^n\|_2^2,$$

where

$$M_2 = \|\hat{u}_t(s)\|_{L^\infty(0,T,L^2(\Omega))}.$$

Similarly, we obtain

$$\begin{aligned} & \left(\left(\hat{u}_t(t_{n+1}) - \frac{P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)}{\tau} \right) | \hat{u}(t_{n+1}) |, \hat{e}^{n+1} \right) \\ & \leq \left\| \hat{u}_t(t_{n+1}) - \frac{P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)}{\tau} \right\|_2 \|\hat{u}(t_{n+1})\|_4 \|\hat{e}^{n+1}\|_4 \\ & \leq (C_{4,\Omega})^2 \left\| \hat{u}_t(t_{n+1}) - \frac{P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)}{\tau} \right\|_2 \|\nabla \hat{u}(t_{n+1})\|_2 \|\nabla \hat{e}^{n+1}\|_2 \\ & \leq (C_{4,\Omega})^2 M_3 J_1 \|\nabla \hat{e}^{n+1}\|_2, \end{aligned} \quad (2.3.13)$$

where

$$M_3 = \|\nabla \hat{u}\|_{L^\infty(0,T,L^2(\Omega))}, \text{ and } J_1 = \left\| \hat{u}_t(t_{n+1}) - \frac{P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)}{\tau} \right\|_2.$$

Putting (2.3.11), (2.3.12), and (2.3.13) together, we infer

$$\begin{aligned} & \frac{1}{2\tau} \left[\left(|\hat{e}^{n+1}|^2, |\hat{u}(t_{n+1})| \right) - \left(|\hat{e}^n|^2, |\hat{u}(t_n)| \right) + 2(\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla \hat{e}^{n+1}) \right] \\ & + (\nabla e^{n+1}, \nabla \hat{e}^{n+1}) + \gamma \|\nabla \hat{e}^{n+1}\|_2^2 + (\nabla \rho^{n+1} + \gamma \nabla \hat{\rho}^{n+1}, \nabla \hat{e}^{n+1}) \\ & \leq (C_{3,\Omega})^3 M_1 \left(\|\nabla \hat{e}^{n+1}\|_2^2 + \|\nabla \hat{\rho}^{n+1}\|_2 \|\nabla \hat{e}^{n+1}\|_2 \right) + \frac{(C_{4,\Omega})^2}{2} M_2 \|\nabla \hat{e}^n\|_2^2 \\ & + (C_{4,\Omega})^2 M_3 J_1 \|\nabla \hat{e}^{n+1}\|_2 + \left(\nabla \hat{u}_t(t_{n+1}) - \frac{\nabla \hat{u}(t_{n+1}) - \nabla \hat{u}(t_n)}{\tau}, \nabla \hat{e}^{n+1} \right). \end{aligned} \quad (2.3.14)$$

Knowing that

$$\begin{aligned} \hat{e}^{n+1} &= \hat{u}^{n+1} - P^h \hat{u}(t_{n+1}) \\ &= \frac{u^{n+1} - u^n}{\tau} - P^h \hat{u}(t_{n+1}) \\ &= \frac{e^{n+1} - e^n}{\tau} + \frac{P^h u(t_{n+1}) - P^h u(t_n)}{\tau} - P^h \hat{u}(t_{n+1}), \end{aligned}$$

hence

$$\begin{aligned} (\nabla e^{n+1}, \nabla \hat{e}^{n+1}) &= \left(\nabla e^{n+1}, \frac{\nabla(P^h u(t_{n+1})) - \nabla(P^h u(t_n))}{\tau} - \nabla(P^h \hat{u}(t_{n+1})) \right) \\ &+ \frac{1}{\tau} (\nabla e^{n+1}, \nabla e^{n+1} - \nabla e^n) \\ &= \left(\nabla e^{n+1}, \frac{\nabla u(t_{n+1}) - \nabla u(t_n)}{\tau} - \nabla \hat{u}(t_{n+1}) \right) \end{aligned} \quad (2.3.15)$$

$$+ \frac{1}{\tau} (\nabla e^{n+1} - \nabla e^n, \nabla e^{n+1}),$$

where we used the definition of P^h .

By substituting (2.3.15) into (2.3.14) yields

$$\begin{aligned} & \frac{1}{2\tau} \left[\left(|\hat{e}^{n+1}|^2, |\hat{u}(t_{n+1})| \right) - \left(|\hat{e}^n|^2, |\hat{u}(t_n)| \right) + 2(\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla \hat{e}^{n+1}) \right. \\ & \quad \left. + 2(\nabla e^{n+1} - \nabla e^n, \nabla e^{n+1}) \right] + \gamma \|\nabla \hat{e}^{n+1}\|_2^2 \\ & \leq (C_{3,\Omega})^3 M_1 \left(\|\nabla \hat{e}^{n+1}\|_2^2 + \|\nabla \hat{\rho}^{n+1}\|_2 \|\nabla \hat{e}^{n+1}\|_2 \right) + \frac{(C_{4,\Omega})^2}{2} M_2 \|\nabla \hat{e}^n\|_2^2 \\ & \quad + (C_{4,\Omega})^2 M_3 J_1 \|\nabla \hat{e}^{n+1}\|_2 + \left(\nabla \hat{u}_t(t_{n+1}) - \frac{\nabla \hat{u}(t_{n+1}) - \nabla \hat{u}(t_n)}{\tau}, \nabla \hat{e}^{n+1} \right) \\ & \quad + \left(\nabla \hat{u}(t_{n+1}) - \frac{\nabla u(t_{n+1}) - \nabla u(t_n)}{\tau}, \nabla e^{n+1} \right) - (\nabla \rho^{n+1} + \gamma \nabla \hat{\rho}^{n+1}, \nabla \hat{e}^{n+1}). \end{aligned}$$

Using Young's inequality, there exist positive constants δ_2 and δ_3 such that

$$\begin{aligned} & \frac{1}{2\tau} \left[\left(|\hat{e}^{n+1}|^2, |\hat{u}(t_{n+1})| \right) - \left(|\hat{e}^n|^2, |\hat{u}(t_n)| \right) + 2(\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla \hat{e}^{n+1}) \right. \\ & \quad \left. + 2(\nabla e^{n+1} - \nabla e^n, \nabla e^{n+1}) \right] + \gamma \|\nabla \hat{e}^{n+1}\|_2^2 \\ & \leq C \left(J_1^2 + J_2^2 + J_3^2 + \|\nabla \rho^{n+1}\|_2^2 + \|\nabla \hat{\rho}^{n+1}\|_2^2 \right) + (C_{3,\Omega})^3 M_1 \|\nabla \hat{e}^{n+1}\|_2^2 \\ & \quad + \frac{(C_{4,\Omega})^2}{2} M_2 \|\nabla \hat{e}^n\|_2^2 + \delta_2 \|\nabla e^{n+1}\|_2^2 + \delta_3 \|\nabla \hat{e}^{n+1}\|_2^2, \end{aligned}$$

where

$$\begin{cases} J_2 = \left\| \nabla \hat{u}_t(t_{n+1}) - \frac{\nabla \hat{u}(t_{n+1}) - \nabla \hat{u}(t_n)}{\tau} \right\|_2, \\ J_3 = \left\| \nabla \hat{u}(t_{n+1}) - \frac{\nabla u(t_{n+1}) - \nabla u(t_n)}{\tau} \right\|_2. \end{cases}$$

By choosing $\delta_2 = (C_{3,\Omega})^3 M_1$, and $\delta_3 = \gamma$, we have

$$\begin{aligned} & \frac{1}{2\tau} \left[\left(|\hat{e}^{n+1}|^2, |\hat{u}(t_{n+1})| \right) - \left(|\hat{e}^n|^2, |\hat{u}(t_n)| \right) + 2(\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla \hat{e}^{n+1}) \right. \\ & \quad \left. + 2(\nabla e^{n+1} - \nabla e^n, \nabla e^{n+1}) \right] \\ & \leq C \left(J_1^2 + J_2^2 + J_3^2 + \|\rho^{n+1}\|_{H_0^1}^2 + \|\hat{\rho}^{n+1}\|_{H_0^1}^2 \right) + \frac{(C_{4,\Omega})^2}{2} M_2 \|\nabla \hat{e}^n\|_2^2 \\ & \quad + (C_{3,\Omega})^3 M_1 \left(\|\nabla e^{n+1}\|_2^2 + \|\nabla \hat{e}^{n+1}\|_2^2 \right). \end{aligned} \tag{2.3.16}$$

Next, we estimate $\|\rho^{n+1}\|_{H_0^1}^2$, $\|\hat{\rho}^{n+1}\|_{H_0^1}^2$, J_1^2 , J_2^2 , and J_3^2 . Using (2.2.3)₂, we get

$$\|\rho^{n+1}\|_{H_0^1}^2 = \|P^h u(t_{n+1}) - u(t_{n+1})\|_{H_0^1}^2 \leq Ch^2 \|u(t_{n+1})\|_{2,\Omega}^2, \tag{2.3.17}$$

and

$$\|\hat{\rho}^{n+1}\|_{H_0^1}^2 = \|P^h \hat{u}(t_{n+1}) - \hat{u}(t_{n+1})\|_{H_0^1}^2 \leq Ch^2 \|\hat{u}(t_{n+1})\|_{2,\Omega}^2. \quad (2.3.18)$$

Invoking the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} J_2^2 &= \left\| \nabla \hat{u}_t(t_{n+1}) - \frac{\nabla \hat{u}(t_{n+1}) - \nabla \hat{u}(t_n)}{\tau} \right\|_2^2 \\ &= \left\| \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \nabla \hat{u}_{tt}(s) ds \right\|_2^2 \\ &\leq \frac{1}{\tau^2} \left(\int_{t_n}^{t_{n+1}} (s - t_n)^2 ds \right) \left(\int_{t_n}^{t_{n+1}} \|\nabla \hat{u}_{tt}(s)\|_2^2 ds \right) \\ &\leq \frac{\tau}{3} \int_{t_n}^{t_{n+1}} \|\nabla \hat{u}_{tt}(s)\|_2^2 ds. \end{aligned} \quad (2.3.19)$$

Similarly, we find the following estimate

$$J_3^2 \leq \frac{\tau}{3} \int_{t_n}^{t_{n+1}} \|\nabla \hat{u}_t(s)\|_2^2 ds. \quad (2.3.20)$$

For J_1 , Minkowski's inequality gives

$$\begin{aligned} J_1 &= \left\| \hat{u}_t(t_{n+1}) - \frac{P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)}{\tau} \right\|_2 \\ &= \left\| \frac{\hat{\rho}(t_{n+1}) - \hat{\rho}(t_n)}{\tau} + \frac{\hat{u}(t_{n+1}) - \hat{u}(t_n)}{\tau} - \hat{u}_t(t_{n+1}) \right\|_2 \\ &\leq \left\| \frac{\hat{\rho}(t_{n+1}) - \hat{\rho}(t_n)}{\tau} \right\|_2 + \left\| \frac{\hat{u}(t_{n+1}) - \hat{u}(t_n)}{\tau} - \hat{u}_t(t_{n+1}) \right\|_2 \\ &\leq \left\| \hat{\rho}_t(t_{n+1}) - \frac{\hat{\rho}(t_{n+1}) - \hat{\rho}(t_n)}{\tau} - \hat{\rho}_t(t_{n+1}) \right\|_2 + \left\| \frac{\hat{u}(t_{n+1}) - \hat{u}(t_n)}{\tau} - \hat{u}_t(t_{n+1}) \right\|_2 \\ &\leq \left\| \hat{\rho}_t(t_{n+1}) - \frac{\hat{\rho}(t_{n+1}) - \hat{\rho}(t_n)}{\tau} \right\|_2 + \left\| \frac{\hat{u}(t_{n+1}) - \hat{u}(t_n)}{\tau} - \hat{u}_t(t_{n+1}) \right\|_2 + \|\hat{\rho}_t(t_{n+1})\|_2. \end{aligned}$$

By the Poincaré inequality, we obtain

$$\begin{aligned} J_1^2 &\leq C \left(J_2^2 + \left\| \hat{\rho}_t(t_{n+1}) - \frac{\hat{\rho}(t_{n+1}) - \hat{\rho}(t_n)}{\tau} \right\|_2^2 + \|\hat{\rho}_t(t_{n+1})\|_2^2 \right) \\ &\leq C \left(\frac{\tau}{3} \int_{t_n}^{t_{n+1}} \|\nabla \hat{u}_{tt}(s)\|_2^2 ds + \left\| \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \hat{\rho}_{tt}(s) ds \right\|_2^2 + \|\hat{\rho}_t(t_{n+1})\|_2^2 \right) \\ &\leq C \left(\frac{\tau}{3} \int_{t_n}^{t_{n+1}} \|\nabla \hat{u}_{tt}(s)\|_2^2 ds + \frac{\tau}{3} \int_{t_n}^{t_{n+1}} \|\hat{\rho}_{tt}(s)\|_2^2 ds + \|\hat{\rho}_t(t_{n+1})\|_2^2 \right). \end{aligned}$$

Applying (2.2.3)₁, we find that

$$\|\hat{\rho}_t(s)\|_2 = \|(P^h \hat{u}(s))_t - \hat{u}_t(s)\|_2 \leq Ch^2 \|\hat{u}_t(s)\|_{2,\Omega},$$

and

$$\|\hat{\rho}_{tt}(s)\|_2 = \|(P^h \hat{u}(s))_{tt} - \hat{u}_{tt}(s)\|_2 \leq Ch^2 \|\hat{u}_{tt}(s)\|_{2,\Omega}.$$

Consequently,

$$J_1^2 \leq C \left(\tau \int_{t_n}^{t_{n+1}} (\|\nabla \hat{u}_{tt}(s)\|_2^2 + h^4 \|\hat{u}_{tt}(s)\|_{2,\Omega}^2) ds + h^4 \|\hat{u}_t(t_{n+1})\|_{2,\Omega}^2 \right). \quad (2.3.21)$$

It follows from (2.3.16)–(2.3.21) that

$$\begin{aligned} & \frac{1}{2\tau} \left[(|\hat{e}^{n+1}|^2, |\hat{u}(t_{n+1})|) - (|\hat{e}^n|^2, |\hat{u}(t_n)|) + 2(\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla \hat{e}^{n+1}) \right. \\ & \quad \left. + 2(\nabla e^{n+1} - \nabla e^n, \nabla e^{n+1}) \right] \\ & \leq C\tau \int_{t_n}^{t_{n+1}} (\|\nabla \hat{u}_t(s)\|_2^2 + \|\nabla \hat{u}_{tt}(s)\|_2^2 + h^4 \|\hat{u}_{tt}(s)\|_{2,\Omega}^2) ds \\ & \quad + Ch^2 (\|u(t_{n+1})\|_{2,\Omega}^2 + \|\hat{u}(t_{n+1})\|_{2,\Omega}^2 + h^2 \|\hat{u}_t(t_{n+1})\|_{2,\Omega}^2) \\ & \quad + \frac{(C_{4,\Omega})^2}{2} M_2 \|\nabla \hat{e}^n\|_2^2 + (C_{3,\Omega})^3 M_1 (\|\nabla e^{n+1}\|_2^2 + \|\nabla \hat{e}^{n+1}\|_2^2). \end{aligned}$$

Employing the elementary equality (2.3.3), we get

$$\begin{aligned} & \frac{1}{2\tau} \left[(|\hat{e}^{n+1}|^2, |\hat{u}(t_{n+1})|) - (|\hat{e}^n|^2, |\hat{u}(t_n)|) + \|\nabla e^{n+1}\|_2^2 - \|\nabla e^n\|_2^2 \right. \\ & \quad \left. + \|\nabla e^{n+1} - \nabla e^n\|_2^2 + \|\nabla \hat{e}^{n+1}\|_2^2 - \|\nabla \hat{e}^n\|_2^2 + \|\nabla \hat{e}^{n+1} - \nabla \hat{e}^n\|_2^2 \right] \\ & \leq C\tau \int_{t_n}^{t_{n+1}} (\|\nabla \hat{u}_t(s)\|_2^2 + \|\nabla \hat{u}_{tt}(s)\|_2^2 + h^4 \|\hat{u}_{tt}(s)\|_{2,\Omega}^2) ds \\ & \quad + Ch^2 (\|u(t_{n+1})\|_{2,\Omega}^2 + \|\hat{u}(t_{n+1})\|_{2,\Omega}^2 + h^2 \|\hat{u}_t(t_{n+1})\|_{2,\Omega}^2) \\ & \quad + \frac{(C_{4,\Omega})^2}{2} M_2 \|\nabla \hat{e}^n\|_2^2 + (C_{3,\Omega})^3 M_1 (\|\nabla e^{n+1}\|_2^2 + \|\nabla \hat{e}^{n+1}\|_2^2). \end{aligned}$$

Summing the last inequality over n , then after some manipulations, we obtain

$$\begin{aligned} & (1 - 2(C_{3,\Omega})^3 M_1 \tau) (\|\nabla e^{n+1}\|_2^2 + \|\nabla \hat{e}^{n+1}\|_2^2) \\ & \leq C\tau^2 \int_{t_1}^{t_{n+1}} (\|\nabla \hat{u}_t(s)\|_2^2 + \|\nabla \hat{u}_{tt}(s)\|_2^2 + h^4 \|\hat{u}_{tt}(s)\|_{2,\Omega}^2) ds \\ & \quad + C\tau h^2 \sum_{i=1}^n (\|u(t_{i+1})\|_{2,\Omega}^2 + \|\hat{u}(t_{i+1})\|_{2,\Omega}^2 + h^2 \|\hat{u}_t(t_{i+1})\|_{2,\Omega}^2) \\ & \quad + ((C_{4,\Omega})^2 M_2 + 2(C_{3,\Omega})^3 M_1) \tau \sum_{i=2}^n (\|\nabla \hat{e}^i\|_2^2 + \|\nabla e^i\|_2^2) + \|\nabla e^1\|_2^2 \\ & \quad + \|\nabla \hat{e}^1\|_2^2 + (|\hat{e}^1|^2, |\hat{u}(t_1)|). \end{aligned} \quad (2.3.22)$$

We now need to bound $\|\nabla e^1\|_2$, $\|\nabla \hat{e}^1\|_2$, and $(|\hat{e}^1|^2, |\hat{u}(t_1)|)$. Indeed,

$$\begin{aligned} (\nabla e^1, \nabla v) &= (\nabla u^1 - \nabla(P^h u(t_1)), \nabla v) \\ &= (\nabla u^0 + \tau \nabla(P^h u_1) - \nabla(P^h u(t_1)), \nabla v) \\ &= (\nabla(P^h u_0) + \tau \nabla(P^h u_1) - \nabla(P^h u(t_1)), \nabla v) \\ &= (\nabla u_0 + \tau \nabla u_1 - \nabla u(t_1), \nabla v) \\ &= (\nabla u(0) + \tau \nabla u_t(0) - \nabla u(t_1), \nabla v), \end{aligned}$$

where we used the definition of P^h , u^1 and u^0 .

Knowing that,

$$\nabla u(t_1) = \nabla u(0) + \tau \nabla u_t(0) + \frac{\tau^2}{2} \nabla u_{tt}(\tau_1), \quad \text{with } 0 \leq \tau_1 \leq t_1,$$

hence

$$(\nabla e^1, \nabla v) = \left(-\frac{\tau^2}{2} \nabla u_{tt}(\tau_1), \nabla v \right).$$

Choosing $v = e^1$, we obtain

$$\|\nabla e^1\|_2^2 \leq \frac{\tau^2}{2} \|\nabla u_{tt}(\tau_1)\|_2 \|\nabla e^1\|_2, \quad \text{that is, } \|\nabla e^1\|_2 \leq C\tau.$$

To bound $\|\nabla \hat{e}^1\|_2$, we use $\hat{u}^1 = \frac{u^1 - u^0}{\tau}$ and $u^1 = u^0 + \tau P^h u_1$ thus

$$\begin{aligned} (\nabla \hat{e}^1, \nabla v) &= (\nabla \hat{u}^1 - \nabla(P^h \hat{u}(t_1)), \nabla v) \\ &= (\nabla(P^h u_1) - \nabla(P^h \hat{u}(t_1)), \nabla v) \\ &= (\nabla(P^h u_1), \nabla v) - (\nabla(P^h \hat{u}(t_1)), \nabla v) \\ &= (\nabla u_1, \nabla v) - (\nabla \hat{u}(t_1), \nabla v) \\ &= (\nabla u_t(0) - \nabla u_t(t_1), \nabla v). \end{aligned}$$

Knowing that

$$\nabla u_t(t_1) = \nabla u_t(0) + \tau \nabla u_{tt}(\tau_2), \quad \text{with } 0 \leq \tau_2 \leq t_1,$$

hence

$$(\nabla \hat{e}^1, \nabla v) = (-\tau \nabla u_{tt}(\tau_2), \nabla v).$$

Choosing $v = \hat{e}^1$, we obtain

$$\|\nabla \hat{e}^1\|_2^2 \leq C\tau \|\nabla u_{tt}(\tau_2)\|_2 \|\nabla \hat{e}^1\|_2, \quad \text{that is, } \|\nabla \hat{e}^1\|_2 \leq C\tau.$$

To estimate the last term in inequality (2.3.22), we apply the Cauchy-Schwarz in-

equality together with the Sobolev-Poincaré inequality, which yields

$$\begin{aligned} \left(|\hat{e}^1|^2, |\hat{u}(t_1)| \right) &\leq \|\hat{u}(t_1)\|_2 \|\hat{e}^1\|_4^2 \\ &\leq C_{4,\Omega} \|\hat{u}(t_1)\|_2 \|\nabla \hat{e}^1\|_2^2 \\ &\leq C\tau^2. \end{aligned}$$

Collecting the previous estimates together with inequality (2.3.22), it follows that

$$\begin{aligned} &\left(1 - 2(C_{3,\Omega})^3 M_1 \tau\right) \left(\|\nabla e^{n+1}\|_2^2 + \|\nabla \hat{e}^{n+1}\|_2^2\right) \\ &\leq C(\tau^2 + h^2) + \left((C_{4,\Omega})^2 M_2 + 2(C_{3,\Omega})^3 M_1\right) \tau \sum_{i=2}^n \left(\|\nabla \hat{e}^i\|_2^2 + \|\nabla e^i\|_2^2\right). \end{aligned} \quad (2.3.23)$$

Assuming that

$$\tau < \left(2(C_{3,\Omega})^3 M_1\right)^{-1},$$

and applying the discrete Gronwall inequality in (2.3.23), we get

$$\|\nabla e^{n+1}\|_2^2 + \|\nabla \hat{e}^{n+1}\|_2^2 \leq C(\tau^2 + h^2).$$

Using the standard decompositions, we find that

$$\begin{aligned} \|\nabla u(t_n) - \nabla u^n\|_2^2 + \|\nabla u_t(t_n) - \nabla \hat{u}^n\|_2^2 &= \|\nabla e^n + \nabla \rho^n\|_2^2 + \|\nabla \hat{e}^n + \nabla \hat{\rho}^n\|_2^2 \\ &\leq C \left(\|\nabla e^n\|_2^2 + \|\nabla \hat{e}^n\|_2^2 + \|\nabla \rho^n\|_2^2 + \|\nabla \hat{\rho}^n\|_2^2 \right) \\ &\leq C(h^2 + \tau^2). \end{aligned}$$

Using the equivalence of norms in finite dimension, we obtain

$$\max_{1 \leq n \leq N} \left\{ \|\nabla u(t_n) - \nabla u^n\|_2^2 + \|\nabla u_t(t_n) - \nabla \hat{u}^n\|_2^2 \right\} \leq C(h^2 + \tau^2).$$

According to the norm definition, we infer

$$\|\nabla u(t_n) - \nabla u^n\|_{L^\infty(0,T),L^2(\Omega)}^2 + \|\nabla u_t(t_n) - \nabla \hat{u}^n\|_{L^\infty(0,T),L^2(\Omega)}^2 \leq C(h^2 + \tau^2).$$

This completes the proof of Theorem 2.3.1. $\square$

2.4 NUMERICAL RESULTS

We introduce the nonhomogeneous form of the nonlinear problem (2.1.1)

$$\begin{cases} |u_t| u_{tt} - \Delta u - \gamma \Delta u_t - \Delta u_{tt} = f, & \text{in } Q, \\ u = 0, & \text{on } \Sigma, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega. \end{cases} \quad (2.4.1)$$

where $f = f(x, t)$ is sufficiently regular external force.

2.4.1 Variational formulation and Galerkin method

Multiply the equation (2.4.1)₁ by a test function $v \in V$. Next, we apply Green's formulation with the boundary condition to obtain the following variational formulation, for almost all $t \in (0, T)$,

$$(|u_t| u_{tt}, v) + (\nabla u, \nabla v) + \gamma (\nabla u_t, \nabla v) + (\nabla u_{tt}, \nabla v) = (f, v), \quad \forall v \in V. \quad (2.4.2)$$

Let V_m be the vector subspace of V generated by $\{\phi_1, \phi_2, \dots, \phi_m\}$, where $\{\phi_i\}_{i=1}^\infty$ is an orthogonal basis to V . If $u^m(x, t) \in V_m$ then can be represented by

$$u^m(x, t) = \sum_{i=1}^m d_i(t) \phi_i(x), \quad \phi_i(x) \in V_m. \quad (2.4.3)$$

Restricting the variational formulation (2.4.2) to the subspace V_m and taking $u^m(x, t)$ as in (2.4.3) and $v = \phi_j$, we find that

$$\begin{aligned} & \left(\sum_{i=1}^m d'_i(t) \phi_i \left| \sum_{i=1}^m d''_i(t) \phi_i, \phi_j \right. \right) + \sum_{i=1}^m [d''_i(t) + \gamma d'_i(t) + d_i(t)] (\nabla \phi_i, \nabla \phi_j) \\ & = (f, \phi_j), \quad \text{for } j = 1, 2, \dots, m. \end{aligned} \quad (2.4.4)$$

Defining the matrice and vectors of (2.4.4) as

$$A_{ij} = (\nabla \phi_i, \nabla \phi_j), \quad F_j(t) = (f, \phi_j), \quad Q_j(d'(t), d''(t)) = \left(\sum_{i=1}^m d'_i(t) \phi_i \left| \sum_{i=1}^m d''_i(t) \phi_i, \phi_j \right. \right),$$

where $Q_j(d'(t), d''(t))$ is the vector that stores the nonlinearity.

Then, we can rewrite (2.4.4) in the following nonlinear system of ordinary differential equations

$$Ad''(t) + \gamma Ad'(t) + Ad(t) + Q(d'(t), d''(t)) = F(t), \quad \forall t \in [0, T]. \quad (2.4.5)$$

2.4.2 Fully Discrete approximation

In this subsection we introduce and analyze a fully discrete finite element method to (2.4.5).

Consider the discretization of the fixed time interval $[0, T]$, using a uniform time step of size $\tau = \frac{T}{N}$, then $t_n = n\tau$, for $n = 0, 1, 2, \dots, N$ and $d^n = d(t_n)$.

The first and second derivatives in time will be approximated by forward-difference and central difference respectively, given by

$$d'^n = \frac{1}{\tau} (d^{n+1} - d^n), \quad d''^n = \frac{1}{\tau^2} (d^{n+1} - 2d^n + d^{n-1}).$$

Taking $t = t_n$ in (2.4.5) and replacing the approximations and multiplying by τ^3 , we obtain the following iterative method

$$R_1 d^{n+1} + Q(d^{n+1}, d^n, d^{n-1}) = R_2 d^n + R_3 d^{n-1} + \tau^3 F^{n+1}, \quad (2.4.6)$$

where

$$\begin{aligned} d^0 &= P^h u_0, \quad d^1 = d^0 + \tau P^h u_1, \quad R_1 = (\tau + \gamma \tau^2 + \tau^3) A, \\ R_2 &= (2\tau + \gamma \tau^2) A, \quad R_3 = -\tau A, \end{aligned}$$

and the vector $Q(d^{n+1}, d^n, d^{n-1})$ representing the nonlinear term is defined by

$$Q(d^{n+1}, d^n, d^{n-1}) = \left(\left| \sum_{i=1}^m (d^{n+1} - d^n) \phi_i \right| \sum_{i=1}^m (d^{n+1} - 2d^n + d^{n-1}) \phi_i, \phi_j \right).$$

Using the initial conditions $d^0 = u(x, 0)$ and the Taylor approximations, we get

$$d^1 = d^0 + \tau u_t(x, 0),$$

where $u(x, 0)$ and $u_t(x, 0)$ are given by initial conditions.

Taking $n = 1$ in (2.4.6), we obtain

$$R_1 d^2 + Q(d^2, d^1, d^0) = R_2 d^1 + R_3 d^0 + \tau^3 F^2. \quad (2.4.7)$$

For $n = 1$, the terms d^0 and d^1 are known it is sufficient to solve the nonlinear system (2.4.7) to determine d^2 . Thereafter, for $n = 2, 3, \dots, N$ we will get by the same method the values of d^n , which are established by the iterative process (2.4.6) in each instant at the time $t_n = n\tau$.

The resulting nonlinear system is solved using the modified Newton method (see [7, 15]).

2.4.3 Numerical simulations

In this subsection, we consider $\Omega =]0, 1[\times]0, 1[$ and fix $T = 1$, with the following initial conditions

$$u_0(x) = \sin(\pi x) \sin(\pi y), \quad u_1(x) = -\sin(\pi x) \sin(\pi y), \quad (x, y) \in \Omega.$$

Inhomogeneous problem

In this context, by appropriately selecting the external force $f(x, t)$, the exact solution is known and can be defined as follows

$$u(x, t) = \sin(\pi x) \sin(\pi y) \exp(-t).$$

Tables 2.1 and 2.2 present the numerical errors and the corresponding convergence orders, respectively, of the proposed numerical method, evaluated in the $L^\infty(0, T, L^2(\Omega))$ norm. The computations were performed using uniform discretization steps, where $\Delta x = \Delta y = \tau$, for various mesh sizes as listed in the first column of the tables, and for different values of the parameter γ . It is important to note that the observed convergence order $p \approx 1$ in Table 2.2 is consistent with the theoretical estimate provided in Theorem 2.2.1, which demonstrates the effectiveness of solving the nonlinear system using the Modified Newton's Method.

$E_{L^\infty(0,T,L^2(\Omega))}$				
$\Delta x = \Delta y = \tau$ \backslash γ	1/2	1	3/2	2
2^{-3}	0.016268	0.009578	0.004543	0.000664
2^{-4}	0.007774	0.004569	0.002152	0.000293
2^{-5}	0.003700	0.002173	0.001019	0.000132
2^{-6}	0.001790	0.001051	0.000492	0.000062

Table 2.1 – Numerical Errors - Inhomogeneous problem: $\Delta x = \Delta y = \tau$

p				
$\Delta x = \Delta y = \tau$ \backslash γ	1/2	1	3/2	2
2^{-3}	–	–	–	–
2^{-4}	1.065308	1.067846	1.077967	1.180283
2^{-5}	1.071132	1.072190	1.078524	1.150363
2^{-6}	1.047566	1.047926	1.050424	1.090198

Table 2.2 – Orders of convergence - Inhomogeneous problem: $\Delta x = \Delta y = \tau$

Tables 2.3 and 2.4 illustrate the numerical errors and the convergence orders, respectively, of the proposed numerical method, evaluated in the $L^\infty(0, T, L^2(\Omega))$ norm. The computations were performed using squared spatial steps, where $\Delta x^2 = \Delta y^2 = \tau$, for various mesh sizes as listed in the first and the second columns of the tables, and for different values of the parameter γ . This setup aims to investigate the effect of non-uniform discretization on the accuracy and convergence behavior of the method. As shown in Table 2.4, the observed convergence order $p \approx 2$ is consistent with the theoretical estimate provided in Theorem 2.2.1, which demonstrates the effectiveness of solving the nonlinear system using the Modified Newton's Method.

$E_{L^\infty(0,T,L^2(\Omega))}$				
$\Delta x^2 = \Delta y^2 = \tau$ \backslash γ	1/2	1	3/2	2
2^{-4}	0.030934	0.017969	0.008184	0.000655
2^{-6}	0.009394	0.005451	0.002471	0.000182
2^{-8}	0.002465	0.001430	0.000648	0.000047
2^{-10}	0.000624	0.000362	0.000164	0.000012

Table 2.3 – Numerical Errors - Inhomogeneous problem: $\Delta x^2 = \Delta y^2 = \tau$

Homogeneous problem

For $f(x, t) = 0$, we obtain the problem (2.1.1). In this case, the exact solution is not known. So, we will suffice with presenting a graphical representation of the approximate solution.

p					
$\Delta x^2 = \Delta y^2 = \tau$	γ	1/2	1	3/2	2
2^{-4}		—	—	—	—
2^{-6}		1.719382	1.720917	1.727711	1.847556
2^{-8}		1.930152	1.930506	1.931029	1.953206
2^{-10}		1.981970	1.981954	1.982298	1.969626

Table 2.4 – Orders of convergence - Inhomogeneous problem: $\Delta x^2 = \Delta y^2 = \tau$

The Fig.2.1 show the graphs of the approximate solution for the variable $u^m(x, y, 0.5)$ where the mesh was $\Delta x = \Delta y = \tau = 1/32$ and $\Delta x = \Delta y = \tau = 1/64$, and for two values of the parameter γ , namely $\gamma = 1/2$ and $\gamma = 1$.

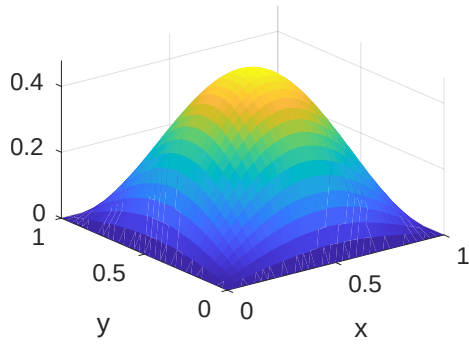
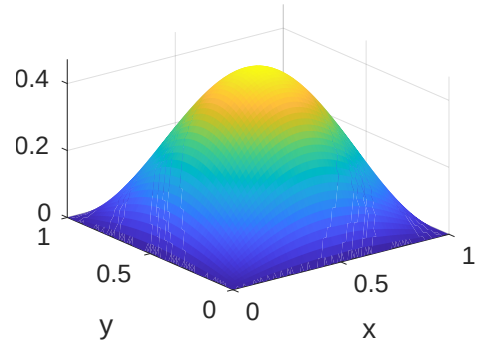
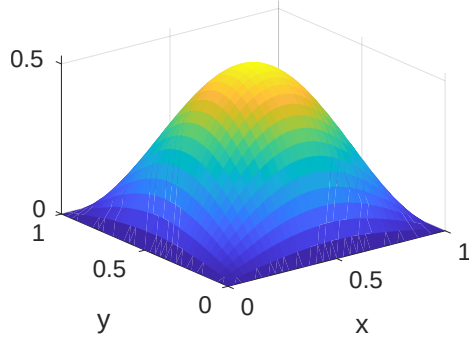
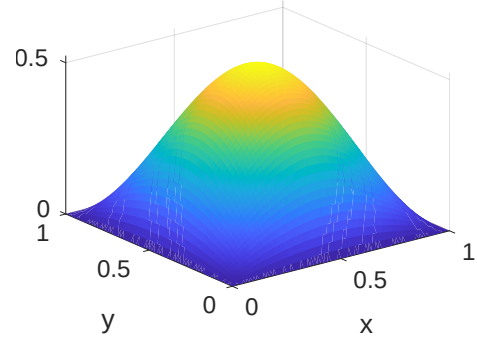
(a) $u^m(x, y, 0.5): \Delta x = \Delta y = \tau = 1/32$, $\gamma = 1/2$.(b) $u^m(x, y, 0.5): \Delta x = \Delta y = \tau = 1/64$, $\gamma = 1/2$.(c) $u^m(x, y, 0.5): \Delta x = \Delta y = \tau = 1/32$, $\gamma = 1$.(d) $u^m(x, y, 0.5): \Delta x = \Delta y = \tau = 1/64$, $\gamma = 1$.

Figure 2.1 – Graphs of the numerical solution for homogeneous problem.

Chapter 3

Discretization for a 2D viscoelastic wave equation with dynamic boundary conditions

3.1 Introduction

In this chapter, we consider the following viscoelastic wave equations with strong damping and dynamic boundary conditions

$$\begin{cases} u_{tt}(x, t) - \Delta u(x, t) - \Delta u_t(x, t) - \Delta u_{tt}(x, t) = f(x, t), & \text{in } Q = \Omega \times [0, T], \\ u(x, t) = 0, & \text{on } \Sigma_0 = \Gamma_0 \times [0, T], \\ \frac{\partial u}{\partial n}(x, t) + \xi(x, t) = 0, & \text{on } \Sigma_1 = \Gamma_1 \times [0, T], \\ \xi_{tt}(x, t) + \xi(x, t) = u_t(x, t) + g(x, t), & \text{on } \Sigma_1 = \Gamma_1 \times [0, T], \end{cases} \quad (3.1.1)$$

with the following initial conditions

$$\begin{cases} u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x), & x \in \Omega, \\ \xi(x, 0) = \xi_0(x), & \xi_t(x, 0) = \xi_1(x), & x \in \Gamma_1, \end{cases} \quad (3.1.2)$$

where $u = u(x, t)$ represents the displacement at time t and the space coordinate x , and $\xi(x, t)$ designates the dynamical control.

Here, Ω is a bounded open set in $\mathbb{R}^2$ with boundary Γ of class C^2 , and $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \neq \emptyset$ and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. $\frac{\partial}{\partial n}$ denotes the unit outer normal derivative, f and g are functions depending on x and t .

The viscoelastic wave equation describes how waves travel through materials that have both elastic and viscous characteristics. Understanding the evolution of waves in materials that can simultaneously dissipate energy and deform is made possible by the use of viscoelastic wave equations, for the physical aspects, see Love [1].

The dynamic boundary conditions constitute the principal difficulty of problem (3.1.1). Moreover, these conditions have numerous practical uses in addition to being significant from a theoretical one.

For example, in one space dimension, the problem (3.1.1) can model the dy-

dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end.

The dynamic boundary conditions, which emerge when we analyse the transverse motion of an extensible membrane whose boundary may be affected by the vibrations only in an area, describe Newton's law for the attached mass (see [17, 18, 32] for more details).

Relativistic contexts, such as in electromagnetic field equations or general relativity, also give rise to wave-like equations with terms involving the Laplacian of the second time derivative of a field, such as the electromagnetic potential. These terms appear in the formulations of electromagnetic wave propagation or gravitational waves. Such equations capture the dynamics of field propagation in space-time, where energy and information transfer are governed by the curvature of the underlying space.

In systems involving viscoelastic materials, where elasticity and viscosity interact, the wave equation may include higher-order time derivatives, leading to terms like Δu_{tt} . These terms account for energy dissipation and the interplay between elasticity and viscosity, which are crucial for modeling materials that deform and dissipate energy over time. Viscoelasticity plays an important role in various engineering applications, including the modeling of vibration damping and the analysis of material fatigue.

In physical contexts like elasticity theory or acoustic wave propagation, the governing equations often involve second-order derivatives with respect to time and space. Modified wave equations incorporating higher-order terms can be found in plate theory or beam theory, where the behavior of materials under external forces is modeled. For example, a more complex equation might look like:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u + \beta \nabla^2 \frac{\partial^2 u}{\partial t^2},$$

which involves higher-order derivatives and is used to understand the behavior of structural vibrations or acoustic waves in materials that combine elasticity with damping properties.

The problem studied here extends the Mooney-Rivlin model for second-grade viscoelastic solids. Specifically, the one-dimensional fourth-order dispersive strongly damped wave equation:

$$u_{tt} - u_{xx} - u_{xtt} - u_{xt} = f(u), \quad x \in (0, l), \quad t > 0,$$

was derived by Hayes and Saccomandi [21] in their study of the propagation of transverse homogeneous waves. The lateral or longitudinal inertia of such a rod can be described by a dispersive term (see [25, 27]). For the n-dimensional case, Shang [26] studied the existence, uniqueness, asymptotic behavior, and blow-up phenomena for solutions of the form:

$$u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f(u), \quad x \in \Omega, \quad t > 0, \quad (3.1.3)$$

where $f(u)$ satisfies polynomial growth assumptions. Later, Liu and Li [19] obtained a unique global strong solution for all $n \geq 1$, and Xu [29] employed the multiplier

method to derive the asymptotic behavior of solutions for the Dirichlet problem of Eq. (3.1.3) under mild assumptions about the nonlinear term.

It is important to note that the problems studied in the aforementioned works were all analyzed with classical Dirichlet boundary conditions. In contrast, the problem investigated in this chapter is based on "dynamic boundary conditions", which introduce a more complex and realistic model for wave propagation in viscoelastic materials. This distinction is crucial, as dynamic boundary conditions account for time-dependent interactions at the boundary, offering a richer description of physical phenomena, particularly in contexts such as structural vibrations and wave propagation in materials with both elastic and viscous properties. This difference in boundary conditions sets our work apart from previous studies and provides a new perspective on the behavior of viscoelastic waves.

The study of different distributed systems with dynamical boundary controls, from a theoretical point of view, was done by many authors see for example ([22, 23, 24], and [28]).

The numerical study of a wave equation with dynamical boundary control is new. Very little attention has been paid to this field. We mention for example ([6] and [7]). It is demonstrated that the semi-discrete energy decays exponentially, uniformly with respect to the mesh parameter. The system also converges, but no error limitations are given.

Motivated by the previous works, in the present chapter we investigate problem (3.1.1) in which we generalize the results obtained in [8] by supposing new boundary conditions and add a viscosity term and strong damping. The content of this chapter is organized as follows: In section 3.2, we introduce preliminaries and main results, followed by the proof of existence and uniqueness of the solution using the Faedo-Galerkin method. In section 3.3, we prove the energy decay and we obtain exponential stability using Martinez lemma ([20]). In section 3.4 a semi-discrete Galerkin approximation to the solution of (3.1.1) is analyzed and in section 3.5 a fully discrete scheme is obtained. Finally, the results of numerical experiments illustrating the theoretical results are presented in section 3.6.

3.2 Notations and main results

Notations. We introduce the following spaces:

$$H_{\Gamma_0}^1(\Omega) = \{u \in H^1(\Omega) / u|_{\Gamma_0} = 0\}, \quad \text{and} \quad V = \left\{u \in H_{\Gamma_0}^1(\Omega) / u|_{\Gamma_1} \in L^2(\Gamma_1)\right\}.$$

We denote by $\|\cdot\|_2$, $\|\cdot\|_{\Gamma_1}$ and $\|\cdot\|_{H_{\Gamma_0}^1}$ the norms in $L^2(\Omega)$, $L^2(\Gamma_1)$ and $H_{\Gamma_0}^1(\Omega)$, and by $(\cdot, \cdot)$ and $(\cdot, \cdot)_1$ the scalar products in $L^2(\Omega)$ and $L^2(\Gamma_1)$, respectively.

Here and throughout this chapter, C_Ω denotes the same constant introduced in Chapter 2.

Weak Formulation. Multiply the first equation of (3.1.1) by a test function $v \in V$ then use Green's formula with the initial and boundary conditions to obtain the weak form of (3.1.1).

Find $u(x, t) \in V$ and $\xi(x, t) \in L^2(\Gamma_1)$ such that, for all $v, w \in V$

$$\begin{cases} (u_{tt}, v) + (\nabla u, \nabla v) + (\nabla u_t, \nabla v) + (\nabla u_{tt}, \nabla v) + (\xi + \xi_t + \xi_{tt}, v)_1 = (f, v), \\ (\xi_{tt}, w)_1 + (\xi, w)_1 = (u_t, w)_1 + (g, w)_1. \end{cases} \quad (3.2.1)$$

In the next theorem we show the existence and uniqueness of the solution of the problem (3.1.1) using the Faedo-Galerkin method.

Theorem 3.2.1. Assume that $u_0, u_1 \in H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega)$, $\xi_0, \xi_1 \in L^2(\Gamma_1)$, $f, \frac{\partial f}{\partial t} \in L^2(Q)$, $g, \frac{\partial g}{\partial t} \in L^2(\Sigma_1)$ and $\frac{\partial u_0}{\partial n}(x) = -\xi_0(x)$, $\frac{\partial u_1}{\partial n}(x) = -\xi_1(x)$ for all $x \in \Gamma_1$. Then for each $T > 0$, problem (3.1.1)–(3.1.2) admits a unique solution (u, ξ) satisfying the following conditions:

- (1) $u \in L^\infty(0, T, H_{\Gamma_0}^1(\Omega))$,
- (2) $u_t \in L^\infty(0, T, L^2(\Omega)) \cap L^\infty(0, T, H_{\Gamma_0}^1(\Omega))$,
- (3) $u_{tt} \in L^\infty(0, T, L^2(\Omega)) \cap L^\infty(0, T, H_{\Gamma_0}^1(\Omega))$,
- (4) $\xi, \xi_{tt} \in L^\infty(0, T, L^2(\Gamma_1))$.

Proof. Introducing the following change of variables

$$y(x, t) = u(x, t) - u_0(x) - tu_1(x) \quad \text{and} \quad \eta(x, t) = \xi(x, t) - \xi_0(x) - t\xi_1(x) \quad (3.2.2)$$

hence, problems (3.1.1) and (3.2.1) become

$$\begin{cases} y_{tt}(x, t) - \Delta y(x, t) - \Delta y_t(x, t) - \Delta y_{tt}(x, t) = f(x, t) + \Delta u_0(x) \\ \quad + (1+t)\Delta u_1(x), & \text{in } Q, \\ y(x, t) = 0, & \text{on } \Sigma_0, \\ \frac{\partial y}{\partial n}(x, t) + \eta(x, t) = -\frac{\partial u_0}{\partial n}(x) - t\frac{\partial u_1}{\partial n}(x) - \xi_0(x) - t\xi_1(x), & \text{on } \Sigma_1, \\ \eta_{tt}(x, t) + \eta(x, t) = y_t(x, t) + g(x, t) - \xi_0(x) - t\xi_1(x) + u_1(x), & \text{on } \Sigma_1. \end{cases} \quad (3.2.3)$$

and

$$\begin{cases} (y_{tt}, v) + (\nabla y + \nabla u_0 + t\nabla u_1, \nabla v) + (\nabla y_t + \nabla u_1, \nabla v) + (\nabla y_{tt}, \nabla v) \\ \quad + (\eta + \eta_t + \eta_{tt} + \xi_0 + (1+t)\xi_1, v)_1 = (f, v), \\ (\eta_{tt} + \eta, w)_1 = (y_t - \xi_0 - t\xi_1 + u_1, w)_1 + (g, w)_1. \end{cases} \quad (3.2.4)$$

Let V_{m_μ} be the subspace spanned respectively by $\{w_1, w_2, \dots, w_{m_\mu}\}$, where $\{w_i\}_{i=1}^\infty$ is a smooth orthogonal basis for V , and Z_{m_ν} the subspace spanned respectively by $\{z_1, z_2, \dots, z_{m_\nu}\}$, where $\{z_i\}_{i=1}^\infty$ is a smooth orthogonal basis for $L^2(\Gamma_1)$.

We look for the approximate solution

$$y^m(x, t) = \sum_{i=1}^{m_\mu} d_{im}(t) w_i(x), \quad \eta^m(x, t) = \sum_{i=1}^{m_\nu} s_{im}(t) z_i(x, y_b) \quad (3.2.5)$$

satisfying $\forall v \in V_{m_\mu}, \forall \psi \in Z_{m_\nu}$

$$\begin{cases} (y_{tt}^m, v) + (\nabla y^m + \nabla u_0 + t \nabla u_1, \nabla v) + (\nabla y_t^m + \nabla u_1, \nabla v) + (\nabla y_{tt}^m, \nabla v) \\ \quad + (\eta^m + \eta_t^m + \eta_{tt}^m + \xi_0 + (1+t)\xi_1, v)_1 = (f, v), \\ (\eta_{tt}^m + \eta^m, \psi)_1 = (y_t^m - \xi_0 - t\xi_1 + u_1, \psi)_1 + (g, \psi)_1. \end{cases} \quad (3.2.6)$$

Note that y_b in (3.2.5) is a fixed point y in the boundary Γ_1 , with the initial conditions $y^m(x, 0) = 0$, $y_t^m(x, 0) = 0$, $\eta^m(x, 0) = 0$, and $\eta_t^m(x, 0) = 0$.

This initial value problem has a local solution on $[0, T_m]$.

The priori estimates established below allow us to conclude that the solution can be extended to $[0, T] \times [0, T]$ for any given $T > 0$.

Estimate I: Taking $v = 2y_t^m$ and $\psi = \eta_{tt}^m + \eta_t^m + \eta^m$, in (3.2.6) to obtain

$$\begin{aligned} & \frac{d}{dt} \left(\|y_t^m\|_2^2 + \|\nabla y^m\|_2^2 + \|\nabla y_t^m\|_2^2 + \|\eta_t^m\|_{\Gamma_1}^2 + \|\eta^m\|_{\Gamma_1}^2 \right) + 2 \|\nabla y_t^m\|_2^2 + 2 \|\eta_{tt}^m + \eta_t^m\|_{\Gamma_1}^2 \\ &= -2 (\nabla u_0 + (1+t) \nabla u_1, \nabla y_t^m) - 2 (\xi_0 + t\xi_1 - u_1 - g, \eta_{tt}^m + \eta_t^m + \eta^m)_1 \\ & \quad - 2 (\xi_0 + (1+t)\xi_1, y_t^m)_1 + 2 (f, y_t^m). \end{aligned}$$

Using Young's inequality, there exist $\delta_0, \delta_1, \delta_2, \delta_3, \delta_4 > 0$ such that

$$\begin{aligned} & \frac{d}{dt} \left(\|y_t^m\|_2^2 + \|\nabla y^m\|_2^2 + \|\nabla y_t^m\|_2^2 + \|\eta_t^m\|_{\Gamma_1}^2 + \|\eta^m\|_{\Gamma_1}^2 \right) + 2 \|\nabla y_t^m\|_2^2 \\ & \quad + 2 \|\eta_{tt}^m + \eta_t^m\|_{\Gamma_1}^2 \\ & \leq C \left(\|u_1\|_{\Gamma_1}^2 + \|\nabla u_0\|_2^2 + \|\nabla u_1\|_2^2 + \|\xi_0\|_{\Gamma_1}^2 + \|\xi_1\|_{\Gamma_1}^2 + \|g\|_{\Gamma_1}^2 + \|f\|_2^2 \right) \\ & \quad + \delta_0 \|\nabla y_t^m\|_2^2 + \delta_1 \|y_t^m\|_2^2 + \delta_2 \|y_t^m\|_{\Gamma_1}^2 + \delta_3 \|\eta_{tt}^m + \eta_t^m\|_{\Gamma_1}^2 + \delta_4 \|\eta_t^m\|_{\Gamma_1}^2. \end{aligned} \quad (3.2.7)$$

Using the embedding

$$H_{\Gamma_0}^1(\Omega) \hookrightarrow L^2(\Gamma_1), \quad (\text{see [16]}) \quad (3.2.8)$$

together with the Poincaré inequality, we find that

$$\|y_t^m\|_{\Gamma_1}^2 \leq C_1 \|y_t^m\|_{H_{\Gamma_0}^1}^2 \leq C_2 \|\nabla y_t^m\|_2^2 \quad \text{and} \quad \|y_t^m\|_2^2 \leq C_\Omega \|\nabla y_t^m\|_2^2.$$

Hence, inequality (3.2.7) gives

$$\begin{aligned} & \frac{d}{dt} \left(\|y_t^m\|_2^2 + \|\nabla y^m\|_2^2 + \|\nabla y_t^m\|_2^2 + \|\eta_t^m\|_{\Gamma_1}^2 + \|\eta^m\|_{\Gamma_1}^2 \right) \\ & \leq C \left(\|u_1\|_{\Gamma_1}^2 + \|\nabla u_0\|_2^2 + \|\nabla u_1\|_2^2 + \|\xi_0\|_{\Gamma_1}^2 + \|\xi_1\|_{\Gamma_1}^2 + \|g\|_{\Gamma_1}^2 + \|f\|_2^2 \right) \\ & \quad + (\delta_0 + \delta_2 C_2 + \delta_1 C_\Omega - 2) \|\nabla y_t^m\|_2^2 + (\delta_3 - 2) \|\eta_{tt}^m + \eta_t^m\|_{\Gamma_1}^2 + \delta_4 \|\eta_t^m\|_{\Gamma_1}^2. \end{aligned}$$

Using the embedding (3.2.8), we get

$$\|u_1(x)\|_{\Gamma_1} \leq C \|u_1(x)\|_{H_{\Gamma_0}^1},$$

and using the assumptions on $u_0, u_1, \xi_0, \xi_1, g$, and f , we find that

$$\begin{aligned} & \frac{d}{dt} \left(\|y_t^m\|_2^2 + \|\nabla y^m\|_2^2 + \|\nabla y_t^m\|_2^2 + \|\eta_t^m\|_{\Gamma_1}^2 + \|\eta^m\|_{\Gamma_1}^2 \right) \\ & \leq C + (\delta_0 + \delta_2 C_2 + \delta_1 C_\Omega - 2) \|\nabla y_t^m\|_2^2 + (\delta_3 - 2) \|\eta_{tt}^m + \eta^m\|_{\Gamma_1}^2 + \delta_4 \|\eta_t^m\|_{\Gamma_1}^2. \end{aligned}$$

Now, choose $\delta_0 = \delta_4 = 1, \delta_3 = 2, \delta_1 = \frac{1}{2C_\Omega}$, and $\delta_2 = \frac{1}{2C_2}$, we get

$$\frac{d}{dt} \left(\|y_t^m\|_2^2 + \|\nabla y^m\|_2^2 + \|\nabla y_t^m\|_2^2 + \|\eta_t^m\|_{\Gamma_1}^2 + \|\eta^m\|_{\Gamma_1}^2 \right) \leq C + \|\eta_t^m\|_{\Gamma_1}^2.$$

Integration from 0 to t yields

$$\begin{aligned} & \|y_t^m\|_2^2 + \|\nabla y^m\|_2^2 + \|\nabla y_t^m\|_2^2 + \|\eta_t^m\|_{\Gamma_1}^2 + \|\eta^m\|_{\Gamma_1}^2 \\ & \leq C + \|y_t^m(0)\|_2^2 + \|\nabla y^m(0)\|_2^2 + \|\nabla y_t^m(0)\|_2^2 + \|\eta_t^m(0)\|_{\Gamma_1}^2 \\ & \quad + \|\eta^m(0)\|_{\Gamma_1}^2 + \int_0^t \|\eta_t^m(s)\|_{\Gamma_1}^2 ds. \end{aligned}$$

Using the initial conditions, we get

$$\|y_t^m\|_2^2 + \|\nabla y^m\|_2^2 + \|\nabla y_t^m\|_2^2 + \|\eta_t^m\|_{\Gamma_1}^2 + \|\eta^m\|_{\Gamma_1}^2 \leq C + \int_0^t \|\eta_t^m(s)\|_{\Gamma_1}^2 ds. \quad (3.2.9)$$

From the inequality (3.2.9), we find that

$$\|\eta_t^m\|_{\Gamma_1}^2 \leq C + \int_0^t \|\eta_t^m(s)\|_{\Gamma_1}^2 ds.$$

Applying the Gronwall's inequality, we obtain

$$\|\eta_t^m\|_{\Gamma_1}^2 \leq C + C \int_0^t e^{t-s} ds \leq C.$$

Hence, it follows from (3.2.9) that

$$\|y_t^m\|_2^2 + \|\nabla y^m\|_2^2 + \|\nabla y_t^m\|_2^2 + \|\eta_t^m\|_{\Gamma_1}^2 + \|\eta^m\|_{\Gamma_1}^2 \leq C. \quad (3.2.10)$$

From (3.2.10), we deduce that

$$\begin{aligned} y^m & \text{ is bounded in } L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \\ y_t^m & \text{ is bounded in } L^\infty(0, T, L^2(\Omega)) \cap L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \\ \eta^m & \text{ is bounded in } L^\infty(0, T, L^2(\Gamma_1)), \\ \eta_t^m & \text{ is bounded in } L^\infty(0, T, L^2(\Gamma_1)). \end{aligned}$$

Consequently the solution can be extended to $[0, T]$.

Also, there exist two sub-sequences still denoted by $(y^m), (\eta^m)$, such that

$$\begin{aligned} y^m &\rightharpoonup y \quad \text{weak-star in } L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \\ y_t^m &\rightharpoonup y_t \quad \text{weak-star in } L^\infty(0, T, L^2(\Omega)), \\ y_{tt}^m &\rightharpoonup y_{tt} \quad \text{weak-star in } L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \\ \eta^m &\rightharpoonup \eta \quad \text{weak-star in } L^\infty(0, T, L^2(\Gamma_1)), \\ \eta_t^m &\rightharpoonup \eta_t \quad \text{weak-star in } L^\infty(0, T, L^2(\Gamma_1)). \end{aligned}$$

Estimate II: Differentiating (3.2.6) with respect to t and let $\hat{y}^m = y_t^m, \hat{\eta}^m = \eta_t^m, \hat{f} = \frac{\partial f}{\partial t}$, and $\hat{g} = \frac{\partial g}{\partial t}$, we obtain

$$\begin{cases} (\hat{y}_{tt}^m, v) + (\nabla \hat{y}^m + \nabla u_1, \nabla v) + (\nabla \hat{y}_t^m, \nabla v) + (\nabla \hat{y}_{tt}^m, \nabla v) \\ \quad + (\hat{\eta}^m + \hat{\eta}_{tt}^m + \hat{\eta}_{tt}^m + \xi_1, v)_1 = (\hat{f}, v), \\ (\hat{\eta}_{tt}^m(x, t) + \hat{\eta}^m(x, t), \psi)_1 = (\hat{y}_t^m(x, t) - \xi_1(x), \psi)_1 + (\hat{g}(x, t), \psi)_1. \end{cases} \quad (3.2.11)$$

Take $v = 2\hat{y}_t^m$ and $\psi = \hat{\eta}_{tt}^m + \hat{\eta}_t^m + \hat{\eta}^m$ in (3.2.11), we find

$$\begin{aligned} \frac{d}{dt} &\left(\|\hat{y}_t^m\|_2^2 + \|\nabla \hat{y}^m\|_2^2 + \|\nabla \hat{y}_t^m\|_2^2 + \|\hat{\eta}^m\|_{\Gamma_1}^2 + \|\hat{\eta}_t^m\|_{\Gamma_1}^2 \right) + 2\|\nabla \hat{y}_t^m\|_2^2 + 2\|\hat{\eta}_{tt}^m + \hat{\eta}^m\|_{\Gamma_1}^2 \\ &= 2(\hat{f}, \hat{y}_t^m) - 2(\nabla u_1, \nabla \hat{y}_t^m) - 2(\xi_1 - \hat{g}, \hat{\eta}_{tt}^m + \hat{\eta}_t^m + \hat{\eta})_1 - 2(\xi_1, \hat{y}_t^m)_1. \end{aligned}$$

Using Young's inequality, there exist $\delta_5, \delta_6, \delta_7, \delta_8, \delta_9 > 0$ such that

$$\begin{aligned} \frac{d}{dt} &\left(\|\hat{y}_t^m\|_2^2 + \|\nabla \hat{y}^m\|_2^2 + \|\nabla \hat{y}_t^m\|_2^2 + \|\hat{\eta}^m\|_{\Gamma_1}^2 + \|\hat{\eta}_t^m\|_{\Gamma_1}^2 \right) \\ &\quad + 2\|\nabla \hat{y}_t^m\|_2^2 + 2\|\hat{\eta}_{tt}^m + \hat{\eta}^m\|_{\Gamma_1}^2 \\ &\leq C \left(\|\nabla u_1\|_2^2 + \|\xi_1\|_{\Gamma_1}^2 + \|\hat{g}\|_{\Gamma_1}^2 + \|\hat{f}\|_2^2 \right) + \delta_5 \|\nabla \hat{y}_t^m\|_2^2 \\ &\quad + \delta_6 \|\hat{\eta}_{tt}^m + \hat{\eta}^m\|_{\Gamma_1}^2 + \delta_7 \|\hat{\eta}_t^m\|_{\Gamma_1}^2 + \delta_8 \|\hat{y}_t^m\|_{\Gamma_1}^2 + \delta_9 \|\hat{y}_t^m\|_2^2. \end{aligned} \quad (3.2.12)$$

Using the embedding (3.2.8) together with the Poincaré inequality, we find

$$\|\hat{y}_t^m\|_{\Gamma_1}^2 \leq C_1 \|\hat{y}_t^m\|_{H_{\Gamma_0}^1}^2 \leq C_2 \|\nabla \hat{y}_t^m\|_2^2 \quad \text{and} \quad \|\hat{y}_t^m\|_2^2 \leq C_\Omega \|\nabla \hat{y}_t^m\|_2^2.$$

Hence, inequality (3.2.12) gives

$$\begin{aligned} \frac{d}{dt} &\left(\|\hat{y}_t^m\|_2^2 + \|\nabla \hat{y}^m\|_2^2 + \|\nabla \hat{y}_t^m\|_2^2 + \|\hat{\eta}^m\|_{\Gamma_1}^2 + \|\hat{\eta}_t^m\|_{\Gamma_1}^2 \right) \\ &\leq C \left(\|\nabla u_1\|_2^2 + \|\xi_1\|_{\Gamma_1}^2 + \|\hat{g}\|_{\Gamma_1}^2 + \|\hat{f}\|_2^2 \right) + (\delta_5 + \delta_8 C_2 + \delta_9 C_\Omega - 2) \|\nabla \hat{y}_t^m\|_2^2 \\ &\quad + (\delta_6 - 2) \|\hat{\eta}_{tt}^m + \hat{\eta}^m\|_{\Gamma_1}^2 + \delta_7 \|\hat{\eta}_t^m\|_{\Gamma_1}^2. \end{aligned}$$

Using the assumptions on u_1, ξ_1, g , and f , we find that

$$\frac{d}{dt} \left(\|\hat{y}_t^m\|_2^2 + \|\nabla \hat{y}^m\|_2^2 + \|\nabla \hat{y}_t^m\|_2^2 + \|\hat{\eta}^m\|_{\Gamma_1}^2 + \|\hat{\eta}_t^m\|_{\Gamma_1}^2 \right)$$

$$\leq C + (\delta_5 + \delta_8 C_2 + \delta_9 C_\Omega - 2) \|\nabla \hat{y}_t^m\|_2^2 + (\delta_6 - 2) \|\hat{\eta}_{tt}^m + \hat{\eta}^m\|_{\Gamma_1}^2 + \delta_7 \|\hat{\eta}_t^m\|_{\Gamma_1}^2.$$

Choosing $\delta_5 = \delta_7 = 1, \delta_6 = 2, \delta_8 = \frac{1}{2C_2}$, and $\delta_9 = \frac{1}{2C_\Omega}$, we get

$$\frac{d}{dt} \left(\|\hat{y}_t^m\|_2^2 + \|\nabla \hat{y}^m\|_2^2 + \|\nabla \hat{y}_t^m\|_2^2 + \|\hat{\eta}^m\|_{\Gamma_1}^2 + \|\hat{\eta}_t^m\|_{\Gamma_1}^2 \right) \leq C + \|\hat{\eta}_t^m\|_{\Gamma_1}^2.$$

Integration from 0 to t yields

$$\begin{aligned} & \|\hat{y}_t^m\|_2^2 + \|\nabla \hat{y}^m\|_2^2 + \|\nabla \hat{y}_t^m\|_2^2 + \|\hat{\eta}^m\|_{\Gamma_1}^2 + \|\hat{\eta}_t^m\|_{\Gamma_1}^2 \\ & \leq C + \|\hat{y}_t^m(0)\|_2^2 + \|\nabla \hat{y}^m(0)\|_2^2 + \|\nabla \hat{y}_t^m(0)\|_2^2 \\ & \quad + \|\hat{\eta}^m(0)\|_{\Gamma_1}^2 + \|\hat{\eta}_t^m(0)\|_{\Gamma_1}^2 + \int_0^t \|\hat{\eta}_t^m(s)\|_{\Gamma_1}^2 ds. \end{aligned} \quad (3.2.13)$$

Since $\|\nabla \hat{y}^m(0)\|_2 = \|\nabla y_t^m(0)\|_2 = 0$ and $\|\hat{\eta}^m(0)\|_{\Gamma_1} = \|\eta_t^m(0)\|_{\Gamma_1} = 0$, it remains to bound $\|\hat{y}_t^m(0)\|_2, \|\nabla \hat{y}_t^m(0)\|_2$, and $\|\hat{\eta}_t^m(0)\|_{\Gamma_1}$.

To bound $\|\hat{\eta}_t^m(0)\|_{\Gamma_1}$, we use (3.2.6)₂ for $\psi = \eta_{tt}^m$ and $t = 0$ we get

$$\|\eta_{tt}^m(0)\|_{\Gamma_1}^2 = (g_0 - \xi_0 + u_1, \eta_{tt}^m(0))_1 \leq C \|g_0 - \xi_0 + u_1\|_{\Gamma_1} \|\eta_{tt}^m(0)\|_{\Gamma_1},$$

where $g_0(x) = g(x, 0)$.

Using the embedding (3.2.8), we get $\|u_1(x)\|_{\Gamma_1} \leq C \|u_1(x)\|_{H_{\Gamma_0}^1}$, and the assumptions on u_1, ξ_0, g give the boundedness of $\|\hat{\eta}_t^m(0)\|_{\Gamma_1}$.

To bound $\|\hat{y}_t^m(0)\|_2$ and $\|\nabla \hat{y}_t^m(0)\|_2$, we use (3.2.6)₁ for $t = 0, v = y_{tt}^m$ and the fact that $y_t^m(0) = y^m(0) = \eta^m(0) = \eta_t^m(0) = 0$, hence

$$\begin{aligned} \|y_{tt}^m(0)\|_2^2 + \|\nabla y_{tt}^m(0)\|_2^2 &= -(\nabla u_0 + \nabla u_1, \nabla y_{tt}^m(0)) - (\eta_{tt}^m(0) + \xi_0 + \xi_1, y_{tt}^m(0))_1 \\ &\quad + (f_0, y_{tt}^m(0)), \end{aligned}$$

where $f_0(x) = f(x, 0)$.

Green's formula yields

$$\|y_{tt}^m(0)\|_2^2 + \|\nabla y_{tt}^m(0)\|_2^2 = (\Delta u_0 + \Delta u_1, y_{tt}^m(0)) - (\eta_{tt}^m(0), y_{tt}^m(0))_1 + (f_0, y_{tt}^m(0)).$$

Using Young's inequality, there exist $\delta_{10}, \delta_{11} > 0$ such that

$$\begin{aligned} \|y_{tt}^m(0)\|_2^2 + \|\nabla y_{tt}^m(0)\|_2^2 &\leq C \left(\|\Delta u_0\|_2^2 + \|\Delta u_1\|_2^2 + \|\eta_{tt}^m(0)\|_{\Gamma_1}^2 + \|f_0\|_2^2 \right) \\ &\quad + \delta_{10} \|y_{tt}^m(0)\|_2^2 + \delta_{11} \|\eta_{tt}^m(0)\|_{\Gamma_1}^2. \end{aligned}$$

Using the embedding (3.2.8), together with the Poincaré inequality, we find

$$\|y_{tt}^m(0)\|_{\Gamma_1}^2 \leq C_1 \|y_{tt}^m(0)\|_{H_{\Gamma_0}^1}^2 \leq C_2 \|\nabla y_{tt}^m(0)\|_2^2.$$

Hence

$$\begin{aligned} \|y_{tt}^m(0)\|_2^2 + \|\nabla y_{tt}^m(0)\|_2^2 &\leq C \left(\|\Delta u_0\|_2^2 + \|\Delta u_1\|_2^2 + \|\eta_{tt}^m(0)\|_{\Gamma_1}^2 + \|f_0\|_2^2 \right) \\ &\quad + \delta_{10} \|y_{tt}^m(0)\|_2^2 + \delta_{11} C_2 \|\nabla y_{tt}^m(0)\|_2^2. \end{aligned}$$

Now, choose $\delta_{10} = \frac{1}{2}$ and $\delta_{11} = \frac{1}{2C_2}$, we get

$$\|y_{tt}^m(0)\|_2^2 + \|\nabla y_{tt}^m(0)\|_2^2 \leq 2C \left(\|\Delta u_0\|_2^2 + \|\Delta u_1\|_2^2 + \|\eta_{tt}^m(0)\|_{\Gamma_1}^2 + \|f_0\|_2^2 \right).$$

Using the assumptions on u_0, u_1, f and the boundedness of $\|\eta_{tt}^m(0)\|_{\Gamma_1}$, we find that

$$\|\hat{y}_t^m(0)\|_2^2 + \|\nabla \hat{y}_t^m(0)\|_2^2 \leq C.$$

Hence inequality (3.2.13) becomes

$$\|\hat{y}_t^m\|_2^2 + \|\nabla \hat{y}_t^m\|_2^2 + \|\nabla \hat{y}_t^m\|_2^2 + \|\hat{\eta}^m\|_{\Gamma_1}^2 + \|\hat{\eta}_t^m\|_{\Gamma_1}^2 \leq C + \int_0^t \|\hat{\eta}_t^m(s)\|_{\Gamma_1}^2 ds. \quad (3.2.14)$$

From the inequality (3.2.14), we find

$$\|\hat{\eta}_t^m\|_{\Gamma_1}^2 \leq C + \int_0^t \|\hat{\eta}_t^m(s)\|_{\Gamma_1}^2 ds.$$

By Gronwall's Lemma, we have

$$\|\hat{\eta}_t^m\|_{\Gamma_1}^2 \leq C + C \int_0^t e^{t-s} ds \leq C.$$

Hence, it follows from (3.2.14) that

$$\|\hat{y}_t^m\|_2^2 + \|\nabla \hat{y}_t^m\|_2^2 + \|\nabla \hat{y}_t^m\|_2^2 + \|\hat{\eta}^m\|_{\Gamma_1}^2 + \|\hat{\eta}_t^m\|_{\Gamma_1}^2 \leq C. \quad (3.2.15)$$

From (3.2.15), we deduce that

$$\begin{aligned} y_{tt}^m &\text{ is bounded in } L^\infty(0, T, L^2(\Omega)) \cap L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \\ \eta_{tt}^m &\text{ is bounded in } L^\infty(0, T, L^2(\Gamma_1)). \end{aligned}$$

Therefore

$$\begin{aligned} y_{tt}^m &\rightharpoonup y_{tt} \text{ weak-star in } L^\infty(0, T, L^2(\Omega)), \\ y_{tt}^m &\rightharpoonup y_{tt} \text{ weak-star in } L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \\ \eta_{tt}^m &\rightharpoonup \eta_{tt} \text{ weak-star in } L^\infty(0, T, L^2(\Gamma_1)), \\ \eta_{tt}^m &\rightharpoonup \eta_{tt} \text{ weak-star in } L^\infty(0, T, L^2(\Gamma_1)). \end{aligned}$$

Using the convergence results, we can pass to the limit in (3.2.6) to obtain, after

reversing the change of variables,

$$\begin{cases} (u_{tt}, v) + (\nabla u, \nabla v) + (\nabla u_t, \nabla v) + (\nabla u_{tt}, \nabla v) + (\xi + \xi_t + \xi_{tt}, v)_1 = (f, v), \\ (\xi_{tt} + \xi, \psi)_1 = (u_t + g, \psi)_1. \end{cases} \quad (3.2.16)$$

Using (3.2.16)₁ for $v \in D(\Omega)$, we get

$$(\nabla u + \nabla u_t + \nabla u_{tt}, \nabla v) - (f, v) = -(u_{tt}, v).$$

Then, we have

$$(\Delta u + \Delta u_t + \Delta u_{tt} + f, v) = (u_{tt}, v),$$

since $u_{tt} \in L^\infty(0, T, L^2(\Omega)) \cap L^\infty(0, T, H_{\Gamma_0}^1(\Omega))$, we obtain

$$\Delta u + \Delta u_t + \Delta u_{tt} + f = u_{tt} \in L^\infty(0, T, L^2(\Omega)) \cap L^\infty(0, T, H_{\Gamma_0}^1(\Omega)),$$

and

$$u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f \quad a.e \quad \text{in } \Omega \times (0, T).$$

Using Green's formula in (3.2.16), we obtain $\forall v \in V$

$$(u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt}, v) + \left(\frac{\partial u}{\partial n} + \xi + \left(\frac{\partial u}{\partial n} + \xi \right)_t + \left(\frac{\partial u}{\partial n} + \xi \right)_{tt}, v \right)_1 = (f, v).$$

Hence

$$\frac{\partial u}{\partial n}(x, t) + \xi(x, t) + \left(\frac{\partial u}{\partial n}(x, t) + \xi(x, t) \right)_t + \left(\frac{\partial u}{\partial n}(x, t) + \xi(x, t) \right)_{tt} = 0,$$

which is equivalent to

$$\begin{cases} z(x, t) + z_t(x, t) + z_{tt}(x, t) = 0, \\ z(x, t) = \frac{\partial u}{\partial n}(x, t) + \xi(x, t). \end{cases}$$

Therefore

$$z(x, t) = z(x, 0) \exp\left(-\frac{t}{2}\right) \cos\left(\frac{t\sqrt{3}}{2}\right) + \frac{2}{\sqrt{3}} \left(z_t(x, 0) + \frac{1}{2} z(x, 0) \right) \exp\left(-\frac{t}{2}\right) \sin\left(\frac{t\sqrt{3}}{2}\right).$$

Note that

$$z(x, 0) = \frac{\partial u}{\partial n}(x, 0) + \xi(x, 0) = \frac{\partial u_0}{\partial n}(x) + \xi_0(x) = 0,$$

and

$$z_t(x, 0) = \frac{\partial u_t}{\partial n}(x, 0) + \xi_t(x, 0) = \frac{\partial u_1}{\partial n}(x) + \xi_1(x) = 0,$$

hence $z(x, t) = 0$, that is

$$\frac{\partial u}{\partial n}(x, t) + \xi(x, t) = 0.$$

And the solution (u, ξ) verifies the boundary condition.

Finally, for the uniqueness, we take (u, ξ) and $(\tilde{u}, \tilde{\xi})$ as two solutions and let $U = u - \tilde{u}$ and $\lambda = \xi - \tilde{\xi}$, then we have

$$\begin{cases} (U_{tt}, v) + (\nabla U, \nabla v) + (\nabla U_t, \nabla v) + (\nabla U_{tt}, \nabla v) + (\lambda + \lambda_t + \lambda_{tt}, v)_1 = 0, & \text{in } Q, \\ (\lambda_{tt} + \lambda, \psi)_1 = (U_t, \psi)_1, & \text{on } \Sigma_1. \end{cases}$$

Take $v = 2U_t$ and $\psi = \lambda + \lambda_t + \lambda_{tt}$ then integrate from 0 to t to obtain

$$\begin{aligned} & \|U_t\|_2^2 + \|\nabla U\|_2^2 + \|\nabla U_t\|_2^2 + \|\lambda\|_{\Gamma_1}^2 + \|\lambda_t\|_{\Gamma_1}^2 + 2 \int_0^t \left(\|\nabla U_t(s)\|_2^2 + \|(\lambda + \lambda_{tt})(s)\|_{\Gamma_1}^2 \right) ds \\ &= \|U_t(0)\|_2^2 + \|\nabla U(0)\|_2^2 + \|\nabla U_t(0)\|_2^2 + \|\lambda(0)\|_{\Gamma_1}^2 + \|\lambda_t(0)\|_{\Gamma_1}^2. \end{aligned}$$

Using the initial conditions, it follows that

$$\begin{aligned} & \|U_t\|_2^2 + \|\nabla U\|_2^2 + \|\nabla U_t\|_2^2 + \|\lambda\|_{\Gamma_1}^2 + \|\lambda_t\|_{\Gamma_1}^2 \\ &+ 2 \int_0^t \left(\|\nabla U_t(s)\|_2^2 + \|(\lambda + \lambda_{tt})(s)\|_{\Gamma_1}^2 \right) ds = 0. \end{aligned}$$

So $U_t = \nabla U = \nabla U_t = 0$ and $\lambda = \lambda_t = 0$, therefore $U = 0$ and $\lambda = 0$. □

3.3 Asymptotic behavior

Consider a problem (3.1.1) in the homogeneous form

$$\begin{cases} u_{tt}(x, t) - \Delta u(x, t) - \Delta u_t(x, t) - \Delta u_{tt}(x, t) = 0, & \text{in } Q, \\ u(x, t) = 0, & \text{on } \Sigma_0, \\ \frac{\partial u}{\partial n}(x, t) + \xi(x, t) = 0, & \text{on } \Sigma_1, \\ \xi_{tt}(x, t) + \xi(x, t) = u_t(x, t), & \text{on } \Sigma_1, \end{cases} \quad (3.3.1)$$

with the following initial conditions

$$\begin{cases} u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x), & x \in \Omega, \\ \xi(x, 0) = \xi_0(x), & \xi_t(x, 0) = \xi_1(x), & x \in \Gamma_1. \end{cases} \quad (3.3.2)$$

In this section, we shall investigate the asymptotic behavior of problem (3.3.1)–(3.3.2). First, we define the energy function of problem (3.3.1)–(3.3.2) as

$$E(t) = \frac{1}{2} \left(\|u_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 + \|\xi_t\|_{\Gamma_1}^2 + \|\xi\|_{\Gamma_1}^2 \right). \quad (3.3.3)$$

Lemma 3.3.1. *Let (u, ξ) be a solution of problem (3.3.1)–(3.3.2). Then the energy function (3.3.3) is a non-increasing function on $[0, T]$ and satisfies*

$$E'(t) = -\|\nabla u_t\|_2^2 - \|\xi_{tt} + \xi\|_{\Gamma_1}^2 \leq 0, \quad \forall \quad 0 \leq t \leq T. \quad (3.3.4)$$

Proof. Multiplying the first equation in (3.3.1) by u_t and integrating over Ω , we get

$$\int_{\Omega} u_{tt} u_t dx - \int_{\Omega} \Delta u u_t dx - \int_{\Omega} \Delta u_t u_t dx - \int_{\Omega} \Delta u_{tt} u_t dx = 0.$$

After some calculations, we get

$$\frac{1}{2} \frac{d}{dt} (\|u_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla u_t\|_2^2) + \|\nabla u_t\|_2^2 + \int_{\Gamma_1} (\xi_{tt} + \xi) u_t d\sigma + \int_{\Gamma_1} \xi_t u_t d\sigma = 0.$$

Then, by using (3.3.1)₄, we obtain

$$\frac{1}{2} \frac{d}{dt} (\|u_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 + \|\xi_t\|_{\Gamma_1}^2 + \|\xi\|_{\Gamma_1}^2) + \|\nabla u_t\|_2^2 + \|\xi_{tt} + \xi\|_{\Gamma_1}^2 = 0.$$

Hence

$$E'(t) = -\|\nabla u_t\|_2^2 - \|\xi_{tt} + \xi\|_{\Gamma_1}^2 \leq 0, \quad \forall 0 \leq t \leq T.$$

□

In the next theorem we show the asymptotic behavior of the problem (3.3.1)–(3.3.2).

Theorem 3.3.1. Assume that $u_0, u_1 \in H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega)$, $\xi_0, \xi_1 \in L^2(\Gamma_1)$ and $\frac{\partial u_0}{\partial n}(x) = -\xi_0(x)$, $\frac{\partial u_1}{\partial n}(x) = -\xi_1(x)$ for all $x \in \Gamma_1$.

Then for some positive constant k , we obtain the following decay property

$$E(t) \leq E(0) e^{1-k\phi(t)}, \quad \forall t \geq 0,$$

where ϕ is function satisfies certain hypotheses, which we will specify in the proof.

Proof. We denote by C various positive constants which may be different at different occurrences.

Let $p \geq 0$ and ϕ is a function satisfying all the hypotheses of lemma 1.3.3.

Multiply the first equation of (3.3.1) by $E^p \phi' u$ and we integrate over $(t, T) \times \Omega$, we obtain

$$\begin{aligned} 0 &= \int_t^T E^p \phi' \int_{\Omega} u (u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt}) dx ds \\ &= \int_t^T E^p \phi' \int_{\Omega} u u_{tt} dx ds + \int_t^T E^p \phi' \int_{\Omega} |\nabla u|^2 dx ds + \int_t^T E^p \phi' \int_{\Omega} \nabla u \nabla u_t dx ds \\ &\quad + \int_t^T E^p \phi' \int_{\Omega} \nabla u \nabla u_{tt} dx ds + \int_t^T E^p \phi' \int_{\Gamma_1} u (\xi + \xi_t + \xi_{tt}) d\sigma ds \\ &= \left[E^p \phi' \int_{\Omega} u u_t dx \right]_t^T - \int_t^T (E^p \phi')' \int_{\Omega} u u_t dx ds - 2 \int_t^T E^p \phi' \int_{\Omega} u_t^2 dx ds \\ &\quad + \int_t^T E^p \phi' \int_{\Omega} u_t^2 dx ds + \int_t^T E^p \phi' \int_{\Omega} |\nabla u|^2 dx ds + \int_t^T E^p \phi' \int_{\Omega} \nabla u \nabla u_t dx ds \\ &\quad + \left[E^p \phi' \int_{\Omega} \nabla u \nabla u_t dx \right]_t^T - \int_t^T (E^p \phi')' \int_{\Omega} \nabla u \nabla u_t dx ds - 2 \int_t^T E^p \phi' \int_{\Omega} |\nabla u_t|^2 dx ds \end{aligned}$$

$$+ \int_t^T E^p \phi' \int_{\Omega} |\nabla u_t|^2 dx ds + \int_t^T E^p \phi' \int_{\Gamma_1} (uu_t + u\xi_t) d\sigma ds.$$

Similarly, we multiply the fourth equation of (3.3.1) by $E^p \phi' \xi$ and we integrate over $(t, T) \times \Gamma_1$, we obtain

$$\begin{aligned} 0 = & \left[E^p \phi' \int_{\Gamma_1} \xi \xi_t d\sigma \right]_t^T - \int_t^T (E^p \phi')' \int_{\Gamma_1} \xi \xi_t d\sigma ds - 2 \int_t^T E^p \phi' \int_{\Gamma_1} \xi_t^2 d\sigma ds \\ & + \int_t^T E^p \phi' \int_{\Gamma_1} \xi_t^2 d\sigma ds + \int_t^T E^p \phi' \int_{\Gamma_1} \xi^2 d\sigma ds - \int_t^T E^p \phi' \int_{\Gamma_1} \xi u_t d\sigma ds. \end{aligned}$$

Taking their sum, we obtain

$$\begin{aligned} 2 \int_t^T E^{p+1} \phi' ds = & - \left[E^p \phi' \left(\int_{\Omega} (uu_t + \nabla u \nabla u_t) dx + \int_{\Gamma_1} \xi \xi_t d\sigma \right) \right]_t^T \\ & + \int_t^T (E^p \phi')' \left(\int_{\Omega} (uu_t + \nabla u \nabla u_t) dx + \int_{\Gamma_1} \xi \xi_t d\sigma \right) ds \\ & + 2 \int_t^T E^p \phi' \int_{\Gamma_1} \xi_t^2 d\sigma ds - \int_t^T E^p \phi' \int_{\Omega} \nabla u \nabla u_t dx ds \quad (3.3.5) \\ & + \int_t^T E^p \phi' \int_{\Gamma_1} (\xi u_t - uu_t - u\xi_t) d\sigma ds \\ & + 2 \int_t^T E^p \phi' \int_{\Omega} (u_t^2 + |\nabla u_t|^2) dx ds. \end{aligned}$$

Assume that ϕ' is a bounded non-negative function on $\mathbb{R}^+$ and ϕ'' is a non-negative function on $\mathbb{R}^+$.

Using the Cauchy–Schwarz, Young and Poincaré’s inequalities and the definition of E , we get

$$\left| E^p(s) \phi'(s) \left(\int_{\Omega} (uu_t + \nabla u \nabla u_t) dx + \int_{\Gamma_1} \xi \xi_t d\sigma \right) \right| \leq C E^{p+1}(s).$$

By recalling (3.3.4), we have

$$\begin{aligned} & \left| \int_t^T (E^p \phi')' \left(\int_{\Omega} (uu_t + \nabla u \nabla u_t) dx + \int_{\Gamma_1} \xi \xi_t d\sigma \right) ds \right| \\ &= \left| \int_t^T (p E' E^{p-1} \phi' + E^p \phi'') \left(\int_{\Omega} (uu_t + \nabla u \nabla u_t) dx + \int_{\Gamma_1} \xi \xi_t d\sigma \right) ds \right| \\ &\leq \left| \int_t^T p E' E^{p-1} \phi' \left(\int_{\Omega} (uu_t + \nabla u \nabla u_t) dx + \int_{\Gamma_1} \xi \xi_t d\sigma \right) ds \right| \\ &\quad + \left| \int_t^T E^p \phi'' \left(\int_{\Omega} (uu_t + \nabla u \nabla u_t) dx + \int_{\Gamma_1} \xi \xi_t d\sigma \right) ds \right| \\ &\leq C \int_t^T -E' E^p ds + C \int_t^T E^{p+1} \phi'' ds \\ &\leq C \int_t^T -E' E^p ds + C \left([E^{p+1} \phi']_t^T - \int_t^T (p+1) E' E^p \phi' ds \right) \end{aligned}$$

$$\begin{aligned}
&\leq C [E^{p+1} \phi']_t^T + C \int_t^T -E' E^p ds \\
&\leq C E^{p+1}(t).
\end{aligned}$$

By the Poincaré's inequality and (3.3.4), we have

$$\begin{aligned}
\int_t^T E^p \phi' \int_{\Omega} (u_t^2 + |\nabla u_t|^2) dx ds &\leq C \int_t^T E^p \phi' \int_{\Omega} |\nabla u_t|^2 dx ds \\
&\leq C \int_t^T E^p (-E') ds \\
&\leq C E^{p+1}(t).
\end{aligned}$$

Using the definition of E and (3.3.4), we get

$$\int_t^T E^p(s) \phi'(s) \int_{\Gamma_1} \xi_t^2 d\sigma ds \leq C \int_t^T E^{p+1}(s) ds \leq C \int_t^T E^{p+1}(t) ds \leq C E^{p+1}(t).$$

Using the Cauchy–Schwarz and Young's inequalities and the definition of E , we get

$$\int_t^T E^p(s) \phi'(s) \int_{\Omega} \nabla u \nabla u_t dx ds \leq C \int_t^T E^{p+1}(s) ds \leq C E^{p+1}(t),$$

and

$$\begin{aligned}
\int_t^T E^p \phi' \int_{\Gamma_1} (\xi u_t - u u_t - u \xi_t) d\sigma ds &\leq C \int_t^T E^p \phi' \int_{\Gamma_1} (\xi^2 + \xi_t^2 + u_t^2 + u^2) d\sigma ds \\
&\leq C \left(\int_t^T E^{p+1} ds + \int_t^T E^p (-E') ds \right) \\
&\quad + C \int_t^T E^p \int_{\Gamma_1} u^2 d\sigma ds \\
&\leq C \left(E^{p+1}(t) + \int_t^T E^p \int_{\Gamma_1} u^2 d\sigma ds \right),
\end{aligned}$$

using the embedding (3.2.8), together with the Poincaré inequality, we find

$$\begin{aligned}
\int_t^T E^p \phi' \int_{\Gamma_1} (\xi u_t - u u_t - u \xi_t) d\sigma ds &\leq C \left(E^{p+1}(t) + \int_t^T E^p \int_{\Omega} |\nabla u|^2 d\sigma ds \right) \\
&\leq C \left(E^{p+1}(t) + \int_t^T E^p (-E') ds \right) \\
&\leq C E^{p+1}(t).
\end{aligned}$$

Combining these estimates, we conclude from (3.3.5) that

$$\int_t^T E^{p+1}(s) \phi'(s) ds \leq E^{p+1}(t).$$

By applying the lemma 1.3.3, we obtain the property of the exponential stability of the solution.

This ends the proof of Theorem 3.3.1. $\square$

3.4 Semi-Discrete Approximation

Assuming that $\overline{\Omega}$ is a polyhedral domain. Let $V^h \subset V$ be a finite-dimensional space, approximating the variational space V , given by

$$V^h = \left\{ v^h \in C(\overline{\Omega}); v_K^h \in \mathbb{P}_1(K), K \in \mathcal{T}_h, v^h = 0 \text{ on } \Gamma_0 \right\},$$

where $\mathbb{P}_1(K)$ represents the space of polynomials of global degree less than or equal to one in K and we denote by $(\mathcal{T}_h)_{h>0}$ a regular family of triangulations of $\overline{\Omega}$ (see for instance [13]), compatible with the partition of the boundary $\Gamma = \partial\Omega$ into Γ_0 and Γ_1 , i.e. the finite element space V^h is composed of continuous and piecewise affine functions.

Let h_K be the diameter of an element $K \in \mathcal{T}_h$ and let $h = \max_{K \in \mathcal{T}_h} h_K$ denote the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $u_0^h = P^h u_0$, $u_1^h = P^h u_1$, $\xi_0^h = P^h \xi_0$, and $\xi_1^h = P^h \xi_1$, where P^h is the classical finite element interpolation operator defined by

$$(\nabla(P^h \psi), \nabla \chi) = (\nabla \psi, \nabla \chi), \quad \forall \chi \in V^h,$$

satisfying the following estimations (see [13])

$$\begin{cases} \|\psi - P^h \psi\|_2 \leq Ch \|\nabla \psi\|_2, & \forall \psi \in V, \\ \|\psi - P^h \psi\|_{H_{\Gamma_0}^1} \leq Ch \|\psi\|_*, & \forall \psi \in V, \end{cases} \quad (3.4.1)$$

where $\|\psi\|_*^2 = \sum_{|\alpha|=2} \int_{\Omega} |\partial^\alpha \psi|^2 dx$.

The semi-discrete finite element method to (3.1.1) is to find $u^h(t)$, $\xi^h(t) \in V^h$, for $0 \leq t \leq T$, such that for all $v^h, w^h \in V^h$

$$\begin{cases} \left(u_{tt}^h, v^h \right) + \left(\nabla u^h, \nabla v^h \right) + \left(\nabla u_{tt}^h, \nabla v^h \right) + \left(\xi^h + \xi_{tt}^h, v^h \right)_1 = \left(f, v^h \right), \\ \left(\xi_{tt}^h + \xi^h, w^h \right)_1 = \left(u_t^h, w^h \right)_1 + \left(g, w^h \right)_1, \end{cases} \quad (3.4.2)$$

with $u^h(0) = P^h u_0$, $u_t^h(0) = P^h u_1$, $\xi^h(0) = P^h \xi_0$, and $\xi_t^h(0) = P^h \xi_1$.

In the next theorem we derive error bounds for the piecewise linear approximation (3.4.2).

Theorem 3.4.1. *Assume that the assumption of Theorem 3.2.1 holds with $\xi_0, \xi_1 \in H_{\Gamma_0}^2(\Omega)$ and $\xi \in H^1(0, T, H_{\Gamma_0}^2(\Omega))$, then we have the following estimate*

$$\|u_t - u_t^h\|_2^2 + \|\nabla u - \nabla u^h\|_2^2 + \|\nabla u_t - \nabla u_t^h\|_2^2 + \|\xi - \xi^h\|_{\Gamma_1}^2 + \|\xi_t - \xi_t^h\|_{\Gamma_1}^2 \leq Ch^2,$$

holds for $0 \leq t \leq T$.

Proof. In particular, (3.2.1) is true for all $v^h, w^h \in V^h$. Taking $v = v^h, w = w^h \in V^h$ and subtracting, we get

$$\left\{ \begin{array}{l} \left(u_{tt} - u_{tt}^h, v^h \right) + \left(\nabla u - \nabla u^h, \nabla v^h \right) + \left(\nabla u_t - \nabla u_t^h, \nabla v^h \right) \\ \quad + \left(\nabla u_{tt} - \nabla u_{tt}^h, \nabla v^h \right) + \left(\xi - \xi^h + \xi_t - \xi_t^h + \xi_{tt} - \xi_{tt}^h, v^h \right)_1 = 0, \\ \left(\xi_{tt} - \xi_{tt}^h + \xi - \xi^h, w^h \right)_1 = \left(u_t - u_t^h, w^h \right)_1. \end{array} \right. \quad (3.4.3)$$

That is

$$\left\{ \begin{array}{l} \left(u_{tt} - P^h u_{tt} + P^h u_{tt} - u_{tt}^h, v^h \right) + \left(\nabla u - \nabla(P^h u) + \nabla(P^h u) - \nabla u^h, \nabla v^h \right) \\ \quad + \left(\nabla u_t - \nabla(P^h u_t) + \nabla(P^h u_t) - \nabla u_t^h, \nabla v^h \right) + \left(\nabla u_{tt} - \nabla(P^h u_{tt}), \nabla v^h \right) \\ \quad + \left(\nabla(P^h u_{tt}) - \nabla u_{tt}^h, \nabla v^h \right) + \left(\xi - P^h \xi + P^h \xi - \xi^h + \xi_t - P^h \xi_t, v^h \right)_1 \\ \quad + \left(P^h \xi_t - \xi_t^h + \xi_{tt} - P^h \xi_{tt} + P^h \xi_{tt} - \xi_{tt}^h, v^h \right)_1 = 0, \\ \left(\xi_{tt} - P^h \xi_{tt} + P^h \xi_{tt} - \xi_{tt}^h + \xi - P^h \xi + P^h \xi - \xi^h, w^h \right)_1 = \left(u_t - P^h u_t, w^h \right)_1 \\ \quad + \left(P^h u_t - u_t^h, w^h \right)_1. \end{array} \right.$$

Consider the decomposition of the error: $e^h = u^h - P^h u$ and $b^h = \xi^h - P^h \xi$. Therefore

$$\left\{ \begin{array}{l} \left(e_{tt}^h, v^h \right) + \left(\nabla e^h, \nabla v^h \right) + \left(\nabla e_t^h, \nabla v^h \right) + \left(\nabla e_{tt}^h, \nabla v^h \right) + \left(b^h + b_t^h + b_{tt}^h, v^h \right)_1 \\ \quad = \left(u_{tt} - P^h u_{tt}, v^h \right) + \left(\nabla u - \nabla(P^h u), \nabla v^h \right) + \left(\nabla u_t - \nabla(P^h u_t), \nabla v^h \right) \\ \quad + \left(\nabla u_{tt} - \nabla(P^h u_{tt}), \nabla v^h \right) + \left(\xi - P^h \xi + \xi_t - P^h \xi_t + \xi_{tt} - P^h \xi_{tt}, v^h \right)_1, \\ \left(e_t^h, w^h \right)_1 = \left(u_t - P^h u_t, w^h \right)_1 - \left(\xi_{tt} - P^h \xi_{tt} + \xi - P^h \xi, w^h \right)_1 \\ \quad + \left(b^h + b_t^h, w^h \right)_1. \end{array} \right. \quad (3.4.4)$$

Taking $v^h = e_t^h$ and $w^h = b^h + b_t^h + b_{tt}^h$, we find

$$\left\{ \begin{array}{l} \frac{1}{2} \frac{d}{dt} \left(\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 \right) + \|\nabla e_t^h\|_2^2 + \left(b^h + b_t^h + b_{tt}^h, e_t^h \right)_1 \\ \quad = \left(u_{tt} - P^h u_{tt}, e_t^h \right) + \left(\nabla u - \nabla(P^h u), \nabla e_t^h \right) + \left(\nabla u_t - \nabla(P^h u_t), \nabla e_t^h \right) \\ \quad + \left(\nabla u_{tt} - \nabla(P^h u_{tt}), \nabla e_t^h \right) + \left(\xi - P^h \xi + \xi_t - P^h \xi_t + \xi_{tt} - P^h \xi_{tt}, e_t^h \right)_1, \\ \left(e_t^h, b^h + b_t^h + b_{tt}^h \right)_1 = \left(u_t - P^h u_t, b^h + b_t^h + b_{tt}^h \right)_1 - \left(\xi - P^h \xi, b^h + b_t^h + b_{tt}^h \right)_1 \\ \quad - \left(\xi_{tt} - P^h \xi_{tt}, b^h + b_t^h + b_{tt}^h \right)_1 + \|b^h + b_{tt}^h\|_{\Gamma_1}^2 + \frac{1}{2} \frac{d}{dt} \left(\|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 \right). \end{array} \right.$$

Therefore

$$\frac{1}{2} \frac{d}{dt} \left(\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 \right) + \|\nabla e_t^h\|_2^2 + \|b^h + b_{tt}^h\|_{\Gamma_1}^2$$

$$\begin{aligned}
&= \left(u_{tt} - P^h u_{tt}, e_t^h \right) + \left(\nabla u - \nabla(P^h u), \nabla e_t^h \right) + \left(\nabla u_t - \nabla(P^h u_t), \nabla e_t^h \right) \\
&\quad + \left(\nabla u_{tt} - \nabla(P^h u_{tt}), \nabla e_t^h \right) + \left(\xi - P^h \xi + \xi_t - P^h \xi_t + \xi_{tt} - P^h \xi_{tt}, e_t^h \right)_1 \\
&\quad + \left(\xi_{tt} - P^h \xi_{tt} + \xi - P^h \xi, b^h + b_t^h + b_{tt}^h \right)_1 - \left(u_t - P^h u_t, b^h + b_t^h + b_{tt}^h \right)_1.
\end{aligned} \tag{3.4.5}$$

By the definition of P^h , we have

$$\begin{aligned}
\left(\nabla u - \nabla(P^h u), \nabla e_t^h \right) &= \left(\nabla u, \nabla e_t^h \right) - \left(\nabla(P^h u), \nabla e_t^h \right) \\
&= \left(\nabla u, \nabla e_t^h \right) - \left(\nabla u, \nabla e_t^h \right) \\
&= 0.
\end{aligned}$$

Similarly we have

$$\left(\nabla u_t - \nabla(P^h u_t), \nabla e_t^h \right) = 0, \quad \text{and} \quad \left(\nabla u_{tt} - \nabla(P^h u_{tt}), \nabla e_t^h \right) = 0.$$

Combine these results with (3.4.5) to obtain

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 \right) + \|\nabla e_t^h\|_2^2 + \|b^h + b_{tt}^h\|_{\Gamma_1}^2 \\
&= \left(\xi_{tt} - P^h \xi_{tt} + \xi - P^h \xi, b^h + b_t^h + b_{tt}^h \right)_1 - \left(u_t - P^h u_t, b^h + b_t^h + b_{tt}^h \right)_1 \\
&\quad + \left(u_{tt} - P^h u_{tt}, e_t^h \right) + \left(\xi - P^h \xi + \xi_t - P^h \xi_t + \xi_{tt} - P^h \xi_{tt}, e_t^h \right)_1.
\end{aligned}$$

Using Young's inequality, there exist $\delta_{12}, \delta_{13}, \delta_{14}, \delta_{15} > 0$ such that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 \right) + \|\nabla e_t^h\|_2^2 + \|b^h + b_{tt}^h\|_{\Gamma_1}^2 \\
&\leq C \left(\|u_{tt} - P^h u_{tt}\|_2^2 + \|u_t - P^h u_t\|_{\Gamma_1}^2 + \|\xi - P^h \xi\|_{\Gamma_1}^2 + \|\xi_t - P^h \xi_t\|_{\Gamma_1}^2 \right) \\
&\quad + C \left\| \xi_{tt} - P^h \xi_{tt} \right\|_{\Gamma_1}^2 + \delta_{12} \|b^h + b_{tt}^h\|_{\Gamma_1}^2 + \delta_{13} \|b_t^h\|_{\Gamma_1}^2 + \delta_{14} \|e_t^h\|_2^2 + \delta_{15} \|e_t^h\|_{\Gamma_1}^2.
\end{aligned}$$

Using the embedding (3.2.8) and (3.4.1), we find

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 \right) + \|\nabla e_t^h\|_2^2 + \|b^h + b_{tt}^h\|_{\Gamma_1}^2 \\
&\leq Ch^2 (\|\nabla u_{tt}\|_2^2 + \|u_t\|_*^2 + \|\xi\|_*^2 + \|\xi_t\|_*^2 + \|\xi_{tt}\|_*^2) + \delta_{12} \|b^h + b_{tt}^h\|_{\Gamma_1}^2 \\
&\quad + \delta_{13} \|b_t^h\|_{\Gamma_1}^2 + \delta_{14} \|e_t^h\|_2^2 + \delta_{15} C_1 \|e_t^h\|_{H_{\Gamma_0}^1}^2.
\end{aligned}$$

By the Poincaré inequality, we obtain

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 \right) + \|\nabla e_t^h\|_2^2 + \|b^h + b_{tt}^h\|_{\Gamma_1}^2 \\
&\leq Ch^2 (\|\nabla u_{tt}\|_2^2 + \|u_t\|_*^2 + \|\xi\|_*^2 + \|\xi_t\|_*^2 + \|\xi_{tt}\|_*^2) + \delta_{12} \|b^h + b_{tt}^h\|_{\Gamma_1}^2
\end{aligned}$$

$$+ \delta_{13} \|b_t^h\|_{\Gamma_1}^2 + (\delta_{14} C_\Omega + \delta_{15} C_2) \|\nabla e_t^h\|_2^2.$$

Choosing $\delta_{12} = 1$, $\delta_{13} = \frac{1}{2}$, $\delta_{14} = \frac{1}{2C_\Omega}$, and $\delta_{15} = \frac{1}{2C_2}$, we get

$$\begin{aligned} & \frac{d}{dt} \left(\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 \right) \\ & \leq Ch^2 (\|\nabla u_{tt}\|_2^2 + \|u_t\|_*^2 + \|\xi\|_*^2 + \|\xi_t\|_*^2 + \|\xi_{tt}\|_*^2) + \|b_t^h\|_{\Gamma_1}^2. \end{aligned}$$

Integrate from 0 to t , we get

$$\begin{aligned} & \|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 \\ & \leq Ch^2 \int_0^t (\|\nabla u_{tt}(s)\|_2^2 + \|u_t(s)\|_*^2 + \|\xi(s)\|_*^2 + \|\xi_t(s)\|_*^2 + \|\xi_{tt}(s)\|_*^2) ds + \|e_t^h(0)\|_2^2 \\ & \quad + \|\nabla e^h(0)\|_2^2 + \|\nabla e_t^h(0)\|_2^2 + \|b^h(0)\|_{\Gamma_1}^2 + \|b_t^h(0)\|_{\Gamma_1}^2 + \int_0^t \|b_t^h(s)\|_{\Gamma_1}^2 ds. \end{aligned}$$

Note that

$$\|e_t^h(0)\|_2 = \|u_t^h(0) - P^h u_t(0)\|_2 = \|P_E^h u_1 - u_t(0)\|_2 = \|P_E^h u_1 - u_1\|_2 \leq Ch \|\nabla u_1\|_2.$$

Similarly we show $\|\nabla e^h(0)\|_2 \leq Ch \|\Delta u_0\|_2$, $\|\nabla e_t^h(0)\|_2 \leq Ch \|\Delta u_1\|_2$,

$$\|b^h(0)\|_{\Gamma_1} = \|\xi^h(0) - P^h \xi(0)\|_{\Gamma_1} = \|\xi_0 - P^h \xi_0\|_{\Gamma_1} \leq C \|\xi_0 - P^h \xi_0\|_{H_{\Gamma_0}^1} \leq Ch \|\xi_0\|_*,$$

and

$$\|b_t^h(0)\|_{\Gamma_1} = \|\xi_t^h(0) - P^h \xi_t(0)\|_{\Gamma_1} = \|\xi_1 - P^h \xi_1\|_{\Gamma_1} \leq C \|\xi_1 - P^h \xi_1\|_{H_{\Gamma_0}^1} \leq Ch \|\xi_1\|_*.$$

Using the above estimates, we get

$$\begin{aligned} & \|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 \\ & \leq Ch^2 \int_0^t (\|\nabla u_{tt}(s)\|_2^2 + \|u_t(s)\|_*^2 + \|\xi(s)\|_*^2 + \|\xi_t(s)\|_*^2 + \|\xi_{tt}(s)\|_*^2) ds \\ & \quad + Ch^2 (\|\nabla u_1\|_2^2 + \|\Delta u_0\|_2^2 + \|\Delta u_1\|_2^2 + \|\xi_0\|_*^2 + \|\xi_1\|_*^2) + \int_0^t \|b_t^h(s)\|_{\Gamma_1}^2 ds. \end{aligned}$$

Using the regularity on $u_0, u_1, \xi_0, \xi_1, \xi$, and u , we get

$$\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 \leq Ch^2 + \int_0^t \|b_t^h(s)\|_{\Gamma_1}^2 ds.$$

Applying Gronwall's inequality, we obtain

$$\|b_t^h\|_{\Gamma_1}^2 \leq Ch^2,$$

which implies that

$$\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 \leq Ch^2.$$

Finally, we get

$$\begin{aligned} & \|u_t - u_t^h\|_2^2 + \|\nabla u - \nabla u^h\|_2^2 + \|\nabla u_t - \nabla u_t^h\|_2^2 + \|\xi - \xi^h\|_{\Gamma_1}^2 + \|\xi_t - \xi_t^h\|_{\Gamma_1}^2 \\ &= \|u_t - P^h u_t + P^h u_t - u_t^h\|_2^2 + \|\nabla u - \nabla(P^h u) + \nabla(P^h u) - \nabla u^h\|_2^2 \\ & \quad + \|\nabla u_t - \nabla(P^h u_t) + \nabla(P^h u_t) - \nabla u_t^h\|_2^2 + \|\xi - P^h \xi + P^h \xi - \xi^h\|_{\Gamma_1}^2 \\ & \quad + \|\xi_t - P^h \xi_t + P^h \xi_t - \xi_t^h\|_{\Gamma_1}^2 \\ &\leq C \left(\|u_t - P^h u_t\|_2^2 + \|\nabla u - \nabla(P^h u)\|_2^2 + \|\nabla u_t - \nabla(P^h u_t)\|_2^2 + \|\xi - P^h \xi\|_{\Gamma_1}^2 \right) \\ & \quad + C \|\xi_t - P^h \xi_t\|_{\Gamma_1}^2 + C \left(\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + \|\nabla e_t^h\|_2^2 + \|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 \right) \\ &\leq Ch^2 (\|\nabla u_t\|_2^2 + \|\Delta u\|_2^2 + \|\Delta u_t\|_2^2 + \|\xi\|_*^2 + \|\xi_t\|_*^2) + Ch^2 \leq Ch^2. \end{aligned}$$

which completes the proof. $\square$

3.5 Fully-Discrete Approximation

In this section, we first present a fully discrete numerical scheme of the problem (3.1.1) and then perform the error analysis. Let $(n, N) \in \mathbb{N} \times \mathbb{N}^*$. Concerning the discretization in time, we divide the interval $[0, T]$ into N subintervals.

Let $\tau = \frac{T}{N}$ denote the time step. Define $t_n = n\tau$, for $n \in \{0, 1, 2, \dots, N\}$. Given an integer $N > 0$, our numerical scheme can be stated as find u^n and ξ^n , $n = 2, 3, \dots, N$ such that for all $v^h, w^h \in V^h$

$$\begin{cases} \left(\frac{\hat{u}^{n+1} - \hat{u}^n}{\tau}, v^h \right) + (\nabla u^{n+1}, \nabla v^h) + (\nabla \hat{u}^{n+1}, \nabla v^h) + \left(\frac{\nabla \hat{u}^{n+1} - \nabla \hat{u}^n}{\tau}, \nabla v^h \right) \\ \quad + (\xi^{n+1} + \hat{\xi}^{n+1}, v^h)_1 + \left(\frac{\hat{\xi}^{n+1} - \hat{\xi}^n}{\tau}, v^h \right)_1 = (f(t_{n+1}), v^h), \\ \left(\frac{\hat{\xi}^{n+1} - \hat{\xi}^n}{\tau}, w^h \right)_1 + (\xi^{n+1}, w^h)_1 = (\hat{u}^{n+1}, w^h)_1 + (g(t_{n+1}), w^h)_1. \end{cases} \quad (3.5.1)$$

where

$$\hat{u}^{n+1} = \frac{u^{n+1} - u^n}{\tau}, \quad \hat{\xi}^{n+1} = \frac{\xi^{n+1} - \xi^n}{\tau},$$

with the following discrete initial conditions

$$\begin{cases} u^0 = P^h u_0, & u^1 = u^0 + \tau P^h u_1, \\ \xi^0 = P^h \xi_0, & \xi^1 = \xi^0 + \tau P^h \xi_1. \end{cases} \quad (3.5.2)$$

3.5.1 Error Estimate

In the next theorem, we establish the error estimate for the fully discrete finite element scheme (3.5.1)–(3.5.2).

Theorem 3.5.1. *Assume that the assumption of theorem 3.2.1 holds, then if $u \in H^3(0, T, L^2(\Omega)) \cap W^{2,\infty}(0, T, H^3(\Omega))$ and $\xi \in H^2(0, T, H^2(\Omega)) \cap W^{1,\infty}(0, T, H^2(\Omega))$ we have*

$$\begin{aligned} & \|u_t(t_n) - \hat{u}^n\|_2^2 + \|\nabla u(t_n) - \nabla u^n\|_2^2 + \|\nabla u_t(t_n) - \nabla \hat{u}^n\|_2^2 + \|\xi(t_n) - \xi^n\|_{\Gamma_1}^2 \\ & + \|\xi_t(t_n) - \hat{\xi}^n\|_{\Gamma_1}^2 \leq C(h^2 + \tau^2). \end{aligned}$$

Proof. We denote $\hat{u}(t) = u_t(t)$, $\hat{\xi}(t_n) = \xi_t(t_n)$, and then we use the standard decompositions

$$\begin{cases} u^n - u(t_n) = e^n + \rho^n, \\ \xi^n - \xi(t_n) = b^n + \sigma^n, \\ \hat{u}^n - \hat{u}(t_n) = \hat{e}^n + \hat{\rho}^n, \\ \hat{\xi}^n - \hat{\xi}(t_n) = \hat{b}^n + \hat{\sigma}^n, \end{cases} \quad \text{where} \quad \begin{cases} e^n = u^n - P^h u(t_n), & \rho^n = P^h u(t_n) - u(t_n), \\ b^n = \xi^n - P^h \xi(t_n), & \sigma^n = P^h \xi(t_n) - \xi(t_n), \\ \hat{e}^n = \hat{u}^n - P^h \hat{u}(t_n), & \hat{\rho}^n = P^h \hat{u}(t_n) - \hat{u}(t_n), \\ \hat{b}^n = \hat{\xi}^n - P^h \hat{\xi}(t_n), & \hat{\sigma}^n = P^h \hat{\xi}(t_n) - \hat{\xi}(t_n). \end{cases}$$

Using (3.2.1) with $t = t_{n+1}$, $v = v^h$, and $w = w^h$, we get

$$\begin{cases} \left(\hat{u}_t(t_{n+1}), v^h \right) + \left(\nabla u(t_{n+1}), \nabla v^h \right) + \left(\nabla \hat{u}(t_{n+1}), \nabla v^h \right) + \left(\nabla \hat{u}_t(t_{n+1}), \nabla v^h \right) \\ \quad + \left(\xi(t_{n+1}) + \hat{\xi}(t_{n+1}) + \hat{\xi}_t(t_{n+1}), v^h \right)_1 = \left(f(t_{n+1}), v^h \right), \\ \left(\hat{\xi}_t(t_{n+1}), w^h \right)_1 + \left(\xi(t_{n+1}), w^h \right)_1 - \left(\hat{u}(t_{n+1}), w^h \right)_1 = \left(g(t_{n+1}), w^h \right)_1. \end{cases} \quad (3.5.3)$$

Subtracting system (3.5.3) from system (3.5.1) term by term, and applying the definition of P^h , it follows that

$$\begin{cases} \frac{1}{\tau} \left(\hat{e}^{n+1} - \hat{e}^n, v^h \right) + \left(\nabla e^{n+1}, \nabla v^h \right) + \left(\nabla \hat{e}^{n+1}, \nabla v^h \right) + \frac{1}{\tau} \left(\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla v^h \right) \\ \quad + \left(b^{n+1} + \hat{b}^{n+1}, v^h \right)_1 + \frac{1}{\tau} \left(\hat{b}^{n+1} - \hat{b}^n, v^h \right)_1 + \left(\sigma^{n+1} + \hat{\sigma}^{n+1}, v^h \right)_1 \\ = \left(\hat{u}_t(t_{n+1}), v^h \right) - \left(\frac{P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)}{\tau}, v^h \right) + \left(\nabla \hat{u}_t(t_{n+1}), \nabla v^h \right) \\ \quad - \left(\frac{\nabla \hat{u}(t_{n+1}) - \nabla \hat{u}(t_n)}{\tau}, \nabla v^h \right) + \left(\hat{\xi}_t(t_{n+1}) - \frac{P^h \hat{\xi}(t_{n+1}) - P^h \hat{\xi}(t_n)}{\tau}, v^h \right)_1, \\ \frac{1}{\tau} \left(\hat{b}^{n+1} - \hat{b}^n, w^h \right)_1 + \left(b^{n+1}, w^h \right)_1 = \left(\hat{e}^{n+1}, w^h \right)_1 + \left(\hat{\rho}^{n+1} - \sigma^{n+1}, w^h \right)_1 \\ \quad + \left(\hat{\xi}_t(t_{n+1}) - \frac{P^h \hat{\xi}(t_{n+1}) - P^h \hat{\xi}(t_n)}{\tau}, w^h \right)_1. \end{cases} \quad (3.5.4)$$

Now, take $v^h = \hat{e}^{n+1}$ and $w^h = \hat{e}^{n+1}$, we get

$$\begin{aligned} & \frac{1}{\tau} (\hat{e}^{n+1} - \hat{e}^n, \hat{e}^{n+1}) + (\nabla e^{n+1}, \nabla \hat{e}^{n+1}) + \frac{1}{\tau} (\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla \hat{e}^{n+1}) + \|\nabla \hat{e}^{n+1}\|_2^2 + \|\hat{e}^{n+1}\|_{\Gamma_1}^2 \\ &= (\hat{u}_t(t_{n+1}), \hat{e}^{n+1}) - \left(\frac{P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)}{\tau}, \hat{e}^{n+1} \right) + (\nabla \hat{u}_t(t_{n+1}), \nabla \hat{e}^{n+1}) \\ & \quad - \left(\frac{\nabla \hat{u}(t_{n+1}) - \nabla \hat{u}(t_n)}{\tau}, \nabla \hat{e}^{n+1} \right) - (\hat{b}^{n+1}, \hat{e}^{n+1})_1 - (\hat{\sigma}^{n+1} + \hat{\rho}^{n+1}, \hat{e}^{n+1})_1. \end{aligned}$$

Using Young's inequality, there exist $\delta_{16}, \delta_{17}, \delta_{18}, \delta_{19} > 0$ such that

$$\begin{aligned} & \frac{1}{\tau} (\hat{e}^{n+1} - \hat{e}^n, \hat{e}^{n+1}) + (\nabla e^{n+1}, \nabla \hat{e}^{n+1}) + \frac{1}{\tau} (\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla \hat{e}^{n+1}) + \|\nabla \hat{e}^{n+1}\|_2^2 + \|\hat{e}^{n+1}\|_{\Gamma_1}^2 \\ & \leq C \left(I_1^2 + \left\| \hat{u}_t(t_{n+1}) - \frac{1}{\tau} (P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)) \right\|_2^2 + \|\hat{\sigma}^{n+1}\|_{\Gamma_1}^2 + \|\hat{\rho}^{n+1}\|_{\Gamma_1}^2 \right) \\ & \quad + \delta_{16} \|\hat{e}^{n+1}\|_2^2 + \delta_{17} \|\nabla \hat{e}^{n+1}\|_2^2 + \delta_{18} \|\hat{b}^{n+1}\|_{\Gamma_1}^2 + \frac{1}{4\delta_{18}} \|\hat{e}^{n+1}\|_{\Gamma_1}^2 + \delta_{19} \|\hat{e}^{n+1}\|_{\Gamma_1}^2 \\ & \leq C \left(I_1^2 + I_2^2 + I_3^2 + \|\hat{\sigma}^{n+1}\|_{\Gamma_1}^2 + \|\hat{\rho}^{n+1}\|_{\Gamma_1}^2 \right) + \delta_{16} \|\hat{e}^{n+1}\|_2^2 + \left(\delta_{19} + \frac{1}{4\delta_{18}} \right) \|\hat{e}^{n+1}\|_{\Gamma_1}^2 \\ & \quad + \delta_{17} \|\nabla \hat{e}^{n+1}\|_2^2 + \delta_{18} \|\hat{b}^{n+1}\|_{\Gamma_1}^2, \end{aligned}$$

where

$$\begin{cases} I_1 &= \left\| \nabla \hat{u}_t(t_{n+1}) - \frac{1}{\tau} (\nabla \hat{u}(t_{n+1}) - \nabla \hat{u}(t_n)) \right\|_2, \\ I_2 &= \left\| \hat{u}_t(t_{n+1}) - \frac{1}{\tau} (\hat{u}(t_{n+1}) - \hat{u}(t_n)) \right\|_2, \\ I_3 &= \left\| \frac{1}{\tau} (\hat{u}(t_{n+1}) - \hat{u}(t_n)) - \frac{1}{\tau} (P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)) \right\|_2. \end{cases}$$

Choosing $\delta_{16} = \delta_{17} = \delta_{18} = 1$, and $\delta_{19} = \frac{3}{4}$, we get

$$\begin{aligned} & \frac{1}{\tau} (\hat{e}^{n+1} - \hat{e}^n, \hat{e}^{n+1}) + (\nabla e^{n+1}, \nabla \hat{e}^{n+1}) + \frac{1}{\tau} (\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla \hat{e}^{n+1}) \\ & \leq C \left(I_1^2 + I_2^2 + I_3^2 + \|\hat{\sigma}^{n+1}\|_{\Gamma_1}^2 + \|\hat{\rho}^{n+1}\|_{\Gamma_1}^2 \right) + \|\hat{e}^{n+1}\|_2^2 + \|\hat{b}^{n+1}\|_{\Gamma_1}^2. \end{aligned} \tag{3.5.5}$$

Knowing that

$$\begin{aligned} \hat{e}^{n+1} &= \hat{u}^{n+1} - P^h \hat{u}(t_{n+1}) \\ &= \frac{1}{\tau} (u^{n+1} - u^n) - P^h \hat{u}(t_{n+1}) \\ &= \frac{1}{\tau} (e^{n+1} - e^n) + \frac{1}{\tau} (P^h u(t_{n+1}) - P^h u(t_n)) - P^h \hat{u}(t_{n+1}), \end{aligned}$$

hence

$$\begin{aligned} (\nabla e^{n+1}, \nabla \hat{e}^{n+1}) &= \frac{1}{\tau} (\nabla e^{n+1}, \nabla e^{n+1} - \nabla e^n) - (\nabla e^{n+1}, \nabla (P^h \hat{u}(t_{n+1}))) \\ & \quad + \frac{1}{\tau} (\nabla e^{n+1}, \nabla (P^h u(t_{n+1})) - \nabla (P^h u(t_n))) \end{aligned}$$

$$= \left(\nabla e^{n+1}, \frac{1}{\tau} (\nabla u(t_{n+1}) - \nabla u(t_n)) - \nabla \hat{u}(t_{n+1}) \right) + \frac{1}{\tau} (\nabla e^{n+1}, \nabla e^{n+1} - \nabla e^n),$$

where we used the definition of P^h .

By using this result, (3.5.5) becomes

$$\begin{aligned} & \frac{1}{\tau} (\hat{e}^{n+1} - \hat{e}^n, \hat{e}^{n+1}) + \frac{1}{\tau} (\nabla e^{n+1}, \nabla e^{n+1} - \nabla e^n) + \frac{1}{\tau} (\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla \hat{e}^{n+1}) \\ & \leq C \left(I_1^2 + I_2^2 + I_3^2 + \|\hat{\sigma}^{n+1}\|_{\Gamma_1}^2 + \|\hat{\rho}^{n+1}\|_{\Gamma_1}^2 \right) + \|\hat{e}^{n+1}\|_2^2 + \|\hat{b}^{n+1}\|_{\Gamma_1}^2 \\ & - \left(\nabla e^{n+1}, \frac{1}{\tau} (\nabla u(t_{n+1}) - \nabla u(t_n)) - \nabla \hat{u}(t_{n+1}) \right). \end{aligned}$$

Using Young's Inequality with $\delta = 1$, we get

$$\begin{aligned} & \frac{1}{\tau} (\hat{e}^{n+1} - \hat{e}^n, \hat{e}^{n+1}) + \frac{1}{\tau} (\nabla e^{n+1}, \nabla e^{n+1} - \nabla e^n) + \frac{1}{\tau} (\nabla \hat{e}^{n+1} - \nabla \hat{e}^n, \nabla \hat{e}^{n+1}) \\ & \leq C \left(I_1^2 + I_2^2 + I_3^2 + I_4^2 + \|\hat{\sigma}^{n+1}\|_{\Gamma_1}^2 + \|\hat{\rho}^{n+1}\|_{\Gamma_1}^2 \right) + \|\hat{e}^{n+1}\|_2^2 + \|\nabla e^{n+1}\|_2^2 \\ & + \|\hat{b}^{n+1}\|_{\Gamma_1}^2, \end{aligned} \quad (3.5.6)$$

where

$$I_4 = \left\| \frac{1}{\tau} (\nabla u(t_{n+1}) - \nabla u(t_n)) - \nabla \hat{u}(t_{n+1}) \right\|_2.$$

Next, we estimate I_1, I_2, I_3 , and I_4 . Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} I_1 &= \left\| \nabla \hat{u}_t(t_{n+1}) - \frac{1}{\tau} (\nabla \hat{u}(t_{n+1}) - \nabla \hat{u}(t_n)) \right\|_2 = \left\| \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \nabla \hat{u}_{tt}(s) ds \right\|_2 \\ &\leq \frac{1}{\tau} \left(\int_{t_n}^{t_{n+1}} (s - t_n)^2 ds \right)^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \|\nabla \hat{u}_{tt}(s)\|_2^2 ds \right)^{\frac{1}{2}} \\ &\leq \left(\frac{\tau}{3} \right)^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \|\nabla \hat{u}_{tt}(s)\|_2^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Similarly, for I_2, I_3 and I_4 , we have

$$\begin{aligned} I_2 &= \left\| \hat{u}_t(t_{n+1}) - \frac{1}{\tau} (\hat{u}(t_{n+1}) - \hat{u}(t_n)) \right\|_2 = \left\| \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \hat{u}_{tt}(s) ds \right\|_2 \\ &\leq \frac{1}{\tau} \left(\int_{t_n}^{t_{n+1}} (s - t_n)^2 ds \right)^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \|\hat{u}_{tt}(s)\|_2^2 ds \right)^{\frac{1}{2}} \\ &\leq \left(\frac{\tau}{3} \right)^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \|\hat{u}_{tt}(s)\|_2^2 ds \right)^{\frac{1}{2}}, \end{aligned}$$

and

$$I_3 = \left\| \frac{1}{\tau} (\hat{u}(t_{n+1}) - \hat{u}(t_n)) - \frac{1}{\tau} (P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)) \right\|_2$$

$$\begin{aligned}
&= \left\| \frac{1}{\tau} (\hat{\rho}(t_{n+1}) - \hat{\rho}(t_n)) \right\|_2 = \left\| \hat{\rho}_t(t_{n+1}) - \frac{1}{\tau} (\hat{\rho}(t_{n+1}) - \hat{\rho}(t_n)) - \hat{\rho}_t(t_{n+1}) \right\|_2 \\
&\leq \left\| \hat{\rho}_t(t_{n+1}) - \frac{1}{\tau} (\hat{\rho}(t_{n+1}) - \hat{\rho}(t_n)) \right\|_2 + \left\| \hat{\rho}_t(t_{n+1}) \right\|_2 \\
&\leq \left\| \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \hat{\rho}_{tt}(s) ds \right\|_2 + \left\| \hat{\rho}_t(t_{n+1}) \right\|_2 \\
&\leq \left(\frac{\tau}{3} \right)^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \left\| \hat{\rho}_{tt}(s) \right\|_2^2 ds \right)^{\frac{1}{2}} + \left\| \hat{\rho}_t(t_{n+1}) \right\|_2,
\end{aligned}$$

also

$$\begin{aligned}
I_4 &= \left\| \nabla \hat{u}(t_{n+1}) - \frac{1}{\tau} (\nabla u(t_{n+1}) - \nabla u(t_n)) \right\|_2 = \left\| \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \nabla \hat{u}_t(s) ds \right\|_2 \\
&\leq \frac{1}{\tau} \left(\int_{t_n}^{t_{n+1}} (s - t_n)^2 ds \right)^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \left\| \nabla \hat{u}_t(s) \right\|_2^2 ds \right)^{\frac{1}{2}} \\
&\leq \left(\frac{\tau}{3} \right)^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \left\| \nabla \hat{u}_t(s) \right\|_2^2 ds \right)^{\frac{1}{2}}.
\end{aligned}$$

Using (3.4.1)₁, we get

$$\begin{cases} \left\| \hat{\rho}_t(s) \right\|_2 = \left\| (P^h \hat{u}(s))_t - \hat{u}_t(s) \right\|_2 & \leq Ch \left\| \nabla \hat{u}_t(s) \right\|_2, \\ \left\| \hat{\rho}_{tt}(s) \right\|_2 = \left\| (P^h \hat{u}(s))_{tt} - \hat{u}_{tt}(s) \right\|_2 & \leq Ch \left\| \nabla \hat{u}_{tt}(s) \right\|_2, \end{cases}$$

hence

$$I_3 \leq Ch\tau^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \left\| \nabla \hat{u}_{tt}(s) \right\|_2^2 ds \right)^{\frac{1}{2}} + Ch \left\| \nabla u_{tt}(t_{n+1}) \right\|_2.$$

Using the embedding (3.2.8) and (3.4.1)₂, we find

$$\begin{cases} \left\| \hat{\rho}^{n+1} \right\|_{\Gamma_1}^2 = \left\| P^h \hat{u}(t_{n+1}) - \hat{u}(t_{n+1}) \right\|_{\Gamma_1}^2 & \leq Ch^2 \left\| \hat{u}(t_{n+1}) \right\|_*^2, \\ \left\| \hat{\sigma}^{n+1} \right\|_{\Gamma_1}^2 = \left\| P^h \hat{\xi}(t_{n+1}) - \hat{\xi}(t_{n+1}) \right\|_{\Gamma_1}^2 & \leq Ch^2 \left\| \hat{\xi}(t_{n+1}) \right\|_*^2. \end{cases}$$

Using the elementary equality (2.3.3) and combining the above estimates with (3.5.6), to obtain

$$\begin{aligned}
&\frac{1}{2\tau} \left(\left\| \hat{e}^{n+1} - \hat{e}^n \right\|_2^2 + \left\| \hat{e}^{n+1} \right\|_2^2 - \left\| \hat{e}^n \right\|_2^2 \right) + \frac{1}{2\tau} \left(\left\| \nabla e^{n+1} - \nabla e^n \right\|_2^2 + \left\| \nabla e^{n+1} \right\|_2^2 - \left\| \nabla e^n \right\|_2^2 \right) \\
&\quad + \frac{1}{2\tau} \left(\left\| \nabla \hat{e}^{n+1} - \nabla \hat{e}^n \right\|_2^2 + \left\| \nabla \hat{e}^{n+1} \right\|_2^2 - \left\| \nabla \hat{e}^n \right\|_2^2 \right) \\
&\leq C\tau \left(\int_{t_n}^{t_{n+1}} \left\| \hat{u}_{tt}(s) \right\|_2^2 ds + (1 + h^2) \int_{t_n}^{t_{n+1}} \left\| \nabla \hat{u}_{tt}(s) \right\|_2^2 ds + \int_{t_n}^{t_{n+1}} \left\| \nabla \hat{u}_t(s) \right\|_2^2 ds \right) \\
&\quad + Ch^2 \left(\left\| \nabla u_{tt}(t_{n+1}) \right\|_2^2 + \left\| u_t(t_{n+1}) \right\|_*^2 + \left\| \xi_t(t_{n+1}) \right\|_*^2 \right) + \left\| \hat{e}^{n+1} \right\|_2^2 + \left\| \nabla e^{n+1} \right\|_2^2 \\
&\quad + \left\| \hat{b}^{n+1} \right\|_{\Gamma_1}^2.
\end{aligned}$$

Multiply the last inequality by 2τ and sum from $i = 1$ to n

$$\begin{aligned}
& \|\hat{e}^{n+1}\|_2^2 + \sum_{i=1}^n \|\hat{e}^{i+1} - \hat{e}^i\|_2^2 + \|\nabla e^{n+1}\|_2^2 + \sum_{i=1}^n \|\nabla e^{i+1} - \nabla e^i\|_2^2 + \|\nabla \hat{e}^{n+1}\|_2^2 \\
& + \sum_{i=1}^n \|\nabla \hat{e}^{i+1} - \nabla \hat{e}^i\|_2^2 \\
& \leq C\tau^2 \left(\int_{t_1}^{t_{n+1}} (\|\hat{u}_{tt}(s)\|_2^2 + \|\nabla \hat{u}_t(s)\|_2^2) ds + (1+h^2) \int_{t_1}^{t_{n+1}} \|\nabla \hat{u}_{tt}(s)\|_2^2 ds \right) \\
& + Ch^2\tau \sum_{i=1}^n (\|\nabla u_{tt}(t_{i+1})\|_2^2 + \|u_t(t_{i+1})\|_*^2 + \|\xi_t(t_{i+1})\|_*^2) + \|\hat{e}^1\|_2^2 + \|\nabla e^1\|_2^2 \\
& + \|\nabla \hat{e}^1\|_2^2 + 2\tau \sum_{i=1}^n \left(\|\hat{e}^{i+1}\|_2^2 + \|\nabla e^{i+1}\|_2^2 + \|\hat{b}^{i+1}\|_{\Gamma_1}^2 \right). \tag{3.5.7}
\end{aligned}$$

We have to bound $\|\hat{e}^1\|_2$, $\|\nabla e^1\|_2$, and $\|\nabla \hat{e}^1\|_2$. Indeed

$$\begin{aligned}
(\nabla e^1, \nabla v) &= (\nabla u^1 - \nabla(P^h u(t_1)), \nabla v) \\
&= (\nabla u^0 + \tau \nabla(P^h u_1) - \nabla(P^h u(t_1)), \nabla v) \\
&= (\nabla(P^h u_0) + \tau \nabla(P^h u_1) - \nabla(P^h u(t_1)), \nabla v) \\
&= (\nabla u_0 + \tau \nabla u_1 - \nabla u(t_1), \nabla v) \\
&= (\nabla u(0) + \tau \nabla u_t(0) - \nabla u(t_1), \nabla v),
\end{aligned}$$

where we used the definition of P^h , u^1 , and u^0 .

Knowing that,

$$\nabla u(t_1) = \nabla u(0) + \tau \nabla u_t(0) + \frac{\tau^2}{2} \nabla u_{tt}(\tau_1), \quad \text{with } 0 \leq \tau_1 \leq t_1,$$

hence

$$(\nabla e^1, \nabla v) = -\frac{\tau^2}{2} (\nabla u_{tt}(\tau_1), \nabla v).$$

Choose $v = e^1$, we obtain

$$\|\nabla e^1\|_2^2 \leq \frac{\tau^2}{2} \|\nabla u_{tt}(\tau_1)\|_2 \|\nabla e^1\|_2, \quad \text{that is, } \|\nabla e^1\|_2 \leq C\tau.$$

To bound $\|\nabla \hat{e}^1\|_2$, we use $\hat{u}^1 = \frac{1}{\tau}(u^1 - u^0)$ and $u^1 = u^0 + \tau P^h u_1$ thus

$$\begin{aligned}
(\nabla \hat{e}^1, \nabla v) &= (\nabla \hat{u}^1 - \nabla(P^h \hat{u}(t_1)), \nabla v) \\
&= (\nabla(P^h u_1) - \nabla(P^h \hat{u}(t_1)), \nabla v) \\
&= (\nabla(P^h u_1), \nabla v) - (\nabla(P^h \hat{u}(t_1)), \nabla v) \\
&= (\nabla u_1, \nabla v) - (\nabla \hat{u}(t_1), \nabla v) \\
&= (\nabla u_t(0) - \nabla u_t(t_1), \nabla v).
\end{aligned}$$

Knowing that

$$\nabla u_t(t_1) = \nabla u_t(0) + \tau \nabla u_{tt}(\tau_2), \quad \text{with } 0 \leq \tau_2 \leq t_1,$$

hence

$$(\nabla \hat{e}^1, \nabla v) = -\tau (\nabla u_{tt}(\tau_2), \nabla v).$$

Choose $v = \hat{e}^1$, we obtain

$$\|\nabla \hat{e}^1\|_2^2 \leq C\tau \|\nabla u_{tt}(\tau_2)\|_2 \|\nabla \hat{e}^1\|_2, \quad \text{that is, } \|\nabla \hat{e}^1\|_2 \leq C\tau.$$

Using the Poincaré's inequality, we find

$$\|\hat{e}^1\|_2 \leq C \|\nabla \hat{e}^1\|_2 \leq C\tau.$$

Using (3.5.7) and the above estimates, we get

$$\begin{aligned} & \|\hat{e}^{n+1}\|_2^2 + \|\nabla e^{n+1}\|_2^2 + \|\nabla \hat{e}^{n+1}\|_2^2 \\ & \leq C(\tau^2 + h^2) + 2\tau \sum_{i=1}^n \left(\|\hat{e}^{i+1}\|_2^2 + \|\nabla e^{i+1}\|_2^2 + \|\hat{b}^{i+1}\|_{\Gamma_1}^2 \right). \end{aligned} \quad (3.5.8)$$

On the other hand, we use (3.5.4)₂ and taking $w^h = \hat{b}^{n+1}$ to obtain

$$\begin{aligned} (b^{n+1}, \hat{b}^{n+1})_1 + \frac{1}{\tau} (\hat{b}^{n+1} - \hat{b}^n, \hat{b}^{n+1})_1 &= \left(\hat{\xi}_t(t_{n+1}) - \frac{1}{\tau} (P^h \hat{\xi}(t_{n+1}) - P^h \hat{\xi}(t_n)), \hat{b}^{n+1} \right)_1 \\ &+ (\hat{e}^{n+1}, \hat{b}^{n+1})_1 + (\hat{\rho}^{n+1} - \sigma^{n+1}, \hat{b}^{n+1})_1. \end{aligned}$$

Young's inequality gives

$$\begin{aligned} & \frac{1}{\tau} (\hat{b}^{n+1} - \hat{b}^n, \hat{b}^{n+1})_1 + (b^{n+1}, \hat{b}^{n+1})_1 \\ & \leq C \left(I_5^2 + \|\hat{\rho}^{n+1}\|_{\Gamma_1}^2 + \|\sigma^{n+1}\|_{\Gamma_1}^2 \right) + \delta_{20} \|\hat{e}^{n+1}\|_{\Gamma_1}^2 + \left(\delta_{21} + \frac{1}{4\delta_{20}} \right) \|\hat{b}^{n+1}\|_{\Gamma_1}^2, \end{aligned} \quad (3.5.9)$$

where

$$I_5 = \left\| \hat{\xi}_t(t_{n+1}) - \frac{1}{\tau} (P^h \hat{\xi}(t_{n+1}) - P^h \hat{\xi}(t_n)) \right\|_{\Gamma_1}.$$

Note that

$$\xi_t(t_{n+1}) = \frac{1}{\tau} (\xi(t_{n+1}) - \xi(t_n)) + \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \xi_{tt}(s) ds,$$

hence

$$\begin{aligned} \hat{b}^{n+1} &= \hat{\xi}^{n+1} - P^h \xi_t(t_{n+1}) \\ &= \frac{1}{\tau} \left[(\xi^{n+1} - P^h \xi(t_{n+1})) - (\xi^n - P^h \xi(t_n)) - \int_{t_n}^{t_{n+1}} (s - t_n) P^h \xi_{tt}(s) ds \right] \\ &= \frac{1}{\tau} (b^{n+1} - b^n - \theta^{n+1}), \end{aligned}$$

where $\theta^{n+1} = \int_{t_n}^{t_{n+1}} (s - t_n) P^h \xi_{tt}(s) ds$.

Therefore

$$\begin{aligned} (b^{n+1}, \hat{b}^{n+1})_1 &= \frac{1}{\tau} (b^{n+1}, b^{n+1} - b^n - \theta^{n+1})_1 \\ &= \frac{1}{\tau} (b^{n+1}, b^{n+1} - b^n)_1 - \frac{1}{\tau} (b^{n+1}, \theta^{n+1})_1. \end{aligned}$$

Combine this result with (3.5.9) to obtain

$$\begin{aligned} & \frac{1}{\tau} (\hat{b}^{n+1} - \hat{b}^n, \hat{b}^{n+1})_1 + \frac{1}{\tau} (b^{n+1}, b^{n+1} - b^n)_1 \\ & \leq C \left(I_5^2 + \|\hat{\rho}^{n+1}\|_{\Gamma_1}^2 + \|\sigma^{n+1}\|_{\Gamma_1}^2 \right) + \delta_{20} \|\hat{e}^{n+1}\|_{\Gamma_1}^2 + \left(\delta_{21} + \frac{1}{4\delta_{20}} \right) \|\hat{b}^{n+1}\|_{\Gamma_1}^2 \\ & \quad + \frac{1}{\tau} (b^{n+1}, \theta^{n+1})_1 \\ & \leq C \left(I_5^2 + \|\hat{\rho}^{n+1}\|_{\Gamma_1}^2 + \|\sigma^{n+1}\|_{\Gamma_1}^2 \right) + \delta_{20} \|\hat{e}^{n+1}\|_{\Gamma_1}^2 + \left(\delta_{21} + \frac{1}{4\delta_{20}} \right) \|\hat{b}^{n+1}\|_{\Gamma_1}^2 \\ & \quad + \frac{1}{\tau} \left(\delta_{22} \|b^{n+1}\|_{\Gamma_1}^2 + \frac{1}{4\delta_{22}} \|\theta^{n+1}\|_{\Gamma_1}^2 \right), \end{aligned} \tag{3.5.10}$$

where we used Young's inequality.

Next, we estimate I_5 . Note that

$$\begin{aligned} I_5 &\leq \left\| \frac{1}{\tau} (\hat{\xi}(t_{n+1}) - \hat{\xi}(t_n)) - \frac{1}{\tau} (P^h \hat{\xi}(t_{n+1}) - P^h \hat{\xi}(t_n)) \right\|_{\Gamma_1} \\ &\quad + \left\| \hat{\xi}_t(t_{n+1}) - \frac{1}{\tau} (\hat{\xi}(t_{n+1}) - \hat{\xi}(t_n)) \right\|_{\Gamma_1}. \end{aligned}$$

Knowing that

$$\left\| \hat{\xi}_t(t_{n+1}) - \frac{1}{\tau} (\hat{\xi}(t_{n+1}) - \hat{\xi}(t_n)) \right\|_{\Gamma_1} = \left\| \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \hat{\xi}_{tt}(s) ds \right\|_{\Gamma_1},$$

and

$$\begin{aligned} & \left\| \frac{1}{\tau} (\hat{\xi}(t_{n+1}) - \hat{\xi}(t_n)) - \frac{1}{\tau} (P^h \hat{\xi}(t_{n+1}) - P^h \hat{\xi}(t_n)) \right\|_{\Gamma_1} \\ &= \left\| \frac{1}{\tau} (\hat{\sigma}(t_{n+1}) - \hat{\sigma}(t_n)) \right\|_{\Gamma_1} \\ &= \left\| \hat{\sigma}_t(t_{n+1}) - \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \hat{\sigma}_{tt}(s) ds \right\|_{\Gamma_1} \\ &\leq \left\| \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \hat{\sigma}_{tt}(s) ds \right\|_{\Gamma_1} + \|\hat{\sigma}_t(t_{n+1})\|_{\Gamma_1}. \end{aligned}$$

By using Cauchy-Schwarz inequality, we obtain

$$\left\| \hat{\xi}_t(t_{n+1}) - \frac{1}{\tau} (\hat{\xi}(t_{n+1}) - \hat{\xi}(t_n)) \right\|_{\Gamma_1} \leq C \tau^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \|\hat{\xi}_{tt}(s)\|_{\Gamma_1}^2 ds \right)^{\frac{1}{2}},$$

and

$$\left\| \frac{1}{\tau} (\hat{\xi}(t_{n+1}) - \hat{\xi}(t_n)) - \frac{1}{\tau} (P^h \hat{\xi}(t_{n+1}) - P^h \hat{\xi}(t_n)) \right\|_{\Gamma_1} \leq C\tau^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \|\hat{\sigma}_{tt}(s)\|_{\Gamma_1}^2 ds \right)^{\frac{1}{2}} + \|\hat{\sigma}_t(t_{n+1})\|_{\Gamma_1}.$$

Using the embedding (3.2.8) and (3.4.1)₂, we find

$$\|\hat{\sigma}_{tt}(s)\|_{\Gamma_1}^2 \leq Ch^2 \|\hat{\xi}_{tt}(s)\|_*^2, \quad \text{and} \quad \|\hat{\sigma}_t(t_{n+1})\|_{\Gamma_1} \leq Ch \|\hat{\xi}_t(t_{n+1})\|_*.$$

Hence

$$\begin{aligned} & \left\| \frac{1}{\tau} (\hat{\xi}(t_{n+1}) - \hat{\xi}(t_n)) - \frac{1}{\tau} (P^h \hat{\xi}(t_{n+1}) - P^h \hat{\xi}(t_n)) \right\|_{\Gamma_1} \\ & \leq Ch \|\hat{\xi}_t(t_{n+1})\|_* + Ch\tau^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \|\hat{\xi}_{tt}(s)\|_*^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore

$$I_5 \leq C\tau^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \|\hat{\xi}_{tt}(s)\|_{\Gamma_1}^2 ds \right)^{\frac{1}{2}} + Ch\tau^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \|\hat{\xi}_{tt}(s)\|_*^2 ds \right)^{\frac{1}{2}} + Ch \|\hat{\xi}_t(t_{n+1})\|_*.$$

Using the embedding (3.2.8) with the Poincaré inequality, we find

$$\|\hat{e}^{n+1}\|_{\Gamma_1}^2 \leq C_1 \|\hat{e}^{n+1}\|_{H_{\Gamma_0}^1}^2 \leq C_2 \|\nabla \hat{e}^{n+1}\|_2^2.$$

Using the embedding (3.2.8) and (3.4.1)₂, we find

$$\|\sigma^{n+1}\|_{\Gamma_1}^2 = \|P^h \xi(t_{n+1}) - \xi(t_{n+1})\|_{\Gamma_1}^2 \leq Ch^2 \|\xi(t_{n+1})\|_*^2.$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|\theta^{n+1}\|_{\Gamma_1} &= \left\| \int_{t_n}^{t_{n+1}} (s - t_n) P^h \xi_{tt}(s) ds \right\|_{\Gamma_1} \\ &\leq C\tau^{\frac{3}{2}} \left(\int_{t_n}^{t_{n+1}} \|P^h \xi_{tt}(s)\|_{\Gamma_1}^2 ds \right)^{\frac{1}{2}} \\ &\leq C\tau^{\frac{3}{2}} \left(\int_{t_n}^{t_{n+1}} \|P^h \xi_{tt}(s) - \xi_{tt}(s)\|_{\Gamma_1}^2 + \|\xi_{tt}(s)\|_{\Gamma_1}^2 ds \right)^{\frac{1}{2}} \\ &\leq C\tau^{\frac{3}{2}} \left(\int_{t_n}^{t_{n+1}} (\|\xi_{tt}(s)\|_{\Gamma_1}^2 + h^2 \|\xi_{tt}(s)\|_*^2) ds \right)^{\frac{1}{2}}. \end{aligned}$$

Combining the above estimates with (3.5.10) and choosing $\delta_{20} = \delta_{21} = \frac{1}{2}$ and $\delta_{22} = \tau$, we obtain

$$\begin{aligned}
& \frac{1}{\tau} (b^{n+1}, b^{n+1} - b^n)_1 + \frac{1}{\tau} (\hat{b}^{n+1} - \hat{b}^n, \hat{b}^{n+1})_1 \\
& \leq C\tau \int_{t_n}^{t_{n+1}} \left(\|\xi_{tt}(s)\|_{\Gamma_1}^2 + \|\hat{\xi}_{tt}(s)\|_{\Gamma_1}^2 + h^2 \left(\|\xi_{tt}(s)\|_*^2 + \|\hat{\xi}_{tt}(s)\|_*^2 \right) \right) ds + Ch^2 \|\hat{u}(t_{n+1})\|_*^2 \\
& \quad + Ch^2 \left(\|\xi(t_{n+1})\|_*^2 + \|\hat{\xi}(t_{n+1})\|_*^2 \right) + \frac{C_2}{2} \|\nabla \hat{e}^{n+1}\|_2^2 + \|b^{n+1}\|_{\Gamma_1}^2 + \|\hat{b}^{n+1}\|_{\Gamma_1}^2.
\end{aligned}$$

Using the elementary equality (2.3.3) to obtain

$$\begin{aligned}
& \frac{1}{2\tau} \left(\|b^{n+1} - b^n\|_{\Gamma_1}^2 + \|b^{n+1}\|_{\Gamma_1}^2 - \|b^n\|_{\Gamma_1}^2 \right) + \frac{1}{2\tau} \left(\|\hat{b}^{n+1} - \hat{b}^n\|_{\Gamma_1}^2 + \|\hat{b}^{n+1}\|_{\Gamma_1}^2 - \|\hat{b}^n\|_{\Gamma_1}^2 \right) \\
& \leq C\tau \int_{t_n}^{t_{n+1}} \left(\|\xi_{tt}(s)\|_{\Gamma_1}^2 + \|\hat{\xi}_{tt}(s)\|_{\Gamma_1}^2 + h^2 \left(\|\xi_{tt}(s)\|_*^2 + \|\hat{\xi}_{tt}(s)\|_*^2 \right) \right) ds + Ch^2 \|\hat{u}(t_{n+1})\|_*^2 \\
& \quad + Ch^2 \left(\|\xi(t_{n+1})\|_*^2 + \|\hat{\xi}(t_{n+1})\|_*^2 \right) + \frac{C_2}{2} \|\nabla \hat{e}^{n+1}\|_2^2 + \|b^{n+1}\|_{\Gamma_1}^2 + \|\hat{b}^{n+1}\|_{\Gamma_1}^2.
\end{aligned}$$

Multiply the last inequality by 2τ and sum from $i = 1$ to n

$$\begin{aligned}
& \sum_{i=1}^n \|b^{i+1} - b^i\|_{\Gamma_1}^2 + \|b^{n+1}\|_{\Gamma_1}^2 + \sum_{i=1}^n \|\hat{b}^{i+1} - \hat{b}^i\|_{\Gamma_1}^2 + \|\hat{b}^{n+1}\|_{\Gamma_1}^2 \\
& \leq C\tau^2 \int_{t_1}^{t_{n+1}} \left(\|\xi_{tt}(s)\|_{\Gamma_1}^2 + \|\hat{\xi}_{tt}(s)\|_{\Gamma_1}^2 + h^2 \left(\|\xi_{tt}(s)\|_*^2 + \|\hat{\xi}_{tt}(s)\|_*^2 \right) \right) ds \\
& \quad + C\tau h^2 \sum_{i=1}^n \left(\|\hat{u}(t_{i+1})\|_*^2 + \|\xi(t_{i+1})\|_*^2 + \|\hat{\xi}(t_{i+1})\|_*^2 \right) + \|b^1\|_{\Gamma_1}^2 + \|\hat{b}^1\|_{\Gamma_1}^2 \\
& \quad + C_2\tau \sum_{i=1}^n \|\nabla \hat{e}^{i+1}\|_2^2 + 2\tau \sum_{i=1}^n \left(\|b^{i+1}\|_{\Gamma_1}^2 + \|\hat{b}^{i+1}\|_{\Gamma_1}^2 \right). \tag{3.5.11}
\end{aligned}$$

We have to bound $\|b^1\|_{\Gamma_1}$ and $\|\hat{b}^1\|_{\Gamma_1}$.

To bound $\|b^1\|_{\Gamma_1}$, we use the fact

$$\xi(t_1) = \xi_0 + \tau \xi_1 + \frac{\tau^2}{2} \xi_{tt}(\tau_3) \quad \text{with} \quad 0 \leq \tau_3 \leq t_1, \quad \text{and} \quad \xi^1 = P^h \xi_0 + \tau P^h \xi_1,$$

then

$$\begin{aligned}
\|b^1\|_{\Gamma_1} &= \|\xi^1 - P^h \xi(t_1)\|_{\Gamma_1} \\
&\leq \|\xi^1 - \xi(t_1)\|_{\Gamma_1} + \|\xi(t_1) - P^h \xi(t_1)\|_{\Gamma_1} \\
&\leq \|\xi_0 - P^h \xi_0\|_{\Gamma_1} + \tau \|\xi_1 - P^h \xi_1\|_{\Gamma_1} + \frac{\tau^2}{2} \|\xi_{tt}(\tau_3)\|_{\Gamma_1} + \|\xi(t_1) - P^h \xi(t_1)\|_{\Gamma_1} \\
&\leq C(h \|\xi_0\|_* + h\tau \|\xi_1\|_* + \tau^2 \|\xi_{tt}(\tau_3)\|_{\Gamma_1} + h \|\xi(t_1)\|_*),
\end{aligned}$$

hence

$$\|b^1\|_{\Gamma_1} \leq C(h + \tau).$$

To bound $\|\hat{b}^1\|_{\Gamma_1}$, we know that

$$\xi_t(t_1) = \xi_1 + \tau \xi_{tt}(\tau_4) \quad \text{with} \quad 0 \leq \tau_4 \leq t_1, \quad \text{and} \quad \hat{\xi}^1 = \frac{\xi^1 - \xi^0}{\tau} = P^h \xi_1,$$

then

$$\begin{aligned} \|\hat{b}^1\|_{\Gamma_1} &= \|\hat{\xi}^1 - P^h \xi_t(t_1)\|_{\Gamma_1} \\ &\leq \|\hat{\xi}^1 - \xi_t(t_1)\|_{\Gamma_1} + \|\xi_t(t_1) - P^h \xi_t(t_1)\|_{\Gamma_1} \\ &\leq \|\xi_1 - P^h \xi_1\|_{\Gamma_1} + \tau \|\xi_{tt}(\tau_4)\|_{\Gamma_1} + \|\xi_t(t_1) - P^h \xi_t(t_1)\|_{\Gamma_1} \\ &\leq C(h \|\xi_1\|_* + \tau \|\xi_{tt}(\tau_4)\|_{\Gamma_1} + h \|\xi_t(t_1)\|_*), \end{aligned}$$

hence

$$\|\hat{b}^1\|_{\Gamma_1} \leq C(h + \tau).$$

Now, we use (3.5.11) with the above estimates to obtain

$$\begin{aligned} \|b^{n+1}\|_{\Gamma_1}^2 + \|\hat{b}^{n+1}\|_{\Gamma_1}^2 &\leq C(\tau^2 + h^2) + 2\tau \sum_{i=1}^n \left(\|b^{i+1}\|_{\Gamma_1}^2 + \|\hat{b}^{i+1}\|_{\Gamma_1}^2 \right) \\ &\quad + C_2 \tau \sum_{i=1}^n \|\nabla \hat{e}^{i+1}\|_2^2. \end{aligned} \quad (3.5.12)$$

By summing inequalities (3.5.8) and (3.5.12) side by side and performing appropriate algebraic manipulations, we obtain

$$\begin{aligned} (1 - \kappa\tau) \left(\|\hat{e}^{n+1}\|_2^2 + \|\nabla e^{n+1}\|_2^2 + \|\nabla \hat{e}^{n+1}\|_2^2 + \|b^{n+1}\|_{\Gamma_1}^2 + \|\hat{b}^{n+1}\|_{\Gamma_1}^2 \right) \\ \leq C(\tau^2 + h^2) + \kappa\tau \sum_{i=2}^n \left(\|\hat{e}^i\|_2^2 + \|\nabla e^i\|_2^2 + \|\nabla \hat{e}^i\|_2^2 + \|b^i\|_{\Gamma_1}^2 + \|\hat{b}^i\|_{\Gamma_1}^2 \right), \end{aligned} \quad (3.5.13)$$

where $\kappa = \max\{4, C_2\}$.

Assuming that $\tau < \frac{1}{\kappa}$ and applying the discrete Gronwall inequality in (3.5.13), we get

$$\|\hat{e}^{n+1}\|_2^2 + \|\nabla e^{n+1}\|_2^2 + \|\nabla \hat{e}^{n+1}\|_2^2 + \|b^{n+1}\|_{\Gamma_1}^2 + \|\hat{b}^{n+1}\|_{\Gamma_1}^2 \leq C(h^2 + \tau^2).$$

Now

$$\begin{aligned} \|u_t(t_n) - \hat{u}^n\|_2^2 + \|\nabla u(t_n) - \nabla u^n\|_2^2 + \|\nabla u_t(t_n) - \nabla \hat{u}^n\|_2^2 + \|\xi(t_n) - \xi^n\|_{\Gamma_1}^2 \\ + \|\xi_t(t_n) - \hat{\xi}^n\|_{\Gamma_1}^2 \\ = \|\hat{e}^n + \hat{\rho}^n\|_2^2 + \|\nabla e^n + \nabla \rho^n\|_2^2 + \|\nabla \hat{e}^n + \nabla \hat{\rho}^n\|_2^2 + \|b^n + \sigma^n\|_{\Gamma_1}^2 \\ + \|\hat{b}^n + \hat{\sigma}^n\|_{\Gamma_1}^2 \\ \leq C(h^2 + \tau^2 + \|\hat{\rho}^n\|_2^2 + \|\nabla \rho^n\|_2^2 + \|\nabla \hat{\rho}^n\|_2^2 + \|\sigma^n\|_{\Gamma_1}^2 + \|\hat{\sigma}^n\|_{\Gamma_1}^2) \\ \leq C(h^2 + \tau^2). \end{aligned}$$

what concludes the Theorem. □

3.6 NUMERICAL RESULTS

We consider the following problem

$$\begin{cases} u_{tt}(x, t) - \Delta u(x, t) - \Delta u_t(x, t) - \Delta u_{tt}(x, t) = f(x, t), & \text{in } Q, \\ u(x, t) = 0, & \text{on } \Sigma_0, \\ \frac{\partial u}{\partial n}(x, t) + \xi(x, t) = 0, & \text{on } \Sigma_1, \\ \xi_{tt}(x, t) + \xi(x, t) - u_t(x, t) = g(x, t), & \text{on } \Sigma_1, \end{cases} \quad (3.6.1)$$

where f, g are sufficiently regular external forces and are being introduced to validate the problem in a numerical approach.

Let V_{m_μ} be the subspace spanned respectively by $\{w_1, w_2, \dots, w_{m_\mu}\}$, where $\{w_i\}_{i=1}^\infty$ is a smooth orthogonal basis for V , and Z_{m_ν} the subspace spanned respectively by $\{z_1, z_2, \dots, z_{m_\nu}\}$, where $\{z_i\}_{i=1}^\infty$ is a smooth orthogonal basis for $L^2(\Gamma_1)$.

We look for

$$u_m(x, t) = \sum_{i=1}^{m_\mu} d_{im}(t) w_i(x), \quad \xi_m(x, t) = \sum_{i=1}^{m_\nu} s_{im}(t) z_i(x, y_b) \quad (3.6.2)$$

satisfying $\forall v \in V_m, \forall \psi \in Z_m$

$$\begin{cases} (u_{tt}^m, v) + (\nabla u^m, \nabla v) + (\nabla u_t^m, \nabla v) + (\nabla u_{tt}^m, \nabla v) \\ \quad + (\xi^m + \xi_t^m + \xi_{tt}^m, v)_1 = (f, v), \\ (\xi_{tt}^m, \psi)_1 + (\xi^m, \psi)_1 - (u_t^m, \psi)_1 = (g, \psi)_1. \end{cases} \quad (3.6.3)$$

Replacing (3.6.2) in (3.6.3), taking $v = w_j, \psi = z_j$, we get

$$\begin{cases} \sum_{i=1}^{m_\mu} [d_{im}''(t) (w_i, w_j) + (d_{im}''(t) + d_{im}'(t) + d_{im}(t)) (\nabla w_i, \nabla w_j)] \\ \quad + \sum_{i=1}^{m_\nu} [(s_{im}''(t) + s_{im}'(t) + s_{im}(t)) (z_i, w_j)_1] = (f(t), w_j), \\ \quad \text{for } j = 1, 2, \dots, m_\mu, \\ \sum_{i=1}^{m_\nu} [(s_{im}''(t) + s_{im}(t)) (z_i, z_j)_1 - d_{im}'(t) (w_i, z_j)_1] = (g(t), z_j)_1, \\ \quad \text{for } j = 1, 2, \dots, m_\nu. \end{cases} \quad (3.6.4)$$

Defining the matrices of (3.6.4) as

$$\begin{aligned} A_{ij} &= (w_i, w_j), & B_{ij} &= (\nabla w_i, \nabla w_j), & L_{1ij} &= (z_i, w_j)_1, \\ L_{2ij} &= (z_i, z_j)_1, & L_{3ij} &= (w_i, z_j)_1. \end{aligned}$$

We can rewrite (3.6.4) in the following system of ordinary differential equations:

$$\begin{cases} (A + B) d''(t) + B d'(t) + B d(t) + L_1^T s''(t) + L_1^T s'(t) + L_1^T s(t) = F(t), \\ L_2 s''(t) + L_2 s(t) - L_3^T d'(t) = G(t), \end{cases} \quad (3.6.5)$$

for all $t \in [0, T]$.

3.6.1 Fully Discrete approximation

In this subsection, we introduce and analyze a fully discrete finite element method to (3.6.5).

Consider the discretization of the fixed time interval $[0, T]$, using a uniform time step of size $\tau = \frac{T}{N}$, then $t_n = n\tau$, for $n = 0, 1, 2, \dots, N$ and $d^n = d(t_n)$, $s^n = s(t_n)$. The first and second derivatives in time will be approximated by forward-difference and central difference respectively, given by

$$d'^n = \frac{1}{\tau} (d^{n+1} - d^n), \quad d''^n = \frac{1}{\tau^2} (d^{n+1} - 2d^n + d^{n-1}),$$

also

$$s'^n = \frac{1}{\tau} (s^{n+1} - s^n), \quad s''^n = \frac{1}{\tau^2} (s^{n+1} - 2s^n + s^{n-1}).$$

Taking $t = t_n$ in (3.6.5) and replacing the approximations and multiplying by τ^2 , we obtain the following iterative method:

$$\begin{cases} R_1 d^{n+1} + R_2 d^n + R_3 d^{n-1} + R_4 s^{n+1} + R_5 s^n + L_1^T s^{n-1} = \tau^2 F^{n+1}, \\ R_6 s^{n+1} - 2L_2 s^n + L_2 s^{n-1} + R_7 d^{n+1} - R_7 d^n = \tau^2 G^{n+1}, \end{cases} \quad (3.6.6)$$

where $d^0 = P^h u_0$, $d^1 = d^0 + \tau P^h u_1$, $s^0 = P^h \xi_0$, $s^1 = s^0 + \tau P^h \xi_1$ and matrices defined by

$$\begin{aligned} R_1 &= A + (1 + \tau + \tau^2) B, & R_2 &= -2A - (2 + \tau) B, & R_3 &= A + B, & R_4 &= (1 + \tau + \tau^2) L_1^T, \\ R_5 &= -(2 + \tau) L_1^T, & R_6 &= (1 + \tau^2) L_2, & R_7 &= -\tau L_3^T. \end{aligned}$$

Thus, multiplying the second equation of (3.6.6) by R_6^{-1} , we get

$$s^{n+1} = 2R_6^{-1} L_2 s^n - R_6^{-1} L_2 s^{n-1} - R_6^{-1} R_7 d^{n+1} + R_6^{-1} R_7 d^n + \tau^2 R_6^{-1} G^{n+1}. \quad (3.6.7)$$

Replacing (3.6.7) in the first equation of (3.6.6), we obtain

$$\hat{R}_1 d^{n+1} = -\hat{R}_2 d^n - R_3 d^{n-1} - \hat{R}_3 s^n - \hat{R}_4 s^{n-1} - \hat{R}_5 G^{n+1} + \tau^2 F^{n+1}, \quad (3.6.8)$$

where

$$\begin{aligned} \hat{R}_1 &= R_1 - R_4 R_6^{-1} R_7, & \hat{R}_2 &= R_2 + R_4 R_6^{-1} R_7, & \hat{R}_3 &= 2R_4 R_6^{-1} L_2 + R_5, \\ \hat{R}_4 &= L_1^T - R_4 R_6^{-1} L_2, & \hat{R}_5 &= \tau^2 R_4 R_6^{-1}. \end{aligned}$$

Multiplying (3.6.8) by $\hat{R}_1^{-1}$, we get

$$d^{n+1} = Q_1 d^n + Q_2 d^{n-1} + Q_3 s^n + Q_4 s^{n-1} + Q_5 G^{n+1} + Q_6 F^{n+1}, \quad (3.6.9)$$

where

$$\begin{aligned} Q_1 &= -\hat{R}_1^{-1} \hat{R}_2, & Q_2 &= -\hat{R}_1^{-1} R_3, & Q_3 &= -\hat{R}_1^{-1} \hat{R}_3, \\ Q_4 &= -\hat{R}_1^{-1} \hat{R}_4, & Q_5 &= -\hat{R}_1^{-1} \hat{R}_5, & Q_6 &= \tau^2 \hat{R}_1^{-1}. \end{aligned}$$

Using the initial conditions $d^0 = u(x, 0)$, $s^0 = \xi(x, 0)$ and the Taylor approximations, we get

$$d^1 = d^0 + \tau u_t(x, 0), \quad s^1 = s^0 + \tau \xi_t(x, 0) + \tau^2 \xi_{tt}(x, 0),$$

where $u(x, 0)$, $u_t(x, 0)$, $\xi(x, 0)$, and $\xi_t(x, 0)$ are given by initial conditions, while $\xi_{tt}(x, 0)$ is obtained by doing in (3.6.1)₄.

Taking $n = 1$ in (3.6.9), we obtain

$$d^2 = Q_1 \hat{R}_2 d^1 + Q_2 d^0 + Q_3 s^1 + Q_4 s^0 + Q_5 G^2 + Q_6 F^2.$$

Next, we can calculate s^2 by taking $n = 1$ in (3.6.7),

$$s^2 = 2R_6^{-1} L_2 s^1 - R_6^{-1} L_2 s^0 - R_6^{-1} R_7 d^2 + R_6^{-1} R_7 d^1 + \tau^2 R_6^{-1} G^2.$$

Now, we use the values of d^2 and s^2 with the equations (3.6.9) and (3.6.7) to calculate the next iterations for $n = 2, 3, \dots, N$.

3.6.2 Numerical simulations

In this subsection, we consider the temporal interval $[0, 1]$ and $\Omega = (0, 1) \times (0, 1)$, the boundary intervals

$$\begin{cases} I_1 = \{(x, 0) \in \Gamma : 0 \leq x \leq 1\}, \\ I_2 = \{(x, 1) \in \Gamma : 0 \leq x \leq 1\}, \\ I_3 = \{(0, y) \in \Gamma : 0 \leq y \leq 1\}, \\ I_4 = \{(1, y) \in \Gamma : 0 \leq y \leq 1\}, \end{cases}$$

where $\Gamma_0 = I_2 \cup I_3 \cup I_4$, $\Gamma_1 = I_1$ and the following initial conditions

$$\begin{cases} u_0(x, y) = \sin(\pi x)(y-1)^3, & u_1(x, y) = \sin(\pi x)(y-1)^3, & (x, y) \in \Omega, \\ \xi_0(x) = 3 \sin(\pi x), & \xi_1(x) = 3 \sin(\pi x), & x \in [0, 1]. \end{cases}$$

Inhomogeneous problem

In this case, choosing conveniently the external forces $f(x, y, t)$ and $g(x, t)$, the exact solutions are known and given by:

$$u(x, y, t) = \sin(\pi x)(y-1)^3(\cos(t) + \sin(t)),$$

and

$$\xi(x, t) = 3 \sin(\pi x)(\cos(t) + \sin(t)).$$

The Table 3.1 shows the numerical errors and convergence orders of the numerical method associated with each of the variables in the $L^\infty(0, T, L^2(\Omega))$ and $L^\infty(0, T, L^2(\Gamma_1))$ norms, respectively, taking $\Delta x = \Delta y = \tau$ for different meshes, which are shown in the first column of the table. Note that the order of convergence $p \approx 1$ obtained in the table is in agreement with the error estimate in Theorem 3.4.1.

$\Delta x = \Delta y = \tau$	E_u	E_ξ	p_u	p_ξ
2^{-3}	0.02817577	0.07002385	—	—
2^{-4}	0.01496716	0.04044654	0.91265454	0.79183009
2^{-5}	0.00762289	0.02172591	0.97339053	0.89659969
2^{-6}	0.00383493	0.01125769	0.99113781	0.94850578

Table 3.1 – Numerical errors and convergence orders - Inhomogeneous problem: $\Delta x = \Delta y = \tau$

The Table 3.2 shows the numerical errors and convergence orders of the numerical method associated with each of the variables in the $L^\infty(0, T, L^2(\Omega))$ and $L^\infty(0, T, L^2(\Gamma_1))$ norms, respectively, taking $\tau = \Delta x^2 = \Delta y^2$ for different meshes. Note that the order of convergence $p \approx 2$ obtained in the table is in agreement with the error estimate in Theorem 3.4.1.

τ	$\Delta x = \Delta y$	E_u	E_ξ	p_u	p_ξ
2^{-6}	2^{-3}	0.01022686	0.00872158	—	—
2^{-8}	2^{-4}	0.00268693	0.00220796	1.92833261	1.98187548
2^{-10}	2^{-5}	0.00068020	0.00055365	1.98192783	1.99566789
2^{-12}	2^{-6}	0.00017096	0.00013837	1.99230019	2.00044305

Table 3.2 – Numerical errors and convergence orders - Inhomogeneous problem: $\tau = \Delta x^2 = \Delta y^2$

To make a numerical illustration of the solution, we present in Fig.3.1 the graphs of the approximate solutions for each of the variables $u^m(x, y, 0.5)$ and $\xi^m(x, t)$ where the mesh was $\Delta x = \Delta y = \tau = 1/32$.

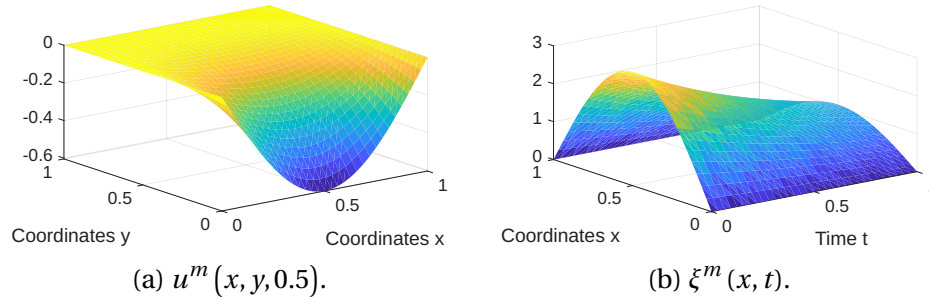


Figure 3.1 – Graphs for inhomogeneous problem: $\Delta x = \Delta y = \tau = 1/32$.

Homogeneous problem

For $f(x, y, t) = 0$ and $g(x, t) = 0$, we have the problem (3.3.1). In this case, the exact solutions are unknown. So, we will suffice with presenting a graphical representation of the approximate solutions.

The Fig.3.2 show the graphs of the approximate solutions for each of the variables $u^m(x, y, 0.5)$ and $\xi^m(x, t)$ where the mesh was $\Delta x = \Delta y = \tau = 1/64$.

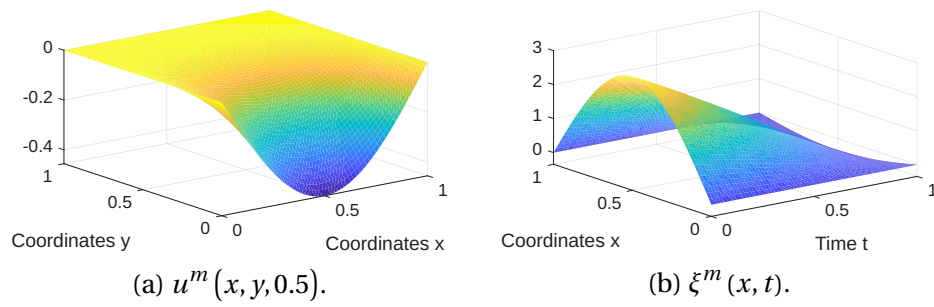


Figure 3.2 – Graphs for homogeneous problem: $\Delta x = \Delta y = \tau = 1/64$.

Chapter 4

Numerical analysis and simulation for a nonlinear viscoelastic wave equation with dynamic boundary conditions

4.1 Introduction

In this chapter, we consider the following nonlinear viscoelastic wave equation with strong damping and dynamic boundary conditions:

$$\begin{cases} u_{tt}(x, t) - \Delta u(x, t) - \Delta u_t(x, t) + |u(x, t)|^{\rho-1} u(x, t) \\ \quad = f(x, t), & \text{in } Q = \Omega \times [0, T], \\ u(x, t) = 0, & \text{on } \Sigma_0 = \Gamma_0 \times [0, T], \\ \frac{\partial u}{\partial n}(x, t) + \xi(x, t) = 0, & \text{on } \Sigma_1 = \Gamma_1 \times [0, T], \\ \xi_t(x, t) + \xi(x, t) = u_t(x, t) + g(x, t), & \text{on } \Sigma_1 = \Gamma_1 \times [0, T], \end{cases} \quad (4.1.1)$$

with the following initial conditions:

$$\begin{cases} u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{for } x \in \Omega, \\ \xi(x, 0) = \xi_0(x), & \text{for } x \in \Gamma_1, \end{cases} \quad (4.1.2)$$

where $\rho > 1$, ensuring a nonlinear damping effect, and $u = u(x, t)$ represents the displacement at time t and space coordinate x , while $\xi(x, t)$ designates the dynamical control. Here, Ω is a bounded open set in $\mathbb{R}^2$ with boundary Γ of class C^2 , and $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \neq \emptyset$ and $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$. The symbol $\frac{\partial}{\partial n}$ denotes the unit outer normal derivative, and f and g are functions depending on x and t .

The viscoelastic equation describes how waves propagate through materials that exhibit both elastic and viscous behavior. This formulation makes it possible to analyze the evolution of waves in materials that simultaneously dissipate energy and deform. For the physical aspects, see Love [1].

The main source of difficulty in problem (4.1.1) lies in the non-standard boundary conditions, known as *dynamic boundary conditions*. Besides their theoretical interest, they also have numerous practical applications. For example, in one spa-

tial dimension, problem (4.1.1) can model the dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached at its free end. Dynamic boundary conditions also appear in the analysis of the transverse motion of an extensible membrane whose boundary is influenced by vibrations only in a certain region. These conditions describe Newton's law for the attached mass (see [17, 18, 32] for more details).

It is worth noting that the problems studied in the aforementioned works were analyzed under classical Dirichlet boundary conditions. In contrast, the problem investigated in this chapter is based on *dynamic boundary conditions*, which provide a more complex and realistic model for wave propagation in viscoelastic materials. This distinction is important, as *dynamic boundary conditions* account for time-dependent interactions at the boundary, providing a more detailed description of physical phenomena, particularly in contexts such as structural vibrations and wave propagation in materials with both elastic and viscous properties. This difference sets our work apart from previous studies and provides a new perspective on the behavior of viscoelastic waves.

Several works have addressed the study of distributed systems with dynamical boundary control, from a theoretical point of view [22, 23, 24, 28]. Regarding the numerical analysis of wave equations with *dynamic boundary conditions*, it has been shown in [6, 7] that the semi-discrete energy decays exponentially and the system converges. Furthermore, finite element error estimates [34] and energy-conserving finite difference schemes for nonlinear wave equations [41], as well as a finite element scheme for 2D wave equations with dynamical boundary control [8], have been provided. These contributions provide a solid foundation for the numerical treatment of such problems and justify the numerical approach adopted in the present paper.

More recently, significant progress has been made in the analysis of nonlinear wave equations with memory and damping mechanisms. For example, Peng and Liu [36] investigated exponential stability for wave equations with infinite memory and frictional damping, while Bashir et al. [30] studied exponential decay and blow-up for equations with strong time-varying delay. Zhou et al. [38] analyzed the dynamics of damped quintic wave equations with time-dependent coefficients, and Jiao, Li, and Zhao [35] proposed implicit-explicit time discretization schemes for semilinear wave equations with nonautonomous damping. Moreover, Eckhardt, Hochbruck, and Verfürth [33] provided a rigorous error analysis of such schemes, and Remil [37] studied the global existence and asymptotic behavior of plate equations with constant delay terms and logarithmic nonlinearities.

Motivated by previous works, in the present chapter we investigate problem (4.1.1), generalizing the results obtained in [8] by introducing a strong nonlinear term and strong damping. The content of this chapter is organized as follows: in Section 4.2, we introduce notations and state the main results, including the proof of existence and uniqueness of the solution using the Faedo-Galerkin method. In Section 4.3, we prove energy decay and establish exponential stability. Section 4.4 is devoted to the analysis of a semi-discrete Galerkin approximation to the solution of (4.1.1), and Section 4.5 deals with the fully discrete scheme. Finally, in Section 4.6, we present numerical experiments that illustrate the theoretical results.

4.2 Notations and main results

Notations. Let p and q be constants defined by

$$p = \rho + 1, \quad \text{and} \quad q = \rho - 1.$$

We define the function $h : \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$h(z) = |z|^q z.$$

We now introduce the following function spaces:

$$H_{\Gamma_0}^1(\Omega) = \{u \in H^1(\Omega) / u|_{\Gamma_0} = 0\}, \quad \text{and} \quad V = \left\{u \in H_{\Gamma_0}^1(\Omega) / u|_{\Gamma_1} \in L^2(\Gamma_1)\right\} \cap L^p(\Omega).$$

We denote by $\|\cdot\|_p$, $\|\cdot\|_{\Gamma_1}$ and $\|\cdot\|_{H_{\Gamma_0}^1}$ the corresponding norms in $L^p(\Omega)$, $L^2(\Gamma_1)$ and $H_{\Gamma_0}^1(\Omega)$, respectively, and by $(\cdot, \cdot)$ and $(\cdot, \cdot)_1$ the scalar products in $L^2(\Omega)$ and $L^2(\Gamma_1)$, respectively.

Here and throughout this chapter, C_Ω denotes the same constant introduced in Chapter 2.

Weak Formulation. Multiplying the first equation of (4.1.1) by a test function $v \in V$, and using Green's formula with the initial and boundary conditions, we obtain the weak form of (4.1.1).

Find $u(t) \in V$ and $\xi(t) \in L^2(\Gamma_1)$ such that, for all $v, w \in V$,

$$\begin{cases} (u_{tt}, v) + (\nabla u, \nabla v) + (\nabla u_t, \nabla v) + (|u|^q u, v) + (\xi + \xi_t, v)_1 = (f, v), \\ (\xi_t, w)_1 + (\xi, w)_1 = (u_t + g, w)_1. \end{cases} \quad (4.2.1)$$

In the next theorem, we show the existence and uniqueness of the solution of problem (4.1.1)–(4.1.2) using the Faedo-Galerkin method.

Theorem 4.2.1. Assume that $u_0, u_1 \in H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega)$, $\xi_0 \in L^2(\Gamma_1)$, $f, \frac{\partial f}{\partial t} \in L^2(Q)$, $g, \frac{\partial g}{\partial t} \in L^2(\Sigma_1)$ and $\frac{\partial u_0}{\partial n}(x) = -\xi_0(x)$ for all $x \in \Gamma_1$. Then for each $T > 0$, problem (4.1.1)–(4.1.2) admits a unique solution (u, ξ) satisfying the following conditions:

- (1) $u \in L^\infty(0, T, L^p(\Omega)) \cap L^\infty(0, T, H_{\Gamma_0}^1(\Omega))$,
- (2) $u_t, u_{tt} \in L^\infty(0, T, L^2(\Omega)) \cap L^2(0, T, H_{\Gamma_0}^1(\Omega))$,
- (3) $\xi, \xi_t \in L^\infty(0, T, L^2(\Gamma_1)) \cap H^1(0, T, L^2(\Gamma_1))$.

Proof. Let V_{m_μ} be the subspace spanned by $\{w_1, w_2, \dots, w_{m_\mu}\}$, where $\{w_i\}_{i=1}^\infty$ is a smooth orthogonal basis for V , and let Z_{m_ν} be the subspace spanned by $\{z_1, z_2, \dots, z_{m_\nu}\}$, where $\{z_i\}_{i=1}^\infty$ is a smooth orthogonal basis for $L^2(\Gamma_1)$.

We look for the approximate solution

$$u_m(x, t) = \sum_{i=1}^{m_\mu} d_{im}(t) w_i(x), \quad \xi_m(x, t) = \sum_{i=1}^{m_\nu} s_{im}(t) z_i(x, y_b), \quad (4.2.2)$$

satisfying $\forall v \in V_{m_\mu}, \forall \psi \in Z_{m_\nu}$,

$$\begin{cases} (u_{tt}^m, v) + (\nabla u^m, \nabla v) + (\nabla u_t^m, \nabla v) + (|u^m|^q u^m, v) \\ \quad + (\xi^m + \xi_t^m, v)_1 = (f, v), \\ (\xi_t^m, \psi)_1 + (\xi^m, \psi)_1 = (u_t^m + g, \psi)_1. \end{cases} \quad (4.2.3)$$

Note that y_b in (4.2.2) is a fixed point y in the boundary Γ_1 .

With the initial conditions

$$\begin{cases} u^m(0) \rightarrow u_0 & \text{in } H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega) & \text{as } m \rightarrow +\infty, \\ u_t^m(0) \rightarrow u_1 & \text{in } H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega) & \text{as } m \rightarrow +\infty, \\ \xi^m(0) \rightarrow \xi_0 & \text{in } L^2(\Gamma_1) & \text{as } m \rightarrow +\infty. \end{cases}$$

This initial value problem has a local solution on $[0, T_m]$. The priori estimates established below allow us to conclude that the solution can be extended to $[0, T] \times [0, T]$ for any given $T > 0$.

The first estimate. Take $v = 2u_t^m$ and $\psi = \xi_t^m + \xi^m$, in (4.2.3) to obtain

$$\begin{aligned} & \frac{d}{dt} \left(\|u_t^m\|_2^2 + \|\nabla u^m\|_2^2 + \frac{2}{p} \|u^m\|_p^p + 2\|\xi^m\|_{\Gamma_1}^2 \right) + 2\|\nabla u_t^m\|_2^2 + 2\|\xi^m\|_{\Gamma_1}^2 + 2\|\xi_t^m\|_{\Gamma_1}^2 \\ & = 2(g, \xi_t^m + \xi^m)_1 + 2(f, u_t^m). \end{aligned}$$

Using Young's inequality, there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} & \frac{d}{dt} \left(\|u_t^m\|_2^2 + \|\nabla u^m\|_2^2 + \frac{2}{p} \|u^m\|_p^p + 2\|\xi^m\|_{\Gamma_1}^2 \right) + 2\|\nabla u_t^m\|_2^2 \\ & \quad + 2\|\xi^m\|_{\Gamma_1}^2 + 2\|\xi_t^m\|_{\Gamma_1}^2 \\ & \leq 2\delta_1 \|u_t^m\|_2^2 + 2\delta_2 \|\xi_t^m + \xi^m\|_{\Gamma_1}^2 + \frac{1}{2\delta_2} \|g\|_{\Gamma_1}^2 + \frac{1}{2\delta_1} \|f\|_2^2 \\ & \leq 2\delta_1 \|u_t^m\|_2^2 + 2\delta_2 \left(\|\xi_t^m\|_{\Gamma_1}^2 + \|\xi^m\|_{\Gamma_1}^2 + \frac{d}{dt} \|\xi^m\|_{\Gamma_1}^2 \right) \\ & \quad + \frac{1}{2\delta_2} \|g\|_{\Gamma_1}^2 + \frac{1}{2\delta_1} \|f\|_2^2. \end{aligned} \quad (4.2.4)$$

Using Poincaré's inequality, we find that

$$\|u_t^m\|_2 \leq C_\Omega \|\nabla u_t^m\|_2.$$

Hence, inequality (4.2.4) gives

$$\begin{aligned} & \frac{d}{dt} \left(\|u_t^m\|_2^2 + \|\nabla u^m\|_2^2 + \frac{2}{p} \|u^m\|_p^p + 2\|\xi^m\|_{\Gamma_1}^2 \right) + 2\|\nabla u_t^m\|_2^2 + 2\|\xi^m\|_{\Gamma_1}^2 + 2\|\xi_t^m\|_{\Gamma_1}^2 \\ & \leq 2\delta_1 C_\Omega^2 \|\nabla u_t^m\|_2^2 + 2\delta_2 \left(\|\xi_t^m\|_{\Gamma_1}^2 + \|\xi^m\|_{\Gamma_1}^2 + \frac{d}{dt} \|\xi^m\|_{\Gamma_1}^2 \right) + \frac{1}{2\delta_2} \|g\|_{\Gamma_1}^2 + \frac{1}{2\delta_1} \|f\|_2^2. \end{aligned}$$

Now, choosing $\delta_1 = \frac{1}{2C_\Omega^2}$ and $\delta_2 = \frac{1}{2}$, we get

$$\begin{aligned} \frac{d}{dt} \left(\|u_t^m\|_2^2 + \|\nabla u^m\|_2^2 + \frac{2}{p} \|u^m\|_p^p + \|\xi^m\|_{\Gamma_1}^2 \right) + \|\nabla u_t^m\|_2^2 + \|\xi^m\|_{\Gamma_1}^2 + \|\xi_t^m\|_{\Gamma_1}^2 \\ \leq \|g\|_{\Gamma_1}^2 + C_\Omega^2 \|f\|_2^2. \end{aligned}$$

Integration from 0 to t yields

$$\begin{aligned} \|u_t^m\|_2^2 + \|\nabla u^m\|_2^2 + \frac{2}{p} \|u^m\|_p^p + \|\xi^m\|_{\Gamma_1}^2 + \int_0^t \|\nabla u_t^m(s)\|_2^2 ds \\ + \int_0^t \left(\|\xi^m(s)\|_{\Gamma_1}^2 + \|\xi_t^m(s)\|_{\Gamma_1}^2 \right) ds \\ \leq \|u_t^m(0)\|_2^2 + \|\nabla u^m(0)\|_2^2 + \frac{2}{p} \|u^m(0)\|_p^p + \|\xi^m(0)\|_{\Gamma_1}^2 \\ + \int_0^t \left(\|g(s)\|_{\Gamma_1}^2 + C_\Omega^2 \|f(s)\|_2^2 \right) ds. \end{aligned} \quad (4.2.5)$$

Using the embedding

$$H_{\Gamma_0}^1(\Omega) \hookrightarrow L^p(\Omega),$$

together with the Poincaré inequality, we find that

$$\|u^m(0)\|_p \leq C_0 \|u^m(0)\|_{H_{\Gamma_0}^1} \leq c_0 \|\nabla u^m(0)\|_2, \quad (4.2.6)$$

where $c_0 = C_0 \sqrt{1 + C_\Omega^2}$.

Using inequalities (4.2.5) and (4.2.6), we obtain

$$\begin{aligned} \|u_t^m\|_2^2 + \|\nabla u^m\|_2^2 + \frac{2}{p} \|u^m\|_p^p + \|\xi^m\|_{\Gamma_1}^2 + \int_0^t \left(\|\nabla u_t^m(s)\|_2^2 + \|\xi^m(s)\|_{\Gamma_1}^2 \right) ds \\ + \int_0^t \|\xi_t^m(s)\|_{\Gamma_1}^2 ds \\ \leq \|u_t^m(0)\|_2^2 + \|\nabla u^m(0)\|_2^2 + \frac{2c_0^p}{p} \|\nabla u^m(0)\|_2^p + \|\xi^m(0)\|_{\Gamma_1}^2 \\ + \int_0^t \left(\|g(s)\|_{\Gamma_1}^2 + C_\Omega^2 \|f(s)\|_2^2 \right) ds. \\ \leq C(\|u_1\|_2, \|\nabla u_0\|_2, \|\xi_0\|_{\Gamma_1}, p, c_0) + \int_0^t \left(\|g(s)\|_{\Gamma_1}^2 + C_\Omega^2 \|f(s)\|_2^2 \right) ds, \end{aligned} \quad (4.2.7)$$

where we used in (4.2.7) the fact that

$$\begin{aligned} u^m(0) &\rightarrow u_0 \quad \text{strongly in } H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega), \\ u_t^m(0) &\rightarrow u_1 \quad \text{strongly in } H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega), \\ \xi^m(0) &\rightarrow \xi_0 \quad \text{strongly in } L^2(\Gamma_1). \end{aligned} \quad (4.2.8)$$

Using the assumptions on u_0, u_1, ξ_0, f , and g , we find that

$$\begin{aligned} & \|u_t^m\|_2^2 + \|\nabla u^m\|_2^2 + \frac{2}{p} \|u^m\|_p^p + \|\xi^m\|_{\Gamma_1}^2 \\ & + \int_0^t \left(\|\nabla u_t^m\|_2^2 + \|\xi^m\|_{\Gamma_1}^2 + \|\xi_t^m\|_{\Gamma_1}^2 \right) ds \leq C. \end{aligned} \quad (4.2.9)$$

From (4.2.9), we deduce that

$$\begin{aligned} u^m & \text{ is bounded in } L^\infty(0, T, L^p(\Omega)) \cap L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \\ u_t^m & \text{ is bounded in } L^\infty(0, T, L^2(\Omega)) \cap L^2(0, T, H_{\Gamma_0}^1(\Omega)), \\ \xi^m & \text{ is bounded in } L^\infty(0, T, L^2(\Gamma_1)) \cap H^1(0, T, L^2(\Gamma_1)). \end{aligned}$$

Consequently, the solution can be extended to $[0, T]$.

Moreover, there exist two sub-sequences still denoted by $(u^m), (\xi^m)$, such that

$$\begin{aligned} u^m & \rightharpoonup u \quad \text{weak-star in } L^\infty(0, T, L^p(\Omega)), \\ u^m & \rightharpoonup u \quad \text{weak-star in } L^\infty(0, T, H_{\Gamma_0}^1(\Omega)), \\ u_t^m & \rightharpoonup u_t \quad \text{weak-star in } L^\infty(0, T, L^2(\Omega)), \\ u_t^m & \rightharpoonup u_t \quad \text{weak-star in } L^2(0, T, H_{\Gamma_0}^1(\Omega)), \\ \xi^m & \rightharpoonup \xi \quad \text{weak-star in } L^\infty(0, T, L^2(\Gamma_1)), \\ \xi^m & \rightharpoonup \xi \quad \text{weak-star in } H^1(0, T, L^2(\Gamma_1)). \end{aligned}$$

The second estimate. Differentiating (4.2.3) with respect to t , and letting $\hat{u}^m = u_t^m, \hat{\xi}^m = \xi_t^m, \hat{f} = \frac{\partial f}{\partial t}$, and $\hat{g} = \frac{\partial g}{\partial t}$, we get

$$\begin{cases} (\hat{u}_{tt}^m, v) + (\nabla \hat{u}^m, \nabla v) + (\nabla \hat{u}_t^m, \nabla v) + \rho (\hat{u}^m |u^m|^q, v) \\ \quad + (\hat{\xi}^m + \xi_t^m, v)_1 = (\hat{f}, v), \\ (\hat{\xi}_t^m, \psi)_1 + (\hat{\xi}^m, \psi)_1 = (\hat{u}_t^m + \hat{g}, \psi)_1. \end{cases} \quad (4.2.10)$$

Taking $v = 2\hat{u}_t^m$ and $\psi = \hat{\xi}_t^m + \hat{\xi}^m$ in (4.2.10), we find

$$\begin{aligned} & \frac{d}{dt} \left(\|\hat{u}_t^m\|_2^2 + \|\nabla \hat{u}^m\|_2^2 + 2\|\hat{\xi}^m\|_{\Gamma_1}^2 \right) + 2\|\nabla \hat{u}_t^m\|_2^2 + 2\|\hat{\xi}^m\|_{\Gamma_1}^2 + 2\|\hat{\xi}_t^m\|_{\Gamma_1}^2 \\ & = 2(\hat{g}, \hat{\xi}_t^m + \hat{\xi}^m)_1 + 2(\hat{f}, \hat{u}_t^m) - 2\rho (\hat{u}^m |u^m|^q, \hat{u}_t^m) \\ & \leq 2(\hat{g}, \hat{\xi}_t^m + \hat{\xi}^m)_1 + 2(\hat{f}, \hat{u}_t^m) + 2\rho \int_\Omega |\hat{u}^m| |u^m|^q |\hat{u}_t^m| dx. \end{aligned}$$

Applying Hölder's inequality, and considering $p_1 > 0$ and $p_2 > 0$, such that $\frac{1}{p_1} + \frac{1}{p_2} =$

$\frac{1}{2}$, it follows that

$$\begin{aligned} & \frac{d}{dt} \left(\|\hat{u}_t^m\|_2^2 + \|\nabla \hat{u}^m\|_2^2 + 2\|\hat{\xi}^m\|_{\Gamma_1}^2 \right) + 2\|\nabla \hat{u}_t^m\|_2^2 + 2\|\hat{\xi}^m\|_{\Gamma_1}^2 + 2\|\hat{\xi}_t^m\|_{\Gamma_1}^2 \\ & \leq 2(\hat{g}, \hat{\xi}_t^m + \hat{\xi}^m)_1 + 2(\hat{f}, \hat{u}_t^m) + 2\rho \|\hat{u}^m\|_{p_1} \|u^m\|_{qp_2}^q \|\hat{u}_t^m\|_2. \end{aligned} \quad (4.2.11)$$

By [14], we can always choose p_1 and p_2 to ensure the continuous embeddings of $H_{\Gamma_0}^1(\Omega) \hookrightarrow L^{p_1}(\Omega)$ and $H_{\Gamma_0}^1(\Omega) \hookrightarrow L^{qp_2}(\Omega)$. Thus, there exist constants $C_2, C_3 > 0$, independent of t , such that

$$\|\hat{u}^m\|_{p_1} \leq C_2 \|\hat{u}^m\|_{H_{\Gamma_0}^1}, \quad \|u^m\|_{qp_2} \leq C_3 \|u^m\|_{H_{\Gamma_0}^1}. \quad (4.2.12)$$

Combining (4.2.11)–(4.2.12), we infer that

$$\begin{aligned} & \frac{d}{dt} \left(\|\hat{u}_t^m\|_2^2 + \|\nabla \hat{u}^m\|_2^2 + 2\|\hat{\xi}^m\|_{\Gamma_1}^2 \right) + 2\|\nabla \hat{u}_t^m\|_2^2 + 2\|\hat{\xi}^m\|_{\Gamma_1}^2 + 2\|\hat{\xi}_t^m\|_{\Gamma_1}^2 \\ & \leq 2(\hat{g}, \hat{\xi}_t^m + \hat{\xi}^m)_1 + 2(\hat{f}, \hat{u}_t^m) + 2\rho C_2 C_3^q \|\hat{u}^m\|_{H_{\Gamma_0}^1} \|u^m\|_{H_{\Gamma_0}^1}^q \|\hat{u}_t^m\|_2. \end{aligned}$$

Using Young's inequality, there exist $\delta_3, \delta_4 > 0$ such that

$$\begin{aligned} & \frac{d}{dt} \left(\|\hat{u}_t^m\|_2^2 + \|\nabla \hat{u}^m\|_2^2 + 2\|\hat{\xi}^m\|_{\Gamma_1}^2 \right) + 2\|\nabla \hat{u}_t^m\|_2^2 + 2\|\hat{\xi}^m\|_{\Gamma_1}^2 + 2\|\hat{\xi}_t^m\|_{\Gamma_1}^2 \\ & \leq C \left(\|\hat{u}^m\|_{H_{\Gamma_0}^1}^2 \|u^m\|_{H_{\Gamma_0}^1}^{2q} + \|\hat{f}\|_2^2 + \|\hat{g}\|_{\Gamma_1}^2 \right) + \delta_3 \|\hat{\xi}_t^m + \hat{\xi}^m\|_{\Gamma_1}^2 + \delta_4 \|\hat{u}_t^m\|_2^2 \\ & \leq C \left(\|\hat{u}^m\|_{H_{\Gamma_0}^1}^2 \|u^m\|_{H_{\Gamma_0}^1}^{2q} + \|\hat{f}\|_2^2 + \|\hat{g}\|_{\Gamma_1}^2 \right) + \delta_3 \left(\|\hat{\xi}_t^m\|_{\Gamma_1}^2 + \|\hat{\xi}^m\|_{\Gamma_1}^2 + \frac{d}{dt} \|\hat{\xi}^m\|_{\Gamma_1}^2 \right) \\ & \quad + \delta_4 \|\hat{u}_t^m\|_2^2. \end{aligned}$$

Using Poincaré's inequality, we find that

$$\begin{aligned} & \frac{d}{dt} \left(\|\hat{u}_t^m\|_2^2 + \|\nabla \hat{u}^m\|_2^2 + 2\|\hat{\xi}^m\|_{\Gamma_1}^2 \right) + 2\|\nabla \hat{u}_t^m\|_2^2 + 2\|\hat{\xi}^m\|_{\Gamma_1}^2 + 2\|\hat{\xi}_t^m\|_{\Gamma_1}^2 \\ & \leq C \left(\|\hat{u}^m\|_{H_{\Gamma_0}^1}^2 \|u^m\|_{H_{\Gamma_0}^1}^{2q} + \|\hat{f}\|_2^2 + \|\hat{g}\|_{\Gamma_1}^2 \right) + \delta_3 \left(\|\hat{\xi}_t^m\|_{\Gamma_1}^2 + \|\hat{\xi}^m\|_{\Gamma_1}^2 + \frac{d}{dt} \|\hat{\xi}^m\|_{\Gamma_1}^2 \right) \\ & \quad + \delta_4 C_\Omega^2 \|\nabla \hat{u}_t^m\|_2^2. \end{aligned}$$

Choosing $\delta_3 = 1$ and $\delta_4 = \frac{1}{C_\Omega^2}$, we get

$$\begin{aligned} & \frac{d}{dt} \left(\|\hat{u}_t^m\|_2^2 + \|\nabla \hat{u}^m\|_2^2 + \|\hat{\xi}^m\|_{\Gamma_1}^2 \right) + \|\nabla \hat{u}_t^m\|_2^2 + \|\hat{\xi}^m\|_{\Gamma_1}^2 + \|\hat{\xi}_t^m\|_{\Gamma_1}^2 \\ & \leq C \left(\|\hat{u}^m\|_{H_{\Gamma_0}^1}^2 \|u^m\|_{H_{\Gamma_0}^1}^{2q} + \|\hat{f}\|_2^2 + \|\hat{g}\|_{\Gamma_1}^2 \right) \\ & \leq C \left(\|\nabla \hat{u}^m\|_2^2 \|\nabla u^m\|_2^{2q} + \|\hat{f}\|_2^2 + \|\hat{g}\|_{\Gamma_1}^2 \right). \end{aligned}$$

Using the boundedness of $\|\nabla u^m\|_2$ and the assumptions on $\hat{f}$ and $\hat{g}$, we get

$$\frac{d}{dt} \left(\|\hat{u}_t^m\|_2^2 + \|\nabla \hat{u}^m\|_2^2 + \|\hat{\xi}^m\|_{\Gamma_1}^2 \right) + \|\nabla \hat{u}_t^m\|_2^2 + \|\hat{\xi}^m\|_{\Gamma_1}^2 + \|\hat{\xi}_t^m\|_{\Gamma_1}^2 \leq C \left(1 + \|\nabla \hat{u}^m\|_2^2 \right).$$

Integrating from 0 to t yields

$$\begin{aligned} & \|\hat{u}_t^m\|_2^2 + \|\nabla \hat{u}^m\|_2^2 + \|\hat{\xi}^m\|_{\Gamma_1}^2 + \int_0^t \left(\|\nabla \hat{u}_t^m(s)\|_2^2 + \|\hat{\xi}^m(s)\|_{\Gamma_1}^2 \right) ds \\ & + \int_0^t \|\hat{\xi}_t^m(s)\|_{\Gamma_1}^2 ds \\ & \leq C + \|\hat{u}_t^m(0)\|_2^2 + \|\nabla \hat{u}^m(0)\|_2^2 + \|\hat{\xi}^m(0)\|_{\Gamma_1}^2 + C \int_0^t \|\nabla \hat{u}^m(s)\|_2^2 ds. \end{aligned} \quad (4.2.13)$$

Since $\|\nabla \hat{u}^m(0)\|_2^2 \leq C$ and $\int_0^t \|\nabla \hat{u}^m(s)\|_2^2 ds \leq C$, it remains to bound $\|\hat{u}_t^m(0)\|_2^2$ and $\|\hat{\xi}^m(0)\|_{\Gamma_1}^2$. To bound $\|\hat{u}_t^m(0)\|_2^2$, we use (4.2.3)₁ with $v = u_{tt}^m$ at $t = 0$ to obtain

$$\begin{aligned} & \|u_{tt}^m(0)\|_2^2 + (\nabla u^m(0), \nabla u_{tt}^m(0)) + (\nabla u_t^m(0), \nabla u_{tt}^m(0)) + (|u^m(0)|^q u^m(0), u_{tt}^m(0)) \\ & + (\xi^m(0) + \xi_t^m(0), u_{tt}^m(0))_1 = (f(0), u_{tt}^m(0)). \end{aligned}$$

Employing Green's formula, we obtain

$$\begin{aligned} & \|u_{tt}^m(0)\|_2^2 - (\Delta u^m(0) + \Delta u_t^m(0), u_{tt}^m(0)) + (|u^m(0)|^q u^m(0), u_{tt}^m(0)) \\ & + \left(\frac{\partial u^m}{\partial n}(0) + \xi^m(0) + \frac{\partial u_t^m}{\partial n}(0) + \xi_t^m(0), u_{tt}^m(0) \right)_1 = (f(0), u_{tt}^m(0)). \end{aligned} \quad (4.2.14)$$

Using the equation (4.1.1)₃, for $t = 0$, we get

$$\frac{\partial u^m}{\partial n}(0) + \xi^m(0) = 0.$$

Deriving the same equation with respect to t , and using it for $t = 0$, we obtain

$$\frac{\partial u_t^m}{\partial n}(0) + \xi_t^m(0) = 0.$$

Hence, equation (4.2.14) becomes

$$\|u_{tt}^m(0)\|_2^2 = (\Delta u^m(0) + \Delta u_t^m(0), u_{tt}^m(0)) - (|u^m(0)|^q u^m(0), u_{tt}^m(0)) + (f(0), u_{tt}^m(0)).$$

By applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} \|u_{tt}^m(0)\|_2 & \leq \|\Delta u^m(0) + \Delta u_t^m(0)\|_2 + \||u^m(0)|^q u^m(0)\|_2 + \|f(0)\|_2 \\ & \leq \|\Delta u^m(0) + \Delta u_t^m(0)\|_2 + \|u^m(0)\|_{2\rho}^\rho + \|f(0)\|_2 \\ & \leq C(\|\Delta u_0\|_2, \|\Delta u_1\|_2) + \|u^m(0)\|_{2\rho}^\rho + \|f(0)\|_2. \end{aligned}$$

Using the embedding $H_{\Gamma_0}^1(\Omega) \hookrightarrow L^{2\rho}(\Omega)$, we find that

$$\begin{aligned} \|u_{tt}^m(0)\|_2 &\leq C(\|\Delta u_0\|_2, \|\Delta u_1\|_2) + C\|u^m(0)\|_{H_{\Gamma_0}^1}^\rho + \|f(0)\|_2 \\ &\leq C\left(\|\Delta u_0\|_2, \|\Delta u_1\|_2, \|u_0\|_{H_{\Gamma_0}^1}, \rho\right) + \|f(0)\|_2. \end{aligned}$$

Using the assumptions on u_0, u_1 , and f , we find that

$$\|u_{tt}^m(0)\|_2 \leq C.$$

To bound $\|\hat{\xi}^m(0)\|_{\Gamma_1}$, we use (4.2.3)₂ with $\psi = \xi_t^m$ at $t = 0$ to obtain

$$\|\xi_t^m(0)\|_{\Gamma_1}^2 = (u_t^m(0) + g(0) - \xi^m(0), \xi_t^m(0))_1.$$

Applying the Cauchy–Schwarz inequality yields

$$\begin{aligned} \|\xi_t^m(0)\|_{\Gamma_1} &\leq \|u_t^m(0) + g(0) - \xi^m(0)\|_{\Gamma_1} \\ &\leq \|u_t^m(0)\|_{\Gamma_1} + \|g(0)\|_{\Gamma_1} + \|\xi^m(0)\|_{\Gamma_1}. \end{aligned}$$

Using the embedding $H_{\Gamma_0}^1(\Omega) \hookrightarrow L^2(\Gamma_1)$ (see [16]), we obtain

$$\begin{aligned} \|\xi_t^m(0)\|_{\Gamma_1} &\leq C\|u_t^m(0)\|_{H_{\Gamma_0}^1} + \|g(0)\|_{\Gamma_1} + \|\xi^m(0)\|_{\Gamma_1} \\ &\leq C\left(\|u_1\|_{H_{\Gamma_0}^1}, \|\xi_0\|_{\Gamma_1}\right) + \|g(0)\|_{\Gamma_1}. \end{aligned}$$

Using the assumptions on u_1, ξ_0 , and g , we find that

$$\|\xi_t^m(0)\|_{\Gamma_1} \leq C.$$

Hence, from (4.2.13) it follows that

$$\|\hat{u}_t^m\|_2^2 + \|\nabla \hat{u}^m\|_2^2 + \|\hat{\xi}^m\|_{\Gamma_1}^2 + \int_0^t \left(\|\nabla \hat{u}_t^m\|_2^2 + \|\hat{\xi}^m\|_{\Gamma_1}^2 + \|\hat{\xi}_t^m\|_{\Gamma_1}^2 \right) ds \leq C. \quad (4.2.15)$$

From (4.2.15), we deduce that

$$\begin{aligned} u_{tt}^m &\text{ is bounded in } L^\infty(0, T, L^2(\Omega)) \cap L^2(0, T, H_{\Gamma_0}^1(\Omega)), \\ \xi_t^m &\text{ is bounded in } L^\infty(0, T, L^2(\Gamma_1)) \cap H^1(0, T, L^2(\Gamma_1)). \end{aligned}$$

Therefore

$$\begin{aligned} u_{tt}^m &\rightharpoonup u_{tt} \text{ weak-star in } L^\infty(0, T, L^2(\Omega)), \\ u_{tt}^m &\rightharpoonup u_{tt} \text{ weak-star in } L^2(0, T, H_{\Gamma_0}^1(\Omega)), \\ \xi_t^m &\rightharpoonup \xi_t \text{ weak-star in } L^\infty(0, T, L^2(\Gamma_1)), \\ \xi_t^m &\rightharpoonup \xi_t \text{ weak-star in } H^1(0, T, L^2(\Gamma_1)). \end{aligned}$$

Analysis of the Nonlinear Term. We know that (u^m) is bounded in $L^\infty(0, T, H_{\Gamma_0}^1(\Omega))$. Hence, (u^m) is also bounded in $L^2(0, T, H_{\Gamma_0}^1(\Omega))$. Moreover, since (u_t^m) is bounded

in $L^\infty(0, T, L^2(\Omega))$, it is bounded in $L^2(0, T, L^2(\Omega))$ as well. Consequently, (u^m) is bounded in $H^1((0, T) \times \Omega)$.

By the compact embedding $H^1((0, T) \times \Omega) \hookrightarrow L^2((0, T) \times \Omega)$, together with the Rellich-Kondrachov theorem, we can extract a subsequence (u^μ) of (u^m) such that

$$u^\mu \rightarrow u \quad \text{strongly in } L^2(\Omega \times (0, T)).$$

Therefore

$$u^\mu \rightarrow u \quad \text{strongly and almost everywhere in } \Omega \times (0, T). \quad (4.2.16)$$

Furthermore, we note that $\{|u^m|^q u^m\}$ is a bounded sequence in $L^\infty(0, T, L^{p/q}(\Omega))$. Hence, by selecting the subsequence (u^μ) , we obtain

$$|u^\mu|^q u^\mu \rightharpoonup \chi \quad \text{weak-star in } L^\infty(0, T, L^{p/q}(\Omega)),$$

for some function $\chi \in L^\infty(0, T, L^{p/q}(\Omega))$. By the pointwise almost everywhere convergence in (4.2.16), we infer that $\chi = |u|^q u$. That is,

$$|u^\mu|^q u^\mu \rightharpoonup |u|^q u \quad \text{weak-star in } L^\infty(0, T, L^{p/q}(\Omega)).$$

Using the convergence results, we can pass to the limit in (4.2.3) to obtain

$$\begin{cases} (u_{tt}, v) + (\nabla u, \nabla v) + (\nabla u_t, \nabla v) + (|u|^q u, v) + (\xi + \xi_t, v)_1 = (f, v), \\ (\xi_t, \psi)_1 + (\xi, \psi)_1 = (u_t + g, \psi)_1. \end{cases} \quad (4.2.17)$$

Using (4.2.17)₁ for $v \in D(\Omega)$, we obtain

$$(\nabla u + \nabla u_t, \nabla v) + (|u|^q u - f, v) = -(u_{tt}, v).$$

Then, we have

$$(\Delta u + \Delta u_t - |u|^q u + f, v) = (u_{tt}, v).$$

Since $u_{tt} \in L^\infty(0, T, L^2(\Omega)) \cap L^2(0, T, H_{\Gamma_0}^1(\Omega))$, it follows that

$$\Delta u + \Delta u_t - |u|^q u + f = u_{tt} \in L^\infty(0, T, L^2(\Omega)) \cap L^2(0, T, H_{\Gamma_0}^1(\Omega)),$$

and

$$u_{tt} - \Delta u - \Delta u_t + |u|^q u = f \quad \text{a.e. in } \Omega \times (0, T).$$

By Green's formula in (4.2.17), we deduce that $\forall v \in V$

$$(u_{tt}, v) - (\Delta u, v) - (\Delta u_t, v) + (|u|^q u, v) + \left(\frac{\partial u}{\partial n} + \xi + \left(\frac{\partial u}{\partial n} + \xi \right)_t, v \right)_1 = (f, v).$$

Hence

$$\frac{\partial u}{\partial n}(x, t) + \xi(x, t) + \left(\frac{\partial u}{\partial n}(x, t) + \xi(x, t) \right)_t = 0.$$

Then

$$z(x, t) + z_t(x, t) = 0,$$

where $z(x, t) = \frac{\partial u}{\partial n}(x, t) + \xi(x, t)$.

Therefore, we obtain

$$z(x, t) = z(x, 0) \exp(-t).$$

Note that

$$z(x, 0) = \frac{\partial u}{\partial n}(x, 0) + \xi(x, 0) = \frac{\partial u_0}{\partial n}(x) + \xi_0(x) = 0.$$

Hence $z(x, t) = 0$, that is,

$$\frac{\partial u}{\partial n}(x, t) + \xi(x, t) = 0.$$

Moreover, the solution (u, ξ) verifies the boundary condition.

Finally, for the uniqueness, we take (u, ξ) and $(\tilde{u}, \tilde{\xi})$ as two solutions and let $Y = u - \tilde{u}$ and $\zeta = \xi - \tilde{\xi}$, then we have

$$\begin{cases} (Y_{tt}, v) + (\nabla Y, \nabla v) + (\nabla Y_t, \nabla v) + (h(u) - h(\tilde{u}), v) + (\zeta + \zeta_t, v)_1 = 0, \\ (\zeta_t + \zeta, \psi)_1 = (Y_t, \psi)_1. \end{cases} \quad (4.2.18)$$

Taking $v = 2Y_t$ and $\psi = \zeta + \zeta_t$ in (4.2.18), we obtain

$$\begin{aligned} \frac{d}{dt} \left(\|Y_t\|_2^2 + \|\nabla Y\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 \right) + 2\|\nabla Y_t\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 + 2\|\zeta_t\|_{\Gamma_1}^2 \\ = -2(h(u) - h(\tilde{u}), Y_t). \end{aligned} \quad (4.2.19)$$

Next, we apply the mean value theorem in h to obtain

$$h(u) - h(\tilde{u}) = (u - \tilde{u}) h'(\alpha u + (1 - \alpha)\tilde{u}), \quad \text{for } \alpha \in (0, 1).$$

Then

$$h(u) - h(\tilde{u}) = (u - \tilde{u}) |\alpha u + (1 - \alpha)\tilde{u}|^q. \quad (4.2.20)$$

Substituting (4.2.20) in (4.2.19), we obtain

$$\begin{aligned} \frac{d}{dt} \left(\|Y_t\|_2^2 + \|\nabla Y\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 \right) + 2\|\nabla Y_t\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 + 2\|\zeta_t\|_{\Gamma_1}^2 \\ = -2(Y |\alpha u + (1 - \alpha)\tilde{u}|^q, Y_t). \end{aligned}$$

Applying Hölder's inequality, and considering $p_1 > 0$ and $p_2 > 0$, such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$, it follows that

$$\begin{aligned} \frac{d}{dt} \left(\|Y_t\|_2^2 + \|\nabla Y\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 \right) + 2\|\nabla Y_t\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 + 2\|\zeta_t\|_{\Gamma_1}^2 \\ \leq 2\|Y\|_{p_1} \|\alpha u + (1 - \alpha)\tilde{u}\|_{q p_2}^q \|Y_t\|_2. \end{aligned}$$

Using the same method as in deriving (4.2.12), we obtain

$$\begin{aligned} \frac{d}{dt} \left(\|Y_t\|_2^2 + \|\nabla Y\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 \right) + 2\|\nabla Y_t\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 + 2\|\zeta_t\|_{\Gamma_1}^2 \\ \leq 2C_2C_3^q \|Y\|_{H_{\Gamma_0}^1} \|\alpha u + (1-\alpha)\tilde{u}\|_{H_{\Gamma_0}^1}^q \|Y_t\|_2. \end{aligned}$$

Using Poincaré's inequality, we find that

$$\begin{aligned} \frac{d}{dt} \left(\|Y_t\|_2^2 + \|\nabla Y\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 \right) + 2\|\nabla Y_t\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 + 2\|\zeta_t\|_{\Gamma_1}^2 \\ \leq 2C_1C_2C_3^q \sqrt{1+C_\Omega^2} \|\nabla Y\|_2 \|\nabla Y_t\|_2 \|\alpha u + (1-\alpha)\tilde{u}\|_{H_{\Gamma_0}^1}^q. \end{aligned}$$

Since $(\alpha u + (1-\alpha)\tilde{u}) \in L^\infty(0, T, H_{\Gamma_0}^1(\Omega))$, there exists $C > 0$ such that

$$\begin{aligned} \frac{d}{dt} \left(\|Y_t\|_2^2 + \|\nabla Y\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 \right) + 2\|\nabla Y_t\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 + 2\|\zeta_t\|_{\Gamma_1}^2 &\leq C \|\nabla Y\|_2 \|\nabla Y_t\|_2 \\ &\leq \frac{C^2}{4} \|\nabla Y\|_2^2 + \|\nabla Y_t\|_2^2, \end{aligned}$$

where we used Young's inequality with $\delta = 1$.

Then

$$\frac{d}{dt} \left(\|Y_t\|_2^2 + \|\nabla Y\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 \right) + \|\nabla Y_t\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 + 2\|\zeta_t\|_{\Gamma_1}^2 \leq \frac{C^2}{4} \|\nabla Y\|_2^2.$$

Integrating the last inequality over $(0, t)$ and using the initial conditions, it follows that

$$\begin{aligned} \|Y_t\|_2^2 + \|\nabla Y\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 + \int_0^t \left(\|\nabla Y_t(s)\|_2^2 + 2\|\zeta(s)\|_{\Gamma_1}^2 + 2\|\zeta_t(s)\|_{\Gamma_1}^2 \right) ds \\ \leq \frac{C^2}{4} \int_0^t \|\nabla Y(s)\|_2^2 ds. \end{aligned}$$

Using Gronwall's inequality, we deduce that

$$\|Y_t\|_2^2 + \|\nabla Y\|_2^2 + 2\|\zeta\|_{\Gamma_1}^2 + \int_0^t \left(\|\nabla Y_t(s)\|_2^2 + 2\|\zeta(s)\|_{\Gamma_1}^2 + 2\|\zeta_t(s)\|_{\Gamma_1}^2 \right) ds = 0.$$

Therefore, $Y_t = \nabla Y = \nabla Y_t = 0$ and $\zeta = \zeta_t = 0$.

Hence, $Y = 0$ and $\zeta = 0$. □

4.3 Asymptotic behavior

In this section, we shall investigate the asymptotic behavior of problem (4.1.1)–(4.1.2).

First, we define the energy function associated with problem (4.1.1)–(4.1.2) as

$$E(t) = \frac{1}{2} \left(\|u_t\|_2^2 + \|\nabla u\|_2^2 + \frac{2}{p} \|u\|_p^p \right) + \|\xi\|_{\Gamma_1}^2. \quad (4.3.1)$$

Lemma 4.3.1. *Let (u, ξ) be a solution of problem (4.1.1)–(4.1.2). Then the energy function defined by (4.3.1) satisfies*

$$E'(t) = -\|\nabla u_t\|_2^2 - \|\xi\|_{\Gamma_1}^2 - \|\xi_t\|_{\Gamma_1}^2 + (f, u_t) + (g, \xi + \xi_t)_1, \quad \forall t \in [0, T]. \quad (4.3.2)$$

Proof. Multiplying the first equation of (4.1.1) by u_t and integrating over Ω , we obtain

$$\frac{1}{2} \frac{d}{dt} \left(\|u_t\|_2^2 + \|\nabla u\|_2^2 + \frac{2}{p} \|u\|_p^p \right) + \|\nabla u_t\|_2^2 + (\xi + \xi_t, u_t)_1 = (f, u_t). \quad (4.3.3)$$

Next, multiplying the fourth equation of (4.1.1) by $\xi_t + \xi$ and integrating over Γ_1 to obtain

$$(\xi + \xi_t, u_t)_1 = \|\xi\|_{\Gamma_1}^2 + \|\xi_t\|_{\Gamma_1}^2 - (g, \xi + \xi_t)_1 + \frac{d}{dt} \|\xi\|_{\Gamma_1}^2. \quad (4.3.4)$$

Substituting (4.3.4) in (4.3.3), we obtain

$$E'(t) = -\|\nabla u_t\|_2^2 - \|\xi\|_{\Gamma_1}^2 - \|\xi_t\|_{\Gamma_1}^2 + (f, u_t) + (g, \xi + \xi_t)_1, \quad \forall t \in [0, T].$$

□

Now, as stated in [31], we define the perturbed energy function by

$$E_\varepsilon(t) = E(t) + \varepsilon \psi(t), \quad \forall \varepsilon > 0, \quad (4.3.5)$$

where

$$\psi(t) = \int_{\Omega} u_t u dx. \quad (4.3.6)$$

Proposition 4.3.1. *There exists a positive constant c_1 such that*

$$|E_\varepsilon(t) - E(t)| \leq \varepsilon c_1 E(t), \quad \forall t \in [0, T] \quad \text{and} \quad \forall \varepsilon > 0.$$

Proof. Using the Cauchy–Schwarz and Young’s inequalities, we get

$$|\psi(t)| \leq \frac{1}{2} (\|u_t\|_2^2 + \|u\|_2^2).$$

By the Poincaré inequality, we have

$$|\psi(t)| \leq \frac{1}{2} (\|u_t\|_2^2 + C_\Omega^2 \|\nabla u\|_2^2) \leq c_1 E(t), \quad (4.3.7)$$

where $c_1 = \max\{1, C_\Omega^2\}$.

From (4.3.5) and (4.3.7), we can write

$$|E_\varepsilon(t) - E(t)| \leq \varepsilon c_1 E(t), \quad \forall \varepsilon > 0.$$

□

Proposition 4.3.2. *There exist two positive constants $c_2 = c_2(\varepsilon)$ and ε_1 such that*

$$E'_\varepsilon(t) \leq -\varepsilon E(t) + c_2 \left(\|f(t)\|_2^2 + \|g(t)\|_{\Gamma_1}^2 \right), \quad \forall t \in [0, T] \quad \text{and} \quad \forall \varepsilon \in (0, \varepsilon_1].$$

Proof. From (4.3.5), (4.3.6), and (4.3.2), we have

$$E'_\varepsilon(t) = -\|\nabla u_t\|_2^2 - \|\xi\|_{\Gamma_1}^2 - \|\xi_t\|_{\Gamma_1}^2 + \varepsilon \|u_t\|_2^2 + \varepsilon (u_{tt}, u) + (f, u_t) + (g, \xi + \xi_t)_1.$$

Using the Cauchy–Schwarz and Young’s inequalities, there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} E'_\varepsilon(t) &\leq -\|\nabla u_t\|_2^2 - \|\xi\|_{\Gamma_1}^2 - \|\xi_t\|_{\Gamma_1}^2 + \varepsilon \|u_t\|_2^2 + \varepsilon (u_{tt}, u) + \delta_1 \|u_t\|_2^2 \\ &\quad + \delta_2 \|\xi + \xi_t\|_{\Gamma_1}^2 + \frac{1}{4\delta_2} \|g\|_{\Gamma_1}^2 + \frac{1}{4\delta_1} \|f\|_2^2 \\ &\leq -\|\nabla u_t\|_2^2 + (2\delta_2 - 1) \|\xi\|_{\Gamma_1}^2 + (2\delta_2 - 1) \|\xi_t\|_{\Gamma_1}^2 + (\varepsilon + \delta_1) \|u_t\|_2^2 + \varepsilon (u_{tt}, u) \\ &\quad + \frac{1}{4\delta_2} \|g\|_{\Gamma_1}^2 + \frac{1}{4\delta_1} \|f\|_2^2. \end{aligned}$$

By the Poincaré inequality, we have

$$\begin{aligned} E'_\varepsilon(t) &\leq ((\varepsilon + \delta_1) C_\Omega^2 - 1) \|\nabla u_t\|_2^2 + (2\delta_2 - 1) \left(\|\xi\|_{\Gamma_1}^2 + \|\xi_t\|_{\Gamma_1}^2 \right) + \varepsilon (u_{tt}, u) \\ &\quad + \frac{1}{4\delta_2} \|g\|_{\Gamma_1}^2 + \frac{1}{4\delta_1} \|f\|_2^2. \end{aligned} \tag{4.3.8}$$

Now, we estimate (u_{tt}, u) . Multiplying the first equation of (4.1.1) by u and integrating over Ω , we obtain

$$(u_{tt}, u) = -\|\nabla u\|_2^2 - (\nabla u_t, \nabla u) - \|u\|_p^p - (\xi + \xi_t, u)_1 + (f, u).$$

Using the Cauchy–Schwarz and Young’s inequalities, there exist $\delta_3, \delta_4, \delta_5 > 0$ such that

$$\begin{aligned} (u_{tt}, u) &\leq -\|\nabla u\|_2^2 - \|u\|_p^p + \delta_3 \|\nabla u_t\|_2^2 + \frac{1}{4\delta_3} \|\nabla u\|_2^2 + \delta_4 \|\xi + \xi_t\|_{\Gamma_1}^2 \\ &\quad + \frac{1}{4\delta_4} \|u\|_{\Gamma_1}^2 + \delta_5 \|u\|_2^2 + \frac{1}{4\delta_5} \|f\|_2^2 \\ &\leq \left(\frac{1}{4\delta_3} - 1 \right) \|\nabla u\|_2^2 - \|u\|_p^p + \delta_3 \|\nabla u_t\|_2^2 + 2\delta_4 \left(\|\xi\|_{\Gamma_1}^2 + \|\xi_t\|_{\Gamma_1}^2 \right) \\ &\quad + \frac{1}{4\delta_4} \|u\|_{\Gamma_1}^2 + \delta_5 \|u\|_2^2 + \frac{1}{4\delta_5} \|f\|_2^2. \end{aligned}$$

Using the embedding $H_{\Gamma_0}^1(\Omega) \hookrightarrow L^2(\Gamma_1)$, together with the Poincaré inequality, we

find that

$$\begin{aligned}
(u_{tt}, u) &\leq \left(\frac{1}{4\delta_3} - 1 \right) \|\nabla u\|_2^2 - \|u\|_p^p + \delta_3 \|\nabla u_t\|_2^2 + 2\delta_4 \left(\|\xi\|_{\Gamma_1}^2 + \|\xi_t\|_{\Gamma_1}^2 \right) \\
&\quad + \frac{1}{4\delta_4} C_4^2 \|u\|_{H_{\Gamma_0}^1}^2 + \delta_5 \|u\|_2^2 + \frac{1}{4\delta_5} \|f\|_2^2 \\
&\leq \left(\frac{1}{4\delta_3} - 1 \right) \|\nabla u\|_2^2 - \|u\|_p^p + \delta_3 \|\nabla u_t\|_2^2 + 2\delta_4 \left(\|\xi\|_{\Gamma_1}^2 + \|\xi_t\|_{\Gamma_1}^2 \right) \\
&\quad + \frac{1}{4\delta_4} C_4^2 (1 + C_\Omega^2) \|\nabla u\|_2^2 + \delta_5 C_\Omega^2 \|\nabla u\|_2^2 + \frac{1}{4\delta_5} \|f\|_2^2.
\end{aligned}$$

Hence

$$\begin{aligned}
(u_{tt}, u) &\leq \left(\frac{1}{4\delta_3} + \frac{1}{4\delta_4} C_4^2 (1 + C_\Omega^2) + \delta_5 C_\Omega^2 - 1 \right) \|\nabla u\|_2^2 - \|u\|_p^p + \delta_3 \|\nabla u_t\|_2^2 \\
&\quad + 2\delta_4 \left(\|\xi\|_{\Gamma_1}^2 + \|\xi_t\|_{\Gamma_1}^2 \right) + \frac{1}{4\delta_5} \|f\|_2^2.
\end{aligned} \tag{4.3.9}$$

From (4.3.8) and (4.3.9), we obtain

$$\begin{aligned}
E'_\varepsilon(t) &\leq \left(\frac{1}{4\delta_3} + \frac{1}{4\delta_4} C_4^2 (1 + C_\Omega^2) + \delta_5 C_\Omega^2 - 1 \right) \varepsilon \|\nabla u\|_2^2 + ((\varepsilon + \delta_1) C_\Omega^2 + \delta_3 \varepsilon - 1) \|\nabla u_t\|_2^2 \\
&\quad - \varepsilon \|u\|_p^p + (2\delta_2 + 2\delta_4 \varepsilon - 1) \left(\|\xi\|_{\Gamma_1}^2 + \|\xi_t\|_{\Gamma_1}^2 \right) + \frac{1}{4\delta_2} \|g\|_{\Gamma_1}^2 + \left(\frac{1}{4\delta_1} + \frac{1}{4\delta_5} \varepsilon \right) \|f\|_2^2.
\end{aligned}$$

Choosing $\delta_1 = \delta_2 = \varepsilon$, $\delta_3 = 1$, $\delta_4 = 2C_4^2(1 + C_\Omega^2)$, and $\delta_5 = \frac{1}{8C_\Omega^2}$, we get

$$\begin{aligned}
E'_\varepsilon(t) &\leq -\frac{\varepsilon}{2} \|\nabla u\|_2^2 + \varepsilon \left(2C_\Omega^2 + 1 - \frac{1}{\varepsilon} \right) \|\nabla u_t\|_2^2 - \varepsilon \|u\|_p^p \\
&\quad + \varepsilon \left(2 + 4C_4^2(1 + C_\Omega^2) - \frac{1}{\varepsilon} \right) \left(\|\xi\|_{\Gamma_1}^2 + \|\xi_t\|_{\Gamma_1}^2 \right) + \frac{1}{4\varepsilon} \|g\|_{\Gamma_1}^2 \\
&\quad + \left(\frac{1}{4\varepsilon} + 2C_\Omega^2 \varepsilon \right) \|f\|_2^2 \\
&\leq -\frac{\varepsilon}{2} \|\nabla u\|_2^2 + \frac{\varepsilon}{2} \left(4 + \frac{2}{C_\Omega^2} - \frac{2}{\varepsilon C_\Omega^2} \right) C_\Omega^2 \|\nabla u_t\|_2^2 - \frac{\varepsilon}{p} \|u\|_p^p \\
&\quad + \varepsilon \left(2 + 4C_4^2(1 + C_\Omega^2) - \frac{1}{\varepsilon} \right) \left(\|\xi\|_{\Gamma_1}^2 + \|\xi_t\|_{\Gamma_1}^2 \right) + c_2 \left(\|g\|_{\Gamma_1}^2 + \|f\|_2^2 \right),
\end{aligned} \tag{4.3.10}$$

where $c_2 = \frac{1}{4\varepsilon} + 2C_\Omega^2 \varepsilon$.

Defining $\varepsilon_1 = \min \left\{ \frac{2}{2 + 5C_\Omega^2}, \frac{1}{3 + 4C_4^2(1 + C_\Omega^2)} \right\}$, and considering $\varepsilon \in (0, \varepsilon_1]$.

We know that for all $\varepsilon \in (0, \varepsilon_1]$, we have

$$4 + \frac{2}{C_\Omega^2} - \frac{2}{\varepsilon C_\Omega^2} \leq -1 \quad \text{and} \quad 2 + 4C_4^2(1 + C_\Omega^2) - \frac{1}{\varepsilon} \leq -1.$$

Thus, from (4.3.10), we deduce that

$$\begin{aligned} E'_\varepsilon(t) &\leq -\frac{\varepsilon}{2} \|\nabla u\|_2^2 - \frac{\varepsilon}{2} C_\Omega^2 \|\nabla u_t\|_2^2 - \frac{\varepsilon}{p} \|u\|_p^p - \varepsilon \left(\|\xi\|_{\Gamma_1}^2 + \|\xi_t\|_{\Gamma_1}^2 \right) + c_2(\varepsilon) \left(\|g\|_{\Gamma_1}^2 + \|f\|_2^2 \right) \\ &\leq -\frac{\varepsilon}{2} \|\nabla u\|_2^2 - \frac{\varepsilon}{2} \|u_t\|_2^2 - \frac{\varepsilon}{p} \|u\|_p^p - \varepsilon \|\xi\|_{\Gamma_1}^2 + c_2 \left(\|g\|_{\Gamma_1}^2 + \|f\|_2^2 \right). \end{aligned}$$

Then

$$E'_\varepsilon(t) \leq -\varepsilon E(t) + c_2 \left(\|g(t)\|_{\Gamma_1}^2 + \|f(t)\|_2^2 \right), \quad \forall t \in [0, T] \quad \text{and} \quad \forall \varepsilon \in (0, \varepsilon_1].$$

□

In the next theorem, we establish the asymptotic behavior of the problem (4.1.1)–(4.1.2).

Theorem 4.3.1. *In addition to assumptions of Theorem 4.2.1, assuming that for a large t the inequality*

$$\int_0^t \exp\left(\frac{\varepsilon}{2}s\right) \left(\|g(s)\|_{\Gamma_1}^2 + \|f(s)\|_2^2 \right) ds \leq \alpha t^\beta, \quad (4.3.11)$$

holds for some positive constants ε, α and β , we obtain the following energy decay:

$$E(t) \leq \left(3E(0) + 2c_2 \alpha t^\beta \right) \exp\left(-\frac{\varepsilon}{2}t\right), \quad \forall t \geq 0 \quad \text{and} \quad \forall \varepsilon \in (0, \varepsilon_0],$$

where ε_0 and c_2 are positive constants to be specified in the proof.

Proof. We define

$$\varepsilon_0 = \min \left\{ \varepsilon_1, \frac{1}{2c_1} \right\}.$$

Let $\varepsilon \in (0, \varepsilon_0]$.

From Proposition 4.3.1, we have

$$(1 - \varepsilon c_1) E(t) \leq E_\varepsilon(t) \leq (1 + \varepsilon c_1) E(t), \quad \forall t \geq 0.$$

Since $\varepsilon \leq \frac{1}{2c_1}$, it follows that

$$\frac{1}{2} E(t) \leq E_\varepsilon(t) \leq \frac{3}{2} E(t) \leq 2E(t), \quad \forall t \geq 0. \quad (4.3.12)$$

Therefore

$$-\varepsilon E(t) \leq -\frac{\varepsilon}{2} E_\varepsilon(t), \quad \forall t \geq 0. \quad (4.3.13)$$

Hence, from (4.3.13) and considering Proposition 4.3.2, we obtain

$$E'_\varepsilon(t) \leq -\frac{\varepsilon}{2} E_\varepsilon(t) + c_2 \left(\|f(t)\|_2^2 + \|g(t)\|_{\Gamma_1}^2 \right).$$

Consequently,

$$\frac{d}{dt} \left(E_\varepsilon(t) \exp\left(\frac{\varepsilon}{2}t\right) \right) \leq c_2 \left(\|f(t)\|_2^2 + \|g(t)\|_{\Gamma_1}^2 \right) \exp\left(\frac{\varepsilon}{2}t\right).$$

Integrating from 0 to t , we obtain

$$\begin{aligned} E_\varepsilon(t) &\leq \left(E_\varepsilon(0) + c_2 \int_0^t \left(\|f(s)\|_2^2 + \|g(s)\|_{\Gamma_1}^2 \right) \exp\left(\frac{\varepsilon}{2}s\right) ds \right) \exp\left(-\frac{\varepsilon}{2}t\right) \\ &\leq \left(E_\varepsilon(0) + c_2 \alpha t^\beta \right) \exp\left(-\frac{\varepsilon}{2}t\right), \end{aligned} \quad (4.3.14)$$

where we used the inequality (4.3.11).

Using (4.3.14) together with (4.3.12), we get

$$E(t) \leq \left(3E(0) + 2c_2 \alpha t^\beta \right) \exp\left(-\frac{\varepsilon}{2}t\right), \quad \forall t \geq 0 \quad \text{and} \quad \forall \varepsilon \in (0, \varepsilon_0].$$

This completes the proof of Theorem 4.3.1. □

4.4 Semi-Discrete Approximation

Assume that $\overline{\Omega}$ is a polyhedral domain. Let $V^h \subset V$ be a finite-dimensional space approximating the variational space V , defined by

$$V^h = \left\{ v \in C(\overline{\Omega}) : v_K^h \in \mathbb{P}_1(K), K \in \mathcal{T}_h, v^h = 0 \text{ on } \Gamma_0 \right\},$$

where $\mathbb{P}_1(K)$ represents the space of polynomials of global degree less than or equal to one in K , and we denote by $(\mathcal{T}_h)_{h>0}$ a regular family of triangulations of $\overline{\Omega}$ (see for instance [13]), compatible with the partition of the boundary $\Gamma = \partial\Omega$ into Γ_0 and Γ_1 , i.e. the finite element space V^h is composed of continuous and piecewise affine functions.

Let h_K be the diameter of an element $K \in \mathcal{T}_h$ and let $h = \max_{K \in \mathcal{T}_h} h_K$ denote the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $u_0^h = P^h u_0$, $u_1^h = P^h u_1$, and $\xi_0^h = P^h \xi_0$, where P^h is the classical finite element interpolation operator defined by

$$\left(\nabla(P^h \psi), \nabla \chi \right) = \left(\nabla \psi, \nabla \chi \right), \quad \forall \chi \in V^h,$$

and satisfying the following estimates (see [13])

$$\begin{cases} \left\| \psi - P^h \psi \right\|_2 \leq Ch \left\| \nabla \psi \right\|_2, & \forall \psi \in V, \\ \left\| \psi - P^h \psi \right\|_{H_{\Gamma_0}^1} \leq Ch \left\| \psi \right\|_*, & \forall \psi \in V, \end{cases} \quad (4.4.1)$$

where $\left\| \psi \right\|_*^2 = \sum_{|\alpha|=2} \int_{\Omega} |\partial^\alpha \psi|^2 dx$.

The semi-discrete finite element method for (4.1.1) is to find $u^h(t), \xi^h(t) \in V^h$,

for $0 \leq t \leq T$, such that for all $v^h, w^h \in V^h$,

$$\begin{cases} \left(u_{tt}^h, v^h \right) + \left(\nabla u^h, \nabla v^h \right) + \left(\nabla u_t^h, \nabla v^h \right) + \left(|u^h|^q u^h, v^h \right) \\ \quad + \left(\xi^h + \xi_t^h, v^h \right)_1 = \left(f, v^h \right), \\ \left(\xi_t^h, w^h \right)_1 + \left(\xi^h, w^h \right)_1 = \left(u_t^h + g, w^h \right)_1, \end{cases} \quad (4.4.2)$$

with $u^h(0) = P^h u_0$, $u_t^h(0) = P^h u_1$, and $\xi^h(0) = P^h \xi_0$.

In the next theorem, we derive error bounds for the piecewise linear approximation (4.4.2).

Theorem 4.4.1. *Assume that the assumptions of Theorem 4.2.1 hold, with $\xi_0 \in H_{\Gamma_0}^2(\Omega)$ and $\xi \in H^1(0, T, H_{\Gamma_0}^2(\Omega))$. Then we have the following estimate:*

$$\|u_t - u_t^h\|_2^2 + \|\nabla u - \nabla u^h\|_2^2 + \|\xi - \xi^h\|_{\Gamma_1}^2 \leq Ch^2,$$

holds for $0 \leq t \leq T$.

Proof. In particular, (4.2.1) is true for all $v^h, w^h \in V^h$. Taking $v = v^h$ and $w = w^h \in V^h$ and subtracting the discrete formulation, we obtain

$$\begin{cases} \left(u_{tt} - u_{tt}^h, v^h \right) + \left(\nabla u - \nabla u^h, \nabla v^h \right) + \left(\nabla u_t - \nabla u_t^h, \nabla v^h \right) \\ \quad + \left(|u|^q u - |u^h|^q u^h, v^h \right) + \left(\xi - \xi^h + \xi_t - \xi_t^h, v^h \right)_1 = 0, \\ \left(\xi_t - \xi_t^h, w^h \right)_1 + \left(\xi - \xi^h, w^h \right)_1 = \left(u_t - u_t^h, w^h \right)_1, \end{cases} \quad (4.4.3)$$

Consider the decomposition of the error: $e^h = u^h - P^h u$ and $b^h = \xi^h - P^h \xi$. Therefore, we obtain

$$\begin{cases} \left(e_{tt}^h, v^h \right) + \left(\nabla e^h, \nabla v^h \right) + \left(\nabla e_t^h, \nabla v^h \right) + \left(|u|^q u - |u^h|^q u^h, v^h \right) \\ \quad + \left(b^h + b_t^h, v^h \right)_1 = \left(u_{tt} - P^h u_{tt}, v^h \right) + \left(\nabla u - P^h(\nabla u), \nabla v^h \right) \\ \quad + \left(\nabla u_t - P^h(\nabla u_t), \nabla v^h \right) + \left(\xi - P^h \xi + \xi_t - P^h \xi_t, v^h \right)_1, \\ \left(e_t^h, w^h \right)_1 = \left(b_t^h + b^h, w^h \right)_1 - \left(\xi_t - P^h \xi_t + \xi - P^h \xi, w^h \right)_1 \\ \quad + \left(u_t - P^h u_t, w^h \right)_1. \end{cases} \quad (4.4.4)$$

Take $v^h = e_t^h$ and $w^h = b^h + b_t^h$. Then, we find that

$$\begin{cases} \frac{1}{2} \frac{d}{dt} \left(\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 \right) + \|\nabla e_t^h\|_2^2 + \left(|u|^q u - |u^h|^q u^h, e_t^h \right) + \left(b^h + b_t^h, e_t^h \right)_1 \\ \quad = \left(u_{tt} - P^h u_{tt}, e_t^h \right) + \left(\nabla u - P^h(\nabla u), \nabla e_t^h \right) + \left(\nabla u_t - P^h(\nabla u_t), \nabla e_t^h \right) \\ \quad + \left(\xi - P^h \xi + \xi_t - P^h \xi_t, e_t^h \right)_1, \\ \left(e_t^h, b^h + b_t^h \right)_1 = \left(u_t - P^h u_t, b^h + b_t^h \right)_1 - \left(\xi_t - P^h \xi_t + \xi - P^h \xi, b^h + b_t^h \right)_1 \\ \quad + \|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 + \frac{d}{dt} \|b^h\|_{\Gamma_1}^2. \end{cases}$$

Therefore, we find that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + 2\|b^h\|_{\Gamma_1}^2 \right) + \|\nabla e_t^h\|_2^2 + \|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 \\
&= \left(u_{tt} - P^h u_{tt}, e_t^h \right) + \left(\nabla u - P^h(\nabla u), \nabla e_t^h \right) + \left(\nabla u_t - P^h(\nabla u_t), \nabla e_t^h \right) \\
&+ \left(\xi_t - P^h \xi_t + \xi - P^h \xi, b^h + b_t^h \right)_1 + \left(\xi - P^h \xi + \xi_t - P^h \xi_t, e_t^h \right)_1 \\
&- \left(u_t - P^h u_t, b^h + b_t^h \right)_1 - \left(|u|^q u - |u^h|^q u^h, e_t^h \right).
\end{aligned} \tag{4.4.5}$$

By the definition of P^h , we have

$$\begin{aligned}
\left(\nabla u - \nabla(P^h u), \nabla e_t^h \right) &= \left(\nabla u, \nabla e_t^h \right) - \left(\nabla(P^h u), \nabla e_t^h \right) \\
&= \left(\nabla u, \nabla e_t^h \right) - \left(\nabla u, \nabla e_t^h \right) \\
&= 0,
\end{aligned}$$

Similarly, we find that

$$\left(\nabla u_t - \nabla(P^h u_t), \nabla e_t^h \right) = 0.$$

Using Young's inequality, there exist $\delta_1, \delta_2, \delta_3, \delta_4 > 0$ such that

$$\begin{aligned}
\left(u_{tt} - P^h u_{tt}, e_t^h \right) &\leq C \|u_{tt} - P^h u_{tt}\|_2^2 + \delta_3 \|e_t^h\|_2^2, \\
\left(u_t - P^h u_t, b^h + b_t^h \right)_1 &\leq C \|u_t - P^h u_t\|_{\Gamma_1}^2 + \delta_1 \|b_t^h\|_{\Gamma_1}^2 + \delta_2 \|b^h\|_{\Gamma_1}^2, \\
\left(\xi - P^h \xi + \xi_t - P^h \xi_t, e_t^h \right)_1 &\leq C \left(\|\xi - P^h \xi\|_{\Gamma_1}^2 + \|\xi_t - P^h \xi_t\|_{\Gamma_1}^2 \right) + \delta_4 \|e_t^h\|_{\Gamma_1}^2, \\
\left(\xi_t - P^h \xi_t + \xi - P^h \xi, b^h + b_t^h \right)_1 &\leq C \left(\|\xi - P^h \xi\|_{\Gamma_1}^2 + \|\xi_t - P^h \xi_t\|_{\Gamma_1}^2 \right) + \delta_1 \|b_t^h\|_{\Gamma_1}^2 \\
&+ \delta_2 \|b^h\|_{\Gamma_1}^2.
\end{aligned}$$

Combine these results with (4.4.5) to obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + 2\|b^h\|_{\Gamma_1}^2 \right) + \|\nabla e_t^h\|_2^2 + \|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 \\
&\leq C \left(\|u_{tt} - P^h u_{tt}\|_2^2 + \|u_t - P^h u_t\|_{\Gamma_1}^2 + \|\xi - P^h \xi\|_{\Gamma_1}^2 + \|\xi_t - P^h \xi_t\|_{\Gamma_1}^2 \right) \\
&+ 2\delta_1 \|b_t^h\|_{\Gamma_1}^2 + 2\delta_2 \|b^h\|_{\Gamma_1}^2 + \delta_3 \|e_t^h\|_2^2 + \delta_4 \|e_t^h\|_{\Gamma_1}^2 - \left(|u|^q u - |u^h|^q u^h, e_t^h \right).
\end{aligned} \tag{4.4.6}$$

Using the first inequality of (4.4.1), we get

$$\|u_{tt} - P^h u_{tt}\|_2^2 \leq Ch^2 \|\nabla u_{tt}\|_2^2,$$

using again the embedding $H_{\Gamma_0}^1(\Omega) \hookrightarrow L^2(\Gamma_1)$ with the second inequality of (4.4.1),

we get

$$\begin{aligned}\|\xi - P^h \xi\|_{\Gamma_1}^2 &\leq Ch^2 \|\xi\|_*^2, \\ \|\xi_t - P^h \xi_t\|_{\Gamma_1}^2 &\leq Ch^2 \|\xi_t\|_*^2, \\ \|u_t - P^h u_t\|_{\Gamma_1}^2 &\leq Ch^2 \|u_t\|_*^2.\end{aligned}$$

Using the embedding $H_{\Gamma_0}^1(\Omega) \hookrightarrow L^2(\Gamma_1)$ together with the Poincaré inequality, we find

$$\|e_t^h\|_{\Gamma_1}^2 \leq C_4^2 \|e_t^h\|_{H_{\Gamma_0}^1}^2 \leq C_4^2 (1 + C_\Omega^2) \|\nabla e_t^h\|_2^2 \quad \text{and} \quad \|e_t^h\|_2^2 \leq C_\Omega^2 \|\nabla e_t^h\|_2^2.$$

Hence, (4.4.6) becomes

$$\begin{aligned}&\frac{1}{2} \frac{d}{dt} \left(\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + 2 \|b^h\|_{\Gamma_1}^2 \right) + \|\nabla e_t^h\|_2^2 + \|b^h\|_{\Gamma_1}^2 + \|b_t^h\|_{\Gamma_1}^2 \\ &\leq Ch^2 (\|\nabla u_{tt}\|_2^2 + \|u_t\|_*^2 + \|\xi\|_*^2 + \|\xi_t\|_*^2) + 2\delta_1 \|b_t^h\|_{\Gamma_1}^2 + 2\delta_2 \|b^h\|_{\Gamma_1}^2 \\ &\quad + \delta_3 C_\Omega^2 \|\nabla e_t^h\|_2^2 + \delta_4 C_4^2 (1 + C_\Omega^2) \|\nabla e_t^h\|_2^2 + \left| (|u|^q u - |u^h|^q u^h, e_t^h) \right|.\end{aligned}\tag{4.4.7}$$

Analysis of the Nonlinear Term. Denoting $h(u) = |u|^q u$. Next, we apply the mean value theorem in h to obtain

$$\begin{aligned}h(u) - h(u^h) &= (u - u^h) h'(\alpha u + (1 - \alpha) u^h) \\ &= (u - u^h) \left| \alpha u + (1 - \alpha) u^h \right|^q, \quad \text{for } \alpha \in (0, 1).\end{aligned}$$

Then

$$\begin{aligned}\left| (|u|^q u - |u^h|^q u^h, e_t^h) \right| &= \left| \left((u - u^h) \left| \alpha u + (1 - \alpha) u^h \right|^q, e_t^h \right) \right| \\ &\leq \int_\Omega |u - u^h| \left| \alpha u + (1 - \alpha) u^h \right|^q |e_t^h| dx.\end{aligned}$$

Applying Hölder's inequality, and considering $p_1 > 0$ and $p_2 > 0$, such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$, it follows that

$$\begin{aligned}\left| (|u|^q u - |u^h|^q u^h, e_t^h) \right| &\leq \|u - u^h\|_{p_1} \left\| \alpha u + (1 - \alpha) u^h \right\|_{qp_2}^q \|e_t^h\|_2 \\ &\leq \|u - u^h\|_{p_1} \left(\alpha^q \|u\|_{qp_2}^q + (1 - \alpha)^q \|u^h\|_{qp_2}^q \right) \|e_t^h\|_2.\end{aligned}$$

By applying the same method by which we derived estimates (4.2.12), we obtain

$$\left| (|u|^q u - |u^h|^q u^h, e_t^h) \right| \leq \gamma C_2 C_3^q \|u - u^h\|_{H_{\Gamma_0}^1} \|e_t^h\|_2,$$

where $\gamma = \gamma(\alpha, q, u, u^h) = \alpha^q \|u\|_{H_{\Gamma_0}^1}^q + (1 - \alpha)^q \|u^h\|_{H_{\Gamma_0}^1}^q$.

Using Poincaré's inequality, we find that

$$\left| \left(|u|^q u - |u^h|^q u^h, e_t^h \right) \right| \leq \gamma C_\Omega C_2 C_3^q \sqrt{1 + C_\Omega^2} \left\| \nabla u - \nabla u^h \right\|_2 \left\| \nabla e_t^h \right\|_2. \quad (4.4.8)$$

Then

$$\begin{aligned} \left| \left(|u|^q u - |u^h|^q u^h, e_t^h \right) \right| &\leq C \left\| \nabla u - P^h(\nabla u) - \nabla e^h \right\|_2 \left\| \nabla e_t^h \right\|_2 \\ &\leq C \left(\left\| \nabla u - P^h(\nabla u) \right\|_2 + \left\| \nabla e^h \right\|_2 \right) \left\| \nabla e_t^h \right\|_2 \\ &\leq Ch \left\| \Delta u \right\|_2 \left\| \nabla e_t^h \right\|_2 + C \left\| \nabla e^h \right\|_2 \left\| \nabla e_t^h \right\|_2. \end{aligned}$$

Using Young's inequality, there exist $\delta_5 > 0$ such that

$$\left| \left(|u|^q u - |u^h|^q u^h, e_t^h \right) \right| \leq Ch^2 \left\| \Delta u \right\|_2^2 + C \left\| \nabla e^h \right\|_2^2 + \delta_5 \left\| \nabla e_t^h \right\|_2^2. \quad (4.4.9)$$

Using (4.4.7) and (4.4.9), we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\left\| e_t^h \right\|_2^2 + \left\| \nabla e^h \right\|_2^2 + 2 \left\| b^h \right\|_{\Gamma_1}^2 \right) + \left\| \nabla e_t^h \right\|_2^2 + \left\| b^h \right\|_{\Gamma_1}^2 + \left\| b_t^h \right\|_{\Gamma_1}^2 \\ &\leq Ch^2 \left(\left\| \Delta u \right\|_2^2 + \left\| \nabla u_{tt} \right\|_2^2 + \left\| u_t \right\|_*^2 + \left\| \xi \right\|_*^2 + \left\| \xi_t \right\|_*^2 \right) + 2\delta_1 \left\| b_t^h \right\|_{\Gamma_1}^2 + 2\delta_2 \left\| b^h \right\|_{\Gamma_1}^2 \\ &\quad + \delta_3 C_1^2 \left\| \nabla e_t^h \right\|_2^2 + \delta_4 C_4^2 (1 + C_1^2) \left\| \nabla e_t^h \right\|_2^2 + \delta_5 \left\| \nabla e_t^h \right\|_2^2 + C \left\| \nabla e^h \right\|_2^2. \end{aligned}$$

Choosing $\delta_1 = \delta_2 = \delta_5 = \frac{1}{4}$, $\delta_3 = \frac{1}{8C_1^2}$, and $\delta_4 = \frac{1}{8C_4^2(1 + C_1^2)}$, we get

$$\begin{aligned} &\frac{d}{dt} \left(\left\| e_t^h \right\|_2^2 + \left\| \nabla e^h \right\|_2^2 + 2 \left\| b^h \right\|_{\Gamma_1}^2 \right) + \left\| \nabla e_t^h \right\|_2^2 + \left\| b^h \right\|_{\Gamma_1}^2 + \left\| b_t^h \right\|_{\Gamma_1}^2 \\ &\leq Ch^2 \left(\left\| \Delta u \right\|_2^2 + \left\| \nabla u_{tt} \right\|_2^2 + \left\| u_t \right\|_*^2 + \left\| \xi \right\|_*^2 + \left\| \xi_t \right\|_*^2 \right) + C \left\| \nabla e^h \right\|_2^2. \end{aligned}$$

Integrating the last inequality from 0 to t , we have

$$\begin{aligned} &\left\| e_t^h \right\|_2^2 + \left\| \nabla e^h \right\|_2^2 + 2 \left\| b^h \right\|_{\Gamma_1}^2 + \int_0^t \left(\left\| \nabla e_t^h(s) \right\|_2^2 + \left\| b^h(s) \right\|_{\Gamma_1}^2 + \left\| b_t^h(s) \right\|_{\Gamma_1}^2 \right) ds \\ &\leq \left\| e_t^h(0) \right\|_2^2 + \left\| \nabla e^h(0) \right\|_2^2 + 2 \left\| b^h(0) \right\|_{\Gamma_1}^2 + Ch^2 \int_0^t \left(\left\| \Delta u(s) \right\|_2^2 + \left\| \nabla u_{tt}(s) \right\|_2^2 \right) ds \\ &\quad + Ch^2 \int_0^t \left(\left\| u_t(s) \right\|_*^2 + \left\| \xi(s) \right\|_*^2 + \left\| \xi_t(s) \right\|_*^2 \right) ds + C \int_0^t \left\| \nabla e^h(s) \right\|_2^2 ds. \end{aligned}$$

Note that

$$\left\| e_t^h(0) \right\|_2^2 = \left\| u_t^h(0) - P^h u_t(0) \right\|_2^2 = \left\| P_E^h u_1 - u_t(0) \right\|_2^2 = \left\| P_E^h u_1 - u_1 \right\|_2^2 \leq Ch \left\| \nabla u_1 \right\|_2.$$

Similarly, we show $\|\nabla e^h(0)\|_2 \leq Ch \|\Delta u_0\|_2$ and

$$\|b^h(0)\|_{\Gamma_1} = \|\xi^h(0) - P^h \xi(0)\|_{\Gamma_1} = \|\xi_0 - P^h \xi_0\|_{\Gamma_1} \leq C \|\xi_0 - P^h \xi_0\|_{H_{\Gamma_0}^1} \leq Ch \|\xi_0\|_*.$$

Using the above estimates, we obtain

$$\begin{aligned} & \|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + 2\|b^h\|_{\Gamma_1}^2 + \int_0^t \left(\|\nabla e_t^h(s)\|_2^2 + \|b^h(s)\|_{\Gamma_1}^2 + \|b_t^h(s)\|_{\Gamma_1}^2 \right) ds \\ & \leq Ch^2 (\|\nabla u_1\|_2^2 + \|\Delta u_0\|_2^2 + \|\xi_0\|_*^2) + Ch^2 \int_0^t (\|\Delta u(s)\|_2^2 + \|\nabla u_{tt}(s)\|_2^2) ds \\ & \quad + Ch^2 \int_0^t (\|u_t(s)\|_*^2 + \|\xi(s)\|_*^2 + \|\xi_t(s)\|_*^2) ds + C \int_0^t \|\nabla e^h(s)\|_2^2 ds. \end{aligned}$$

Using the regularity on u_0, u_1, ξ_0, ξ , and u , we get

$$\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + 2\|b^h\|_{\Gamma_1}^2 \leq Ch^2 + C \int_0^t \|\nabla e^h(s)\|_2^2 ds.$$

Applying Gronwall's inequality, we obtain

$$\|\nabla e^h\|_2^2 \leq Ch^2 + Ch^2 \int_0^t e^{C(t-s)} ds \leq Ch^2.$$

Therefore, we conclude that

$$\|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + 2\|b^h\|_{\Gamma_1}^2 \leq Ch^2.$$

Finally, we get

$$\begin{aligned} & \|u_t - u_t^h\|_2^2 + \|\nabla u - \nabla u^h\|_2^2 + \|\xi - \xi^h\|_{\Gamma_1}^2 \\ & \leq C \left(\|u_t - P^h u_t\|_2^2 + \|\nabla u - \nabla(P^h u)\|_2^2 + \|\xi - P^h \xi\|_{\Gamma_1}^2 + \|e_t^h\|_2^2 + \|\nabla e^h\|_2^2 + \|b^h\|_{\Gamma_1}^2 \right) \\ & \leq Ch^2 (\|\nabla u_t\|_2^2 + \|\Delta u\|_2^2 + \|\xi\|_*^2) + Ch^2 \leq Ch^2, \end{aligned}$$

where we used the regularity on u and ξ . □

4.5 Fully-Discrete Approximation

In this section, we first present a fully discrete numerical scheme for the problem (4.1.1) and then perform the error analysis. Let $(n, N) \in \mathbb{N} \times \mathbb{N}^*$. Concerning the discretization in time, we divide the interval $[0, T]$ into N subintervals.

Let $\tau = \frac{T}{N}$ denote the time step. Define $t_n = n\tau$, for $n \in \{0, 1, 2, \dots, N\}$.

Given an integer $N > 0$, our numerical scheme can be stated as follows: find u^n

and ξ^n , $n = 2, \dots, N$ such that for all $v^h, w^h \in V^h$

$$\begin{cases} \frac{1}{\tau} (\hat{u}^{n+1} - \hat{u}^n, v^h) + (\nabla u^{n+1}, \nabla v^h) + (\nabla \hat{u}^{n+1}, \nabla v^h) + (\xi^{n+1}, v^h)_1 \\ \quad + (\hat{\xi}^{n+1}, v^h)_1 + (h(u^{n+1}), v^h) = (f(t_{n+1}), v^h), \\ (\hat{\xi}^{n+1}, w^h)_1 + (\xi^{n+1}, w^h)_1 = (\hat{u}^{n+1}, w^h)_1 + (g(t_{n+1}), w^h)_1. \end{cases} \quad (4.5.1)$$

where

$$\hat{u}^{n+1} = \frac{u^{n+1} - u^n}{\tau}, \quad \hat{\xi}^{n+1} = \frac{\xi^{n+1} - \xi^n}{\tau},$$

with the following discrete initial conditions

$$u^0 = P^h u_0, \quad u^1 = u^0 + \tau P^h u_1, \quad \xi^0 = P^h \xi_0. \quad (4.5.2)$$

4.5.1 Error Estimate

The following theorem provides the main error estimate for the fully discrete finite element scheme (4.5.1)–(4.5.2).

Theorem 4.5.1. *Assume that the assumptions of Theorem 4.2.1 hold. Let $u \in H^3(0, T, L^2(\Omega)) \cap W^{2,\infty}(0, T, H^3(\Omega))$ and $\xi \in H^2(0, T, H^2(\Omega)) \cap W^{1,\infty}(0, T, H^2(\Omega))$. Then for sufficiently small τ , we have the following estimate:*

$$\|u_t(t_n) - \hat{u}^n\|_2^2 + \|\nabla u(t_n) - \nabla \hat{u}^n\|_2^2 + \|\xi(t_n) - \xi^n\|_{\Gamma_1}^2 \leq C(h^2 + \tau^2),$$

where C is a positive constant independent of h and τ .

Proof. We denote $\hat{u}(t) = u_t(t)$, $\hat{\xi}(t_n) = \xi_t(t_n)$, and then we use the standard decompositions

$$\begin{cases} u^n - u(t_n) = e^n + \rho^n, \\ \xi^n - \xi(t_n) = b^n + \sigma^n, \\ \hat{u}^n - \hat{u}(t_n) = \hat{e}^n + \hat{\rho}^n, \\ \hat{\xi}^n - \hat{\xi}(t_n) = \hat{b}^n + \hat{\sigma}^n, \end{cases} \quad \text{where} \quad \begin{cases} e^n = u^n - P^h u(t_n), & \rho^n = P^h u(t_n) - u(t_n), \\ b^n = \xi^n - P^h \xi(t_n), & \sigma^n = P^h \xi(t_n) - \xi(t_n), \\ \hat{e}^n = \hat{u}^n - P^h \hat{u}(t_n), & \hat{\rho}^n = P^h \hat{u}(t_n) - \hat{u}(t_n), \\ \hat{b}^n = \hat{\xi}^n - P^h \hat{\xi}(t_n), & \hat{\sigma}^n = P^h \hat{\xi}(t_n) - \hat{\xi}(t_n). \end{cases}$$

Using (4.2.1) with $t = t_{n+1}$, $v = v^h$, and $w = w^h$, we obtain

$$\begin{cases} (\hat{u}_t(t_{n+1}), v^h) + (\nabla u(t_{n+1}), \nabla v^h) + (\nabla \hat{u}(t_{n+1}), \nabla v^h) + (\xi(t_{n+1}), v^h)_1 \\ \quad + (\hat{\xi}(t_{n+1}), v^h)_1 + (h(u(t_{n+1})), v^h) = (f(t_{n+1}), v^h), \\ (\hat{\xi}(t_{n+1}), w^h)_1 + (\xi(t_{n+1}), w^h)_1 = (\hat{u}(t_{n+1}), w^h)_1 + (g(t_{n+1}), w^h)_1. \end{cases} \quad (4.5.3)$$

Subtracting system (4.5.3) from system (4.5.1) term by term, and applying the defi-

dition of P^h , it follows that

$$\left\{ \begin{array}{l} \frac{1}{\tau} (\hat{e}^{n+1} - \hat{e}^n, v^h) + (\nabla e^{n+1}, \nabla v^h) + (\nabla \hat{e}^{n+1}, \nabla v^h) + (b^{n+1} + \hat{b}^{n+1}, v^h)_1 \\ = \left(\hat{u}_t(t_{n+1}) - \frac{1}{\tau} (P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)), v^h \right) - (h(u^{n+1}) - h(u(t_{n+1}))), v^h \\ - (\sigma^{n+1} + \hat{\sigma}^{n+1}, v^h)_1, \\ (\hat{b}^{n+1} + b^{n+1}, w^h)_1 = (\hat{e}^{n+1}, w^h)_1 + (\hat{\rho}^{n+1}, w^h)_1 - (\sigma^{n+1} + \hat{\sigma}^{n+1}, w^h)_1. \end{array} \right.$$

Now, take $v^h = \hat{e}^{n+1}$ and $w^h = b^{n+1} + \hat{b}^{n+1}$ to obtain

$$\begin{aligned} & \frac{1}{\tau} (\hat{e}^{n+1} - \hat{e}^n, \hat{e}^{n+1}) + (\nabla e^{n+1}, \nabla \hat{e}^{n+1}) + 2(b^{n+1}, \hat{b}^{n+1})_1 \\ & + \|\nabla \hat{e}^{n+1}\|_2^2 + \|b^{n+1}\|_{\Gamma_1}^2 + \|\hat{b}^{n+1}\|_{\Gamma_1}^2 + (h(u^{n+1}) - h(u(t_{n+1}))), \hat{e}^{n+1} \\ & = \left(\hat{u}_t(t_{n+1}) - \frac{1}{\tau} (P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)), \hat{e}^{n+1} \right) + (\hat{\rho}^{n+1}, b^{n+1} + \hat{b}^{n+1})_1 \\ & - (\sigma^{n+1} + \hat{\sigma}^{n+1}, \hat{e}^{n+1})_1 - (\sigma^{n+1} + \hat{\sigma}^{n+1}, b^{n+1} + \hat{b}^{n+1})_1. \end{aligned} \quad (4.5.4)$$

Note that

$$\xi_t(t_{n+1}) = \frac{1}{\tau} (\xi(t_{n+1}) - \xi(t_n)) + \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \xi_{tt}(s) ds,$$

hence

$$\begin{aligned} \hat{b}^{n+1} &= \hat{\xi}^{n+1} - P^h \xi_t(t_{n+1}) \\ &= \frac{1}{\tau} \left[(\xi^{n+1} - P^h \xi(t_{n+1})) - (\xi^n - P^h \xi(t_n)) - \int_{t_n}^{t_{n+1}} (s - t_n) P^h \xi_{tt}(s) ds \right] \\ &= \frac{1}{\tau} (b^{n+1} - b^n - \theta^{n+1}), \end{aligned}$$

where $\theta^{n+1} = \int_{t_n}^{t_{n+1}} (s - t_n) P^h \xi_{tt}(s) ds$.

Therefore, we have

$$\begin{aligned} (b^{n+1}, \hat{b}^{n+1})_1 &= \frac{1}{\tau} (b^{n+1}, b^{n+1} - b^n - \theta^{n+1})_1 \\ &= \frac{1}{\tau} (b^{n+1}, b^{n+1} - b^n)_1 - \frac{1}{\tau} (b^{n+1}, \theta^{n+1})_1. \end{aligned} \quad (4.5.5)$$

Knowing that

$$\begin{aligned} \hat{e}^{n+1} &= \hat{u}^{n+1} - P^h \hat{u}(t_{n+1}) \\ &= \frac{1}{\tau} (u^{n+1} - u^n) - P^h \hat{u}(t_{n+1}) \\ &= \frac{1}{\tau} (e^{n+1} - e^n) + \frac{1}{\tau} (P^h u(t_{n+1}) - P^h u(t_n)) - P^h \hat{u}(t_{n+1}), \end{aligned}$$

hence

$$\begin{aligned}
(\nabla e^{n+1}, \nabla \hat{e}^{n+1}) &= \frac{1}{\tau} (\nabla e^{n+1}, \nabla e^{n+1} - \nabla e^n) - (\nabla e^{n+1}, \nabla (P^h \hat{u}(t_{n+1}))) \\
&\quad + \frac{1}{\tau} (\nabla e^{n+1}, \nabla (P^h u(t_{n+1})) - \nabla (P^h u(t_n))) \\
&= \left(\nabla e^{n+1}, \frac{1}{\tau} (\nabla u(t_{n+1}) - \nabla u(t_n)) - \nabla \hat{u}(t_{n+1}) \right) \\
&\quad + \frac{1}{\tau} (\nabla e^{n+1}, \nabla e^{n+1} - \nabla e^n),
\end{aligned} \tag{4.5.6}$$

where we used the definition of P^h .

Combine (4.5.5), (4.5.6) with (4.5.4) to obtain

$$\begin{aligned}
&\frac{1}{\tau} (\hat{e}^{n+1} - \hat{e}^n, \hat{e}^{n+1}) + \frac{1}{\tau} (\nabla e^{n+1}, \nabla e^{n+1} - \nabla e^n) + \frac{2}{\tau} (b^{n+1}, b^{n+1} - b^n)_1 \\
&\quad + \|\nabla \hat{e}^{n+1}\|_2^2 + \|b^{n+1}\|_{\Gamma_1}^2 + \|\hat{b}^{n+1}\|_{\Gamma_1}^2 + (h(u^{n+1}) - h(u(t_{n+1})), \hat{e}^{n+1}) \\
&= \left(\hat{u}_t(t_{n+1}) - \frac{1}{\tau} (P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)), \hat{e}^{n+1} \right) + (\hat{\rho}^{n+1}, b^{n+1} + \hat{b}^{n+1})_1 \\
&\quad + \left(\nabla \hat{u}(t_{n+1}) - \frac{1}{\tau} (\nabla u(t_{n+1}) - \nabla u(t_n)), \nabla e^{n+1} \right) - (\sigma^{n+1} + \hat{\sigma}^{n+1}, \hat{e}^{n+1})_1 \\
&\quad - (\sigma^{n+1} + \hat{\sigma}^{n+1}, b^{n+1} + \hat{b}^{n+1})_1 + \frac{2}{\tau} (b^{n+1}, \theta^{n+1})_1.
\end{aligned} \tag{4.5.7}$$

For the nonlinear term in (4.5.7), taking the inequality (4.4.8) in discrete time, we obtain

$$\begin{aligned}
|(h(u^{n+1}) - h(u(t_{n+1}))), \hat{e}^{n+1}| &\leq \gamma c_3 \|\nabla u^{n+1} - \nabla u(t_{n+1})\|_2 \|\nabla \hat{e}^{n+1}\|_2 \\
&\leq \gamma c_3 \|\nabla e^{n+1} + \nabla \rho^{n+1}\|_2 \|\nabla \hat{e}^{n+1}\|_2 \\
&\leq \gamma c_3 (\|\nabla e^{n+1}\|_2 + \|\nabla \rho^{n+1}\|_2) \|\nabla \hat{e}^{n+1}\|_2,
\end{aligned} \tag{4.5.8}$$

where $\gamma = \alpha^q \|u^{n+1}\|_{H_{\Gamma_0}^1}^q + (1 - \alpha)^q \|u(t_{n+1})\|_{H_{\Gamma_0}^1}^q$ and $c_3 = C_\Omega C_2 C_3^q \sqrt{1 + C_\Omega^2}$.

From (4.5.7) and (4.5.8), we obtain that

$$\begin{aligned}
&\frac{1}{\tau} (\hat{e}^{n+1} - \hat{e}^n, \hat{e}^{n+1}) + \frac{1}{\tau} (\nabla e^{n+1}, \nabla e^{n+1} - \nabla e^n) + \frac{2}{\tau} (b^{n+1}, b^{n+1} - b^n)_1 \\
&\quad + \|\nabla \hat{e}^{n+1}\|_2^2 + \|b^{n+1}\|_{\Gamma_1}^2 + \|\hat{b}^{n+1}\|_{\Gamma_1}^2 \\
&\leq \left(\hat{u}_t(t_{n+1}) - \frac{1}{\tau} (P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)), \hat{e}^{n+1} \right) + (\hat{\rho}^{n+1}, b^{n+1} + \hat{b}^{n+1})_1 \\
&\quad + \left(\nabla \hat{u}(t_{n+1}) - \frac{1}{\tau} (\nabla u(t_{n+1}) - \nabla u(t_n)), \nabla e^{n+1} \right) - (\sigma^{n+1} + \hat{\sigma}^{n+1}, \hat{e}^{n+1})_1 \\
&\quad - (\sigma^{n+1} + \hat{\sigma}^{n+1}, b^{n+1} + \hat{b}^{n+1})_1 + \frac{2}{\tau} (b^{n+1}, \theta^{n+1})_1 \\
&\quad + \gamma c_3 (\|\nabla e^{n+1}\|_2 + \|\nabla \rho^{n+1}\|_2) \|\nabla \hat{e}^{n+1}\|_2.
\end{aligned}$$

Using Young's inequality, there exist $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8 > 0$ such that

$$\begin{aligned}
& \frac{1}{\tau} (\hat{e}^{n+1} - \hat{e}^n, \hat{e}^{n+1}) + \frac{1}{\tau} (\nabla e^{n+1}, \nabla e^{n+1} - \nabla e^n) + \frac{2}{\tau} (b^{n+1}, b^{n+1} - b^n)_1 \\
& \quad + \|\nabla \hat{e}^{n+1}\|_2^2 + \|b^{n+1}\|_{\Gamma_1}^2 + \|\hat{b}^{n+1}\|_{\Gamma_1}^2 \\
& \leq C \left(I_1^2 + \left\| \hat{u}_t(t_{n+1}) - \frac{1}{\tau} (P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)) \right\|_2^2 + \|\sigma^{n+1}\|_{\Gamma_1}^2 + \|\hat{\sigma}^{n+1}\|_{\Gamma_1}^2 \right) \\
& \quad + C \left(\|\hat{\rho}^{n+1}\|_{\Gamma_1}^2 + \|\nabla \rho^{n+1}\|_2^2 \right) + \gamma c_3 \left(\delta_1 \|\nabla e^{n+1}\|_2^2 + \frac{1}{4\delta_1} \|\nabla \hat{e}^{n+1}\|_2^2 + \delta_2 \|\nabla \hat{e}^{n+1}\|_2^2 \right) \\
& \quad + \delta_3 \|\hat{e}^{n+1}\|_2^2 + \delta_4 \|\nabla e^{n+1}\|_2^2 + \delta_5 \|\hat{e}^{n+1}\|_{\Gamma_1}^2 + \delta_6 \|b^{n+1}\|_{\Gamma_1}^2 + \delta_7 \|\hat{b}^{n+1}\|_{\Gamma_1}^2 \\
& \quad + \frac{2}{\tau} \left(\delta_8 \|b^{n+1}\|_{\Gamma_1}^2 + \frac{1}{4\delta_8} \|\theta^{n+1}\|_{\Gamma_1}^2 \right) \\
& \leq C \left(I_1^2 + I_2^2 + I_3^2 + \|\sigma^{n+1}\|_{\Gamma_1}^2 + \|\hat{\sigma}^{n+1}\|_{\Gamma_1}^2 + \|\hat{\rho}^{n+1}\|_{\Gamma_1}^2 + \|\nabla \rho^{n+1}\|_2^2 \right) + \delta_3 \|\hat{e}^{n+1}\|_2^2 \\
& \quad + (\gamma c_3 \delta_1 + \delta_4) \|\nabla e^{n+1}\|_2^2 + \left(\delta_2 + \frac{1}{4\delta_1} \right) \gamma c_3 \|\nabla \hat{e}^{n+1}\|_2^2 + \delta_5 \|\hat{e}^{n+1}\|_{\Gamma_1}^2 + \delta_7 \|\hat{b}^{n+1}\|_{\Gamma_1}^2 \\
& \quad + \left(\delta_6 + \frac{2\delta_8}{\tau} \right) \|b^{n+1}\|_{\Gamma_1}^2 + \frac{1}{2\delta_8 \tau} \|\theta^{n+1}\|_{\Gamma_1}^2,
\end{aligned}$$

where

$$\begin{cases} I_1 = \left\| \nabla \hat{u}(t_{n+1}) - \frac{1}{\tau} (\nabla u(t_{n+1}) - \nabla u(t_n)) \right\|_2, \\ I_2 = \left\| \hat{u}_t(t_{n+1}) - \frac{1}{\tau} (\hat{u}(t_{n+1}) - \hat{u}(t_n)) \right\|_2, \\ I_3 = \left\| \frac{1}{\tau} (\hat{u}(t_{n+1}) - \hat{u}(t_n)) - \frac{1}{\tau} (P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)) \right\|_2. \end{cases}$$

Using the embedding $H_{\Gamma_0}^1(\Omega) \hookrightarrow L^2(\Gamma_1)$, together with the Poincaré inequality, we find that

$$\begin{aligned}
& \frac{1}{\tau} (\hat{e}^{n+1} - \hat{e}^n, \hat{e}^{n+1}) + \frac{1}{\tau} (\nabla e^{n+1}, \nabla e^{n+1} - \nabla e^n) + \frac{2}{\tau} (b^{n+1}, b^{n+1} - b^n)_1 \\
& \quad + \|\nabla \hat{e}^{n+1}\|_2^2 + \|b^{n+1}\|_{\Gamma_1}^2 + \|\hat{b}^{n+1}\|_{\Gamma_1}^2 \\
& \leq C \left(I_1^2 + I_2^2 + I_3^2 + \|\sigma^{n+1}\|_{\Gamma_1}^2 + \|\hat{\sigma}^{n+1}\|_{\Gamma_1}^2 + \|\hat{\rho}^{n+1}\|_{\Gamma_1}^2 + \|\nabla \rho^{n+1}\|_2^2 \right) + \delta_3 \|\hat{e}^{n+1}\|_2^2 \\
& \quad + (\gamma c_3 \delta_1 + \delta_4) \|\nabla e^{n+1}\|_2^2 + \left[\left(\delta_2 + \frac{1}{4\delta_1} \right) \gamma c_3 + \delta_5 C_4^2 (1 + C_\Omega^2) \right] \|\nabla \hat{e}^{n+1}\|_2^2 \\
& \quad + \left(\delta_6 + \frac{2\delta_8}{\tau} \right) \|b^{n+1}\|_{\Gamma_1}^2 + \delta_7 \|\hat{b}^{n+1}\|_{\Gamma_1}^2 + \frac{1}{2\delta_8 \tau} \|\theta^{n+1}\|_{\Gamma_1}^2.
\end{aligned}$$

By choosing $\delta_1 = \frac{3}{4} \gamma c_3, \delta_2 = \frac{1}{3 \gamma c_3}, \delta_3 = \delta_4 = \frac{1}{2}, \delta_5 = \frac{1}{3 C_4^2 (1 + C_\Omega^2)}, \delta_6 = \delta_7 = 1$, and

$\delta_8 = \frac{\tau}{4}$, we get

$$\begin{aligned} & \frac{1}{\tau} (\hat{e}^{n+1} - \hat{e}^n, \hat{e}^{n+1}) + \frac{1}{\tau} (\nabla e^{n+1}, \nabla e^{n+1} - \nabla e^n) + \frac{2}{\tau} (b^{n+1}, b^{n+1} - b^n)_1 \\ & \leq C \left(I_1^2 + I_2^2 + I_3^2 + \|\sigma^{n+1}\|_{\Gamma_1}^2 + \|\hat{\sigma}^{n+1}\|_{\Gamma_1}^2 + \|\hat{\rho}^{n+1}\|_{\Gamma_1}^2 + \|\nabla \rho^{n+1}\|_2^2 \right) \\ & \quad + \frac{1}{2} \|\hat{e}^{n+1}\|_2^2 + \left(\frac{3}{4} (\gamma c_3)^2 + \frac{1}{2} \right) \|\nabla e^{n+1}\|_2^2 + \frac{1}{2} \|b^{n+1}\|_{\Gamma_1}^2 + \frac{2}{\tau^2} \|\theta^{n+1}\|_{\Gamma_1}^2. \end{aligned} \quad (4.5.9)$$

Next, we estimate I_1, I_2 , and I_3 . By applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} I_1 &= \left\| \nabla \hat{u}(t_{n+1}) - \frac{1}{\tau} (\nabla u(t_{n+1}) - \nabla u(t_n)) \right\|_2 = \left\| \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \nabla \hat{u}_t(s) ds \right\|_2 \\ &\leq \frac{1}{\tau} \left(\int_{t_n}^{t_{n+1}} (s - t_n)^2 ds \right)^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \|\nabla \hat{u}_t(s)\|_2^2 ds \right)^{\frac{1}{2}} \\ &\leq \left(\frac{\tau}{3} \right)^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \|\nabla \hat{u}_t(s)\|_2^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Similarly, for I_2 and I_3 , we have

$$\begin{aligned} I_2 &= \left\| \hat{u}_t(t_{n+1}) - \frac{1}{\tau} (\hat{u}(t_{n+1}) - \hat{u}(t_n)) \right\|_2 = \left\| \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \hat{u}_{tt}(s) ds \right\|_2 \\ &\leq \frac{1}{\tau} \left(\int_{t_n}^{t_{n+1}} (s - t_n)^2 ds \right)^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \|\hat{u}_{tt}(s)\|_2^2 ds \right)^{\frac{1}{2}} \\ &\leq \left(\frac{\tau}{3} \right)^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \|\hat{u}_{tt}(s)\|_2^2 ds \right)^{\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} I_3 &= \left\| \frac{1}{\tau} (\hat{u}(t_{n+1}) - \hat{u}(t_n)) - \frac{1}{\tau} (P^h \hat{u}(t_{n+1}) - P^h \hat{u}(t_n)) \right\|_2 = \left\| \frac{1}{\tau} (\hat{\rho}(t_{n+1}) - \hat{\rho}(t_n)) \right\|_2 \\ &= \left\| \hat{\rho}_t(t_{n+1}) - \frac{1}{\tau} (\hat{\rho}(t_{n+1}) - \hat{\rho}(t_n)) - \hat{\rho}_t(t_{n+1}) \right\|_2 \\ &\leq \left\| \hat{\rho}_t(t_{n+1}) - \frac{1}{\tau} (\hat{\rho}(t_{n+1}) - \hat{\rho}(t_n)) \right\|_2 + \|\hat{\rho}_t(t_{n+1})\|_2 \\ &\leq \left\| \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (s - t_n) \hat{\rho}_{tt}(s) ds \right\|_2 + \|\hat{\rho}_t(t_{n+1})\|_2 \\ &\leq \left(\frac{\tau}{3} \right)^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \|\hat{\rho}_{tt}(s)\|_2^2 ds \right)^{\frac{1}{2}} + \|\hat{\rho}_t(t_{n+1})\|_2. \end{aligned}$$

Using (4.4.1)₁, we obtain

$$\begin{cases} \|\hat{\rho}_t(s)\|_2 = \|(P^h \hat{u}(s))_t - \hat{u}_t(s)\|_2 & \leq Ch \|\nabla \hat{u}_t(s)\|_2, \\ \|\hat{\rho}_{tt}(s)\|_2 = \|(P^h \hat{u}(s))_{tt} - \hat{u}_{tt}(s)\|_2 & \leq Ch \|\nabla \hat{u}_{tt}(s)\|_2. \end{cases}$$

Hence

$$I_3 \leq Ch\tau^{\frac{1}{2}} \left(\int_{t_n}^{t_{n+1}} \|\nabla \hat{u}_{tt}(s)\|_2^2 ds \right)^{\frac{1}{2}} + Ch \|\nabla u_{tt}(t_{n+1})\|_2.$$

Using the embedding $H_{\Gamma_0}^1(\Omega) \hookrightarrow L^2(\Gamma_1)$ and (4.4.1)₂, we find

$$\begin{cases} \|\hat{\rho}^{n+1}\|_{\Gamma_1}^2 = \|P^h \hat{u}(t_{n+1}) - \hat{u}(t_{n+1})\|_{\Gamma_1}^2 & \leq Ch^2 \|\hat{u}(t_{n+1})\|_*^2, \\ \|\sigma^{n+1}\|_{\Gamma_1}^2 = \|P^h \xi(t_{n+1}) - \xi(t_{n+1})\|_{\Gamma_1}^2 & \leq Ch^2 \|\xi(t_{n+1})\|_*^2, \\ \|\hat{\sigma}^{n+1}\|_{\Gamma_1}^2 = \|P^h \hat{\xi}(t_{n+1}) - \hat{\xi}(t_{n+1})\|_{\Gamma_1}^2 & \leq Ch^2 \|\hat{\xi}(t_{n+1})\|_*^2. \end{cases}$$

Using the definition of P^h , we have

$$\|\nabla \rho^{n+1}\|_2^2 = \|\nabla(P^h u(t_{n+1})) - \nabla u(t_{n+1})\|_2^2 \leq Ch^2 \|\Delta u(t_{n+1})\|_2^2.$$

Using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} \|\theta^{n+1}\|_{\Gamma_1} &= \left\| \int_{t_n}^{t_{n+1}} (s - t_n) P^h \xi_{tt}(s) ds \right\|_{\Gamma_1} \\ &\leq C\tau^{\frac{3}{2}} \left(\int_{t_n}^{t_{n+1}} \|P^h \xi_{tt}(s)\|_{\Gamma_1}^2 ds \right)^{\frac{1}{2}} \\ &\leq C\tau^{\frac{3}{2}} \left(\int_{t_n}^{t_{n+1}} \|P^h \xi_{tt}(s) - \xi_{tt}(s)\|_{\Gamma_1}^2 + \|\xi_{tt}(s)\|_{\Gamma_1}^2 ds \right)^{\frac{1}{2}} \\ &\leq C\tau^{\frac{3}{2}} \left(\int_{t_n}^{t_{n+1}} h^2 \|\xi_{tt}(s)\|_*^2 + \|\xi_{tt}(s)\|_{\Gamma_1}^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Using the elementary equality

$$(a, b) = \frac{1}{2} (\|a\|^2 + \|b\|^2 - \|a - b\|^2),$$

we combine the above estimates with (4.5.9) to obtain

$$\begin{aligned} &\frac{1}{2\tau} \left(\|\hat{e}^{n+1} - \hat{e}^n\|_2^2 + \|\hat{e}^{n+1}\|_2^2 - \|\hat{e}^n\|_2^2 \right) + \frac{1}{2\tau} \left(\|\nabla e^{n+1} - \nabla e^n\|_2^2 + \|\nabla e^{n+1}\|_2^2 - \|\nabla e^n\|_2^2 \right) \\ &\quad + \frac{1}{\tau} \left(\|b^{n+1} - b^n\|_{\Gamma_1}^2 + \|b^{n+1}\|_{\Gamma_1}^2 - \|b^n\|_{\Gamma_1}^2 \right) \\ &\leq C\tau \int_{t_n}^{t_{n+1}} \left(\|\hat{u}_{tt}(s)\|_2^2 + \|\nabla \hat{u}_t(s)\|_2^2 + \|\xi_{tt}(s)\|_{\Gamma_1}^2 + h^2 (\|\nabla \hat{u}_{tt}(s)\|_2^2 + \|\xi_{tt}(s)\|_*^2) \right) ds \\ &\quad + Ch^2 (\|\nabla u_{tt}(t_{n+1})\|_2^2 + \|\Delta u(t_{n+1})\|_2^2 + \|u_t(t_{n+1})\|_*^2 + \|\xi(t_{n+1})\|_*^2 + \|\xi_t(t_{n+1})\|_*^2) \\ &\quad + \frac{1}{2} \|\hat{e}^{n+1}\|_2^2 + \left(\frac{3}{4} (\gamma c_3)^2 + \frac{1}{2} \right) \|\nabla e^{n+1}\|_2^2 + \frac{1}{2} \|b^{n+1}\|_{\Gamma_1}^2. \end{aligned}$$

Multiply the last inequality by 2τ and sum from $i = 1$ to n

$$\begin{aligned}
& \sum_{i=1}^n \left\| \hat{e}^{i+1} - \hat{e}^i \right\|_2^2 + \sum_{i=1}^n \left\| \nabla e^{i+1} - \nabla e^i \right\|_2^2 + 2 \sum_{i=1}^n \left\| b^{i+1} - b^i \right\|_{\Gamma_1}^2 + \left\| \hat{e}^{n+1} \right\|_2^2 \\
& + \left\| \nabla e^{n+1} \right\|_2^2 + 2 \left\| b^{n+1} \right\|_{\Gamma_1}^2 \\
& \leq C\tau^2 \int_{t_1}^{t_{n+1}} \left(\left\| \hat{u}_{tt}(s) \right\|_2^2 + \left\| \nabla \hat{u}_t(s) \right\|_2^2 + \left\| \xi_{tt}(s) \right\|_{\Gamma_1}^2 + h^2 \left\| \nabla \hat{u}_{tt}(s) \right\|_2^2 \right) ds \\
& + C(h\tau)^2 \int_{t_1}^{t_{n+1}} \left\| \xi_{tt}(s) \right\|_*^2 ds + Ch^2\tau \sum_{i=1}^n \left(\left\| \nabla u_{tt}(t_{i+1}) \right\|_2^2 + \left\| \Delta u(t_{i+1}) \right\|_2^2 \right) \\
& + Ch^2\tau \sum_{i=1}^n \left(\left\| u_t(t_{i+1}) \right\|_*^2 + \left\| \xi(t_{i+1}) \right\|_*^2 + \left\| \xi_t(t_{i+1}) \right\|_*^2 \right) + \left\| \hat{e}^1 \right\|_2^2 + \left\| \nabla e^1 \right\|_2^2 \\
& + 2 \left\| b^1 \right\|_{\Gamma_1}^2 + \tau \sum_{i=1}^n \left(\left\| \hat{e}^{i+1} \right\|_2^2 + \left(\frac{3}{2} (\gamma c_3)^2 + 1 \right) \left\| \nabla e^{i+1} \right\|_2^2 + \left\| b^{i+1} \right\|_{\Gamma_1}^2 \right). \tag{4.5.10}
\end{aligned}$$

We have to bound $\left\| \hat{e}^1 \right\|_2$, $\left\| \nabla e^1 \right\|_2$, and $\left\| b^1 \right\|_{\Gamma_1}$. Indeed,

$$\begin{aligned}
(\nabla e^1, \nabla v) &= \left(\nabla u^1 - \nabla(P^h u(t_1)), \nabla v \right) \\
&= \left(\nabla u^0 + \tau \nabla(P^h u_1) - \nabla(P^h u(t_1)), \nabla v \right) \\
&= \left(\nabla(P^h u_0) + \tau \nabla(P^h u_1) - \nabla(P^h u(t_1)), \nabla v \right) \\
&= (\nabla u_0 + \tau \nabla u_1 - \nabla u(t_1), \nabla v) \\
&= (\nabla u(0) + \tau \nabla u_t(0) - \nabla u(t_1), \nabla v),
\end{aligned}$$

where we used the definition of P^h , u^1 , and u^0 .

Knowing that,

$$\nabla u(t_1) = \nabla u(0) + \tau \nabla u_t(0) + \frac{\tau^2}{2} \nabla u_{tt}(\tau_1), \quad \text{with } 0 \leq \tau_1 \leq t_1,$$

hence

$$(\nabla e^1, \nabla v) = -\frac{\tau^2}{2} (\nabla u_{tt}(\tau_1), \nabla v).$$

Choosing $v = e^1$, we obtain

$$\left\| \nabla e^1 \right\|_2^2 \leq \frac{\tau^2}{2} \left\| \nabla u_{tt}(\tau_1) \right\|_2 \left\| \nabla e^1 \right\|_2, \quad \text{that is, } \left\| \nabla e^1 \right\|_2 \leq C\tau.$$

To bound $\left\| \nabla \hat{e}^1 \right\|_2$, we use $\hat{u}^1 = \frac{1}{\tau} (u^1 - u^0)$ and $u^1 = u^0 + \tau P^h u_1$. Thus,

$$\begin{aligned}
(\nabla \hat{e}^1, \nabla v) &= \left(\nabla \hat{u}^1 - \nabla(P^h \hat{u}(t_1)), \nabla v \right) \\
&= \left(\nabla(P^h u_1) - \nabla(P^h \hat{u}(t_1)), \nabla v \right) \\
&= \left(\nabla(P^h u_1), \nabla v \right) - \left(\nabla(P^h \hat{u}(t_1)), \nabla v \right)
\end{aligned}$$

$$\begin{aligned}
&= (\nabla u_1, \nabla v) - (\nabla \hat{u}(t_1), \nabla v) \\
&= (\nabla u_t(0) - \nabla u_t(t_1), \nabla v).
\end{aligned}$$

Knowing that

$$\nabla u_t(t_1) = \nabla u_t(0) + \tau \nabla u_{tt}(\tau_2), \quad \text{with } 0 \leq \tau_2 \leq t_1,$$

hence

$$(\nabla \hat{e}^1, \nabla v) = -\tau (\nabla u_{tt}(\tau_2), \nabla v).$$

Choosing $v = \hat{e}^1$, we obtain

$$\|\nabla \hat{e}^1\|_2^2 \leq C\tau \|\nabla u_{tt}(\tau_2)\|_2 \|\nabla \hat{e}^1\|_2, \quad \text{that is, } \|\nabla \hat{e}^1\|_2 \leq C\tau.$$

Using Poincaré's inequality, we find

$$\|\hat{e}^1\|_2 \leq C \|\nabla \hat{e}^1\|_2 \leq C\tau.$$

To bound $\|b^1\|_{\Gamma_1}$, we use the fact

$$\xi(t_1) = \xi_0 + \tau \xi_t(\tau_3) \quad \text{with } 0 \leq \tau_3 \leq t_1, \quad \text{and } \xi^1 = \xi^0 + \tau \hat{\xi}^1,$$

then

$$\|b^1\|_{\Gamma_1} = \|\xi^1 - P^h \xi(t_1)\|_{\Gamma_1} = \tau \|\hat{\xi}^1 - P^h \xi_t(\tau_3)\|_{\Gamma_1} \leq C\tau.$$

Using (4.5.10) together with the above estimates, we obtain

$$\begin{aligned}
&\|\hat{e}^{n+1}\|_2^2 + \|\nabla e^{n+1}\|_2^2 + \|b^{n+1}\|_{\Gamma_1}^2 \\
&\leq C(\tau^2 + h^2) + k\tau \sum_{i=1}^n \left(\|\hat{e}^{i+1}\|_2^2 + \|\nabla e^{i+1}\|_2^2 + \|b^{i+1}\|_{\Gamma_1}^2 \right),
\end{aligned} \tag{4.5.11}$$

where $k = \frac{3}{2}(\gamma c_3)^2 + 1$.

The inequality (4.5.11) leads to

$$\begin{aligned}
&(1 - k\tau) \left(\|\hat{e}^{n+1}\|_2^2 + \|\nabla e^{n+1}\|_2^2 + \|b^{n+1}\|_{\Gamma_1}^2 \right) \\
&\leq C(\tau^2 + h^2) + k\tau \sum_{i=2}^n \left(\|\hat{e}^i\|_2^2 + \|\nabla e^i\|_2^2 + \|b^i\|_{\Gamma_1}^2 \right).
\end{aligned} \tag{4.5.12}$$

For sufficiently small τ , satisfying $1 - k\tau > 0$, we can rewrite (4.5.12) as follows:

$$\begin{aligned}
&\|\hat{e}^{n+1}\|_2^2 + \|\nabla e^{n+1}\|_2^2 + \|b^{n+1}\|_{\Gamma_1}^2 \\
&\leq C(\tau^2 + h^2) + \frac{k\tau}{1 - k\tau} \sum_{i=2}^n \left(\|\hat{e}^i\|_2^2 + \|\nabla e^i\|_2^2 + \|b^i\|_{\Gamma_1}^2 \right).
\end{aligned} \tag{4.5.13}$$

From (4.5.13), applying the discrete Gronwall inequality yields

$$\|\hat{e}^{n+1}\|_2^2 + \|\nabla e^{n+1}\|_2^2 + \|b^{n+1}\|_{\Gamma_1}^2 \leq C(h^2 + \tau^2).$$

Now, it follows that

$$\begin{aligned}
& \|u_t(t_n) - \hat{u}^n\|_2^2 + \|\nabla u(t_n) - \nabla u^n\|_2^2 + \|\xi(t_n) - \xi^n\|_{\Gamma_1}^2 \\
&= \|\hat{e}^n + \hat{\rho}^n\|_2^2 + \|\nabla e^n + \nabla \rho^n\|_2^2 + \|b^n + \sigma^n\|_{\Gamma_1}^2 \\
&\leq C \left(\|\hat{e}^n\|_2^2 + \|\nabla e^n\|_2^2 + \|b^n\|_{\Gamma_1}^2 + \|\hat{\rho}^n\|_2^2 + \|\nabla \rho^n\|_2^2 + \|\sigma^n\|_{\Gamma_1}^2 \right) \\
&\leq C \left(h^2 + \tau^2 + \|\hat{\rho}^n\|_2^2 + \|\nabla \rho^n\|_2^2 + \|\sigma^n\|_{\Gamma_1}^2 \right) \\
&\leq C (h^2 + \tau^2).
\end{aligned}$$

which completes the proof of the Theorem. $\square$

4.6 NUMERICAL RESULTS

We consider the following problem:

$$\begin{cases} u_{tt}(x, t) - \Delta u(x, t) - \Delta u_t(x, t) + |u(x, t)|^{\rho-1} u(x, t) = f(x, t), & \text{in } Q, \\ u(x, t) = 0, & \text{on } \Sigma_0, \\ \frac{\partial u}{\partial n}(x, t) + \xi(x, t) = 0, & \text{on } \Sigma_1, \\ \xi_t(x, t) + \xi(x, t) = u_t(x, t) + g(x, t), & \text{on } \Sigma_1, \end{cases} \quad (4.6.1)$$

with the following initial conditions:

$$\begin{cases} u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{for } x \in \Omega, \\ \xi(x, 0) = \xi_0(x), & \text{for } x \in \Gamma_1, \end{cases} \quad (4.6.2)$$

where f and g are sufficiently regular external forces, introduced to validate the problem numerically.

Let V_{m_μ} be the subspace spanned by $\{w_1, w_2, \dots, w_{m_\mu}\}$, where $\{w_i\}_{i=1}^\infty$ is a smooth orthogonal basis for V , and let Z_{m_ν} be the subspace spanned by $\{z_1, z_2, \dots, z_{m_\nu}\}$, where $\{z_i\}_{i=1}^\infty$ is a smooth orthogonal basis for $L^2(\Gamma_1)$.

We look for the approximate solution

$$u_m(x, t) = \sum_{i=1}^{m_\mu} d_{im}(t) w_i(x), \quad \xi_m(x, t) = \sum_{i=1}^{m_\nu} s_{im}(t) z_i(x, y_b), \quad (4.6.3)$$

satisfying $\forall v \in V_{m_\mu}, \forall \psi \in Z_{m_\nu}$,

$$\begin{cases} (u_{tt}^m, v) + (\nabla u^m, \nabla v) + (\nabla u_t^m, \nabla v) + (|u^m|^q u^m, v) \\ \quad + (\xi^m + \xi_t^m, v)_1 = (f, v), \\ (\xi_t^m, \psi)_1 + (\xi^m, \psi)_1 - (u_t^m, \psi)_1 = (g, \psi)_1. \end{cases} \quad (4.6.4)$$

Substituting (4.6.3) in (4.6.4) and taking $v = w_j, \psi = z_j$, we obtain

$$\left\{ \begin{array}{l} \sum_{i=1}^{m_\mu} [d''_{im}(t)(w_i, w_j) + (d'_{im}(t) + d_{im}(t))(\nabla w_i, \nabla w_j)] \\ \quad + \sum_{i=1}^{m_\nu} [(s'_{im}(t) + s_{im}(t))(z_i, w_j)_1] + \left(\left| \sum_{i=1}^{m_\mu} d_{im}(t) w_i \right|^q \sum_{i=1}^{m_\mu} d_{im}(t) w_i, w_j \right) \\ = (f(t), w_j), \quad \text{for } j = 1, 2, \dots, m_\mu, \\ \sum_{i=1}^{m_\nu} [(s'_{im}(t) + s_{im}(t))(z_i, z_j)_1 - d'_{im}(t)(w_i, z_j)_1] = (g(t), z_j)_1, \\ \quad \text{for } j = 1, 2, \dots, m_\nu. \end{array} \right. \quad (4.6.5)$$

Defining the matrices and vectors of (4.6.5) as follows:

$$\begin{aligned} A_{ij} &= (w_i, w_j), & B_{ij} &= (\nabla w_i, \nabla w_j), & L_{1ij} &= (z_i, w_j)_1, & L_{2ij} &= (z_i, z_j)_1, \\ L_{3ij} &= (w_i, z_j)_1, & F_j(t) &= (f(t), w_j), & G_j(t) &= (g(t), z_j)_1, \end{aligned}$$

$$Q_j(d(t)) = \left(\left| \sum_{i=1}^{m_\mu} d_{im}(t) w_i \right|^q \sum_{i=1}^{m_\mu} d_{im}(t) w_i, w_j \right),$$

where $Q_j(d(t))$ is the vector that stores the nonlinearity.

We can rewrite (4.6.5) in the following system of ordinary differential equations:

$$\left\{ \begin{array}{l} Ad''(t) + Bd'(t) + Bd(t) + Q(d(t)) + L_1^T s'(t) + L_1^T s(t) = F(t), \\ L_2 s'(t) + L_2 s(t) - L_3^T d'(t) = G(t), \end{array} \right. \quad (4.6.6)$$

for all $t \in [0, T]$.

4.6.1 Fully Discrete approximation

In this subsection, we introduce and analyze a fully discrete finite element method for (4.6.6).

Consider the discretization of the fixed time interval $[0, T]$ using a uniform time step of size $\tau = \frac{T}{N}$. Then, $t_n = n\tau$ for $n = 0, 1, 2, \dots, N$, with $d^n = d(t_n)$ and $s^n = s(t_n)$. The first and second time derivatives will be approximated by forward-difference and central-difference schemes, respectively, given by

$$d'^n = \frac{1}{\tau} (d^{n+1} - d^n), \quad d''^n = \frac{1}{\tau^2} (d^{n+1} - 2d^n + d^{n-1}).$$

Also, we define the approximation of s'^n as follows:

$$s'^n = \frac{1}{\tau} (s^{n+1} - s^n).$$

By taking $t = t_n$ in (4.6.6) and substituting the approximations, we obtain the following iterative method:

$$\begin{cases} R_1 d^{n+1} + R_2 d^n + A d^{n-1} + \tau^2 Q(d^{n+1}) + R_3 s^{n+1} + R_4 s^n = \tau^2 F^{n+1}, \\ R_5 s^{n+1} - L_2 s^n - L_3^T d^{n+1} + L_3^T d^n = \tau G^{n+1}, \end{cases} \quad (4.6.7)$$

where $d^0 = P^h u_0$, $d^1 = d^0 + \tau P^h u_1$, $s^0 = P^h \xi_0$, and the matrices defined by

$$\begin{aligned} R_1 &= A + (\tau + \tau^2) B, & R_2 &= -2A - \tau B, & R_3 &= (\tau + \tau^2) L_1^T, \\ R_4 &= -\tau L_1^T, & R_5 &= (1 + \tau) L_2. \end{aligned}$$

Thus, multiplying the second equation of (4.6.7) by R_5^{-1} , we get

$$s^{n+1} = R_5^{-1} L_2 s^n + R_5^{-1} L_3^T d^{n+1} - R_5^{-1} L_3^T d^n + \tau R_5^{-1} G^{n+1}. \quad (4.6.8)$$

Substituting (4.6.8) in the first equation of (4.6.7), we obtain

$$\hat{R}_1 d^{n+1} = -\hat{R}_2 d^n - A d^{n-1} - \tau^2 Q(d^{n+1}) - \hat{R}_3 s^n - \hat{R}_4 G^{n+1} + \tau^2 F^{n+1}, \quad (4.6.9)$$

where

$$\hat{R}_1 = R_1 + R_3 R_5^{-1} L_3^T, \quad \hat{R}_2 = R_2 - R_3 R_5^{-1} L_3^T, \quad \hat{R}_3 = R_3 R_5^{-1} L_2 + R_4, \quad \hat{R}_4 = \tau R_3 R_5^{-1}.$$

Multiplying (4.6.9) by $\hat{R}_1^{-1}$, we get

$$\begin{aligned} d^{n+1} + \tau^2 \hat{R}_1^{-1} Q(d^{n+1}) &= -\hat{R}_1^{-1} \hat{R}_2 d^n - \hat{R}_1^{-1} A d^{n-1} - \hat{R}_1^{-1} \hat{R}_3 s^n - \hat{R}_1^{-1} \hat{R}_4 G^{n+1} \\ &\quad + \tau^2 \hat{R}_1^{-1} F^{n+1}. \end{aligned} \quad (4.6.10)$$

Using the initial conditions $d^0 = u(x, 0)$ and $s^0 = \xi(x, 0)$, along with the Taylor approximations, we obtain

$$d^1 = d^0 + \tau u_t(x, 0) + \tau^2 u_{tt}(x, 0), \quad s^1 = s^0 + \tau \xi_t(x, 0),$$

where $u(x, 0)$, $u_t(x, 0)$, and $\xi(x, 0)$ are given by the initial conditions, while $u_{tt}(x, 0)$ and $\xi_t(x, 0)$ are obtained by setting $t = 0$ in (4.6.1)₁ and (4.6.1)₄, respectively.

By taking $n = 1$ in (4.6.10), we obtain

$$d^2 + \tau^2 \hat{R}_1^{-1} Q(d^2) = -\hat{R}_1^{-1} \hat{R}_2 d^1 - \hat{R}_1^{-1} A d^0 - \hat{R}_1^{-1} \hat{R}_3 s^1 - \hat{R}_1^{-1} \hat{R}_4 G^2 + \tau^2 \hat{R}_1^{-1} F^2. \quad (4.6.11)$$

By applying the Modified Newton's Method (see [7, 15]), we can solve the nonlinear equation (4.6.11) to determine d^2 . Next, by taking $n = 1$ in (4.6.8), we obtain

$$s^2 = R_5^{-1} L_2 s^1 + R_5^{-1} L_3^T d^2 - R_5^{-1} L_3^T d^1 + \tau R_5^{-1} G^2.$$

Now, using the values of d^2 and s^2 along with equations (4.6.8) and (4.6.10), and applying the modified Newton's method, we calculate the subsequent iterations for $n = 2, 3, \dots, N$.

4.6.2 Numerical simulations

In this subsection, we consider the temporal interval $[0, 1]$ and the spatial domain $\Omega = (0, 1) \times (0, 1)$, with the following boundary intervals:

$$\begin{cases} I_1 = \{(x, 0) \in \Gamma : 0 \leq x \leq 1\}, \\ I_2 = \{(x, 1) \in \Gamma : 0 \leq x \leq 1\}, \\ I_3 = \{(0, y) \in \Gamma : 0 \leq y \leq 1\}, \\ I_4 = \{(1, y) \in \Gamma : 0 \leq y \leq 1\}, \end{cases}$$

where $\Gamma_0 = I_2 \cup I_3 \cup I_4$ and $\Gamma_1 = I_1$.

In addition, we define the initial conditions as follows:

$$\begin{cases} u_0(x, y) = \sin(\pi x)(y-1)^3, & u_1(x, y) = -\sin(\pi x)(y-1)^3, & (x, y) \in \Omega, \\ \xi_0(x) = 3\sin(\pi x), & & x \in [0, 1]. \end{cases}$$

Inhomogeneous problem

In this case, choosing conveniently the external forces $f(x, y, t)$ and $g(x, t)$, the exact solutions are known and given by

$$u(x, y, t) = \sin(\pi x)(y-1)^3 \exp(-t) \quad \text{and} \quad \xi(x, t) = 3\sin(\pi x) \exp(-t).$$

Tables 4.1, 4.2 and 4.3 show the numerical errors and convergence orders of the numerical method associated with each of the variables in the $L^\infty(0, T, L^2(\Omega))$ and $L^\infty(0, T, L^2(\Gamma_1))$ norms, respectively, taking $\tau = \Delta x = \Delta y$ for different meshes. Note that the obtained convergence order $p \approx 1$ agrees with the error estimate in Theorem 4.4.1.

ρ	$\Delta x = \Delta y = \tau$	E_u	E_ξ	p_u	p_ξ
2	2^{-3}	0.006121	0.046111	—	—
	2^{-4}	0.003123	0.025440	0.970835	0.858012
	2^{-5}	0.001566	0.013342	0.995848	0.931124
	2^{-6}	0.000782	0.006830	1.001844	0.966017

Table 4.1 – Numerical errors and convergence orders - Inhomogeneous problem: $\rho = 2$ and $\Delta x = \Delta y = \tau$

ρ	$\Delta x = \Delta y = \tau$	E_u	E_ξ	p_u	p_ξ
3	2^{-3}	0.006196	0.046012	—	—
	2^{-4}	0.003152	0.025401	0.975070	0.857125
	2^{-5}	0.001578	0.013325	0.998170	0.930750
	2^{-6}	0.000788	0.006822	1.001830	0.965869

Table 4.2 – Numerical errors and convergence orders - Inhomogeneous problem: $\rho = 3$ and $\Delta x = \Delta y = \tau$

ρ	$\Delta x = \Delta y = \tau$	E_u	E_ξ	p_u	p_ξ
4	2^{-3}	0.006231	0.045965	—	—
	2^{-4}	0.003166	0.025381	0.976802	0.856787
	2^{-5}	0.001585	0.013317	0.998178	0.930480
	2^{-6}	0.000791	0.006818	1.002733	0.965849

Table 4.3 – Numerical errors and convergence orders - Inhomogeneous problem: $\rho = 4$ and $\Delta x = \Delta y = \tau$

Tables 4.4, 4.5 and 4.6 show the numerical errors and convergence orders of the numerical method associated with each of the variables in the $L^\infty(0, T, L^2(\Omega))$ and $L^\infty(0, T, L^2(\Gamma_1))$ norms, respectively, taking $\tau = \Delta x^2 = \Delta y^2$ for different meshes. Note that the obtained convergence order $p \approx 2$ agrees with the error estimate in Theorem 4.4.1.

ρ	τ	$\Delta x = \Delta y$	E_u	E_ξ	p_u	p_ξ
2	2^{-4}	2^{-2}	0.003537	0.026046	—	—
	2^{-6}	2^{-3}	0.000827	0.006976	2.096567	1.900590
	2^{-8}	2^{-4}	0.000193	0.001774	2.099286	1.975394
	2^{-10}	2^{-5}	0.000046	0.000445	2.068895	1.995129

Table 4.4 – Numerical errors and convergence orders - Inhomogeneous problem: $\rho = 2$ and $\tau = \Delta x^2 = \Delta y^2$

ρ	τ	$\Delta x = \Delta y$	E_u	E_ξ	p_u	p_ξ
3	2^{-4}	2^{-2}	0.003782	0.025778	—	—
	2^{-6}	2^{-3}	0.000883	0.006903	2.098664	1.900845
	2^{-8}	2^{-4}	0.000206	0.001755	2.099769	1.975752
	2^{-10}	2^{-5}	0.000049	0.000441	2.071791	1.992620

Table 4.5 – Numerical errors and convergence orders - Inhomogeneous problem: $\rho = 3$ and $\tau = \Delta x^2 = \Delta y^2$

ρ	τ	$\Delta x = \Delta y$	E_u	E_ξ	p_u	p_ξ
4	2^{-4}	2^{-2}	0.003899	0.025653	—	—
	2^{-6}	2^{-3}	0.000909	0.006868	2.100752	1.901166
	2^{-8}	2^{-4}	0.000212	0.001746	2.100216	1.975836
	2^{-10}	2^{-5}	0.000051	0.000438	2.055495	1.995051

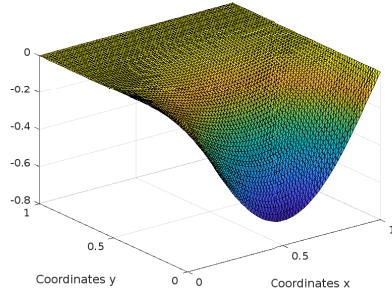
Table 4.6 – Numerical errors and convergence orders - Inhomogeneous problem: $\rho = 4$ and $\tau = \Delta x^2 = \Delta y^2$

Homogeneous problem

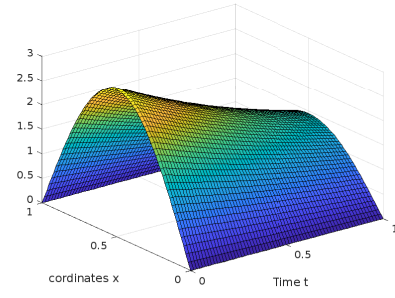
For $f(x, y, t) = 0$ and $g(x, t) = 0$, the exact solutions are unknown. Therefore, we will limit ourselves to presenting a graphical representation of the approximate

solutions.

The Fig. 4.1 shows the graphs of the approximate solutions for the variables $u^m(x, y, 0.5)$ and $\xi^m(x, t)$, where the mesh sizes are $\Delta x = \Delta y = \tau = 1/64$ and $\rho = 2$.



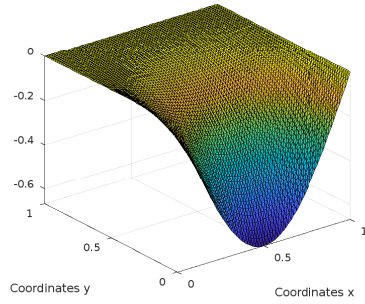
(a) $u^m(x, y, 0.5)$.



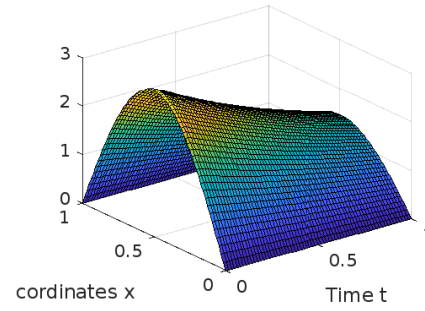
(b) $\xi^m(x, t)$.

Figure 4.1 – Graphs for homogeneous problem: $\rho = 2$ and $\Delta x = \Delta y = \tau = 1/64$.

The Fig. 4.2 shows the graphs of the approximate solutions for the variables $u^m(x, y, 0.5)$ and $\xi^m(x, t)$, where the mesh sizes are $\Delta x = \Delta y = \tau = 1/64$ and $\rho = 3$.



(a) $u^m(x, y, 0.5)$.



(b) $\xi^m(x, t)$.

Figure 4.2 – Graphs for homogeneous problem: $\rho = 3$ and $\Delta x = \Delta y = \tau = 1/64$.

Conclusion

In conclusion, this thesis has established a comprehensive theoretical and numerical framework for nonlinear wave equations with viscoelastic effects under various boundary conditions. The global existence and uniqueness of weak solutions have been rigorously proven, and the long-term behavior of solutions has been analyzed, showing exponential energy decay under appropriate assumptions on the nonlinear and viscoelastic terms.

From a numerical perspective, finite element methods have been employed for spatial discretization, resulting in both semi-discrete and fully discrete schemes. The numerical results demonstrate that these schemes are highly effective and efficient. Nonetheless, the methods have been validated on representative test cases, and their application to more complex or higher-dimensional systems may require further adaptations or increased computational resources.

Overall, this work provides a solid foundation for future extensions to more intricate models, other types of damping, or alternative boundary conditions. It highlights the potential of combining rigorous theoretical analysis with robust numerical schemes to address challenging problems in nonlinear partial differential equations.

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