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Variational Problem and Maximal Monotone Operators

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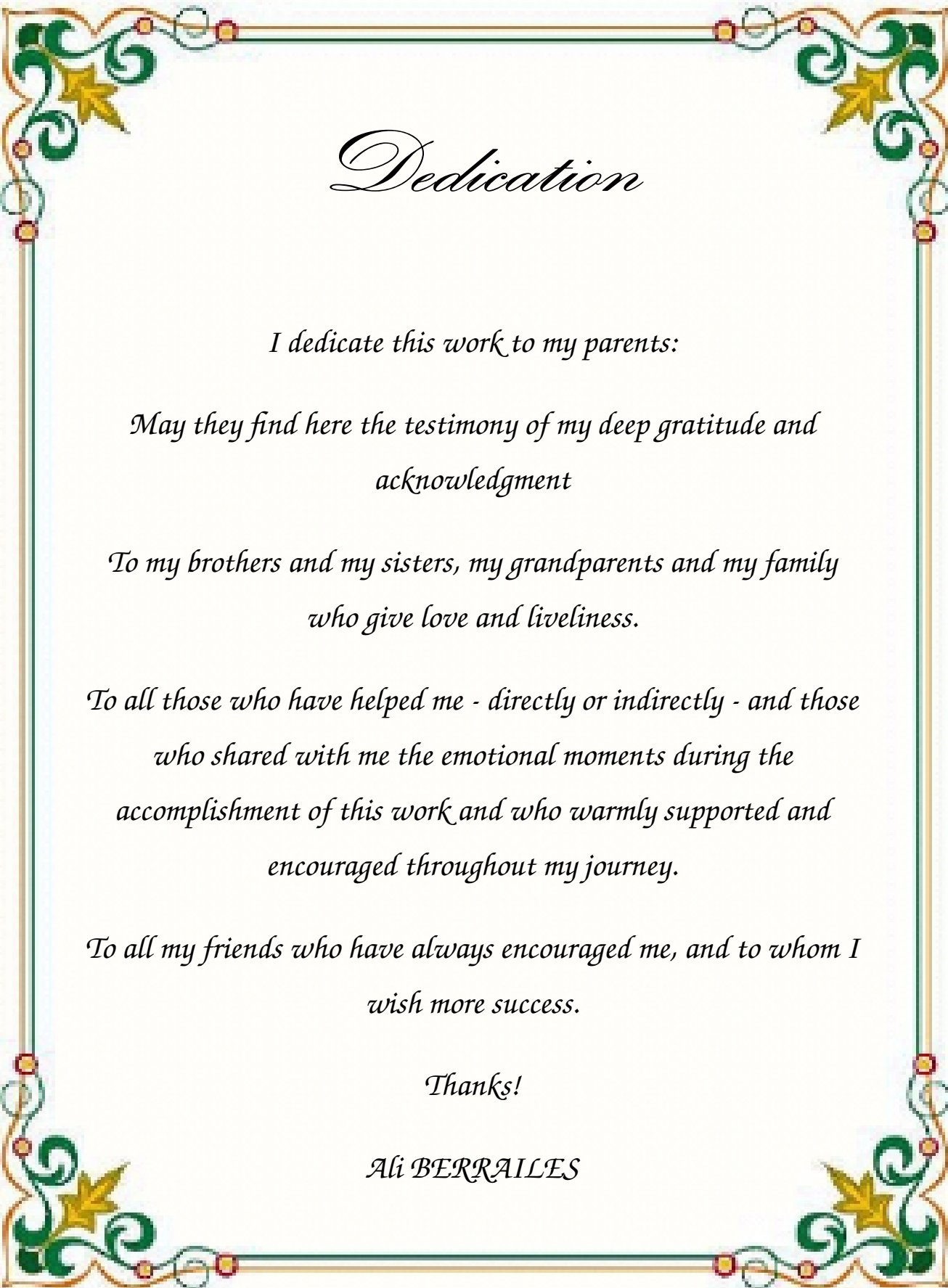
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O Allah, send your blessings on your noble messenger, his family, and companions, and bless us in our life.



Dedication

I dedicate this work to my parents:

*May they find here the testimony of my deep gratitude and
acknowledgment*

*To my brothers and my sisters, my grandparents and my family
who give love and liveliness.*

*To all those who have helped me - directly or indirectly - and those
who shared with me the emotional moments during the
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ملخص

تعد قضايا المشاكل التفاضلية والاحتواءات التفاضلية من الموضوعات الأساسية في الرياضيات التطبيقية، نظراً لارتباطها الوثيق بنظرية المؤثرات الرتيبة الأعظمية ودورها المحوري في تحليل العديد من النماذج الرياضية في مختلف المجالات. يشمل هذا الإطار مشكلة إيجاد مجموعة الأصفار لمجموع المؤثرات الرتيبة الأعظمية، والتي تشكل الأساس لعدد كبير من التطبيقات في التحليل المحدب والتحسين ونظرية النقطة الثابتة.

تهدف هذه الأطروحة إلى دراسة مشكلة إيجاد مجموعة الأصفار لمجموع عائلة منتهية من المؤثرات الأعظمية الرتيبة، مع التركيز على تطوير خوارزميات فعالة وتحقيق خصائص تقاربها. أولاً، قمنا بمراجعة أهم الخوارزميات المعروفة في الأبحاث العلمية للتعامل مع حالة مؤثر رتيب أعظمي واحد، ثم حالة مجموع مؤثرين رتيبين أعظميين، مع عرض أبرز التعديلات والتحسينات التي اقترحها الباحثون لتسريع التقارب وتعزيز الاستقرار العددي. بعد ذلك، تناولنا المشكلة الأساسية المتمثلة في إيجاد أصفار مجموعة منتهية من المؤثرات الرتيبة الأعظمية، وقسمناها إلى عدة حالات خاصة. مجموع مؤثرين رتيبين أعظميين، اقترحنا طريقة محسنة لإيجاد الحلول بناءً على تقريب يوسفيد، مع تحليل نظري لخصائص الطريقة وإثبات تقاربها. كما قدمنا خوارزمية جديدة للتعامل مع حالة مجموع اثنين من المؤثرات الرتيبة الأعظمية المرتبطة بمجموعة محدودة من المؤثرات غير التوسعية، حيث أثبتنا التقارب القوي لتسلسلاتها التكرارية في ظل الظروف المناسبة.

في مرحلة لاحقة، قمنا بتعميم الدراسة على حالة مجموع عائلة منتهية من المؤثرات الرتيبة الأعظمية، واقترحنا خوارزمية عامة تستند إلى دالة شعاعية لحساب مجموعة أصفار المشكلة، مع تحليل نظري مفصل لخصائصها وإثبات نتائج التقارب المناسبة. في نهاية الأطروحة، تناولنا مشكلة الفرق بين مؤثرين رتيبين أعظميين واقترحنا طريقة جديدة لإيجاد حلولها، ودراسة خصائص تقاربها وإبراز مساهمتها في توسيع نطاق التطبيقات الممكنة لنظرية المؤثرات الرتيبة الأعظمية.

تساهم النتائج التي تم الحصول عليها في هذه الأطروحة في إثراء البحوث المتعلقة بالخوارزميات لحل مشاكل الشمولية المرتبطة بالتأثيرات العادية القصوى وتفتح آفاقاً جديدة لتطبيقاتها في مجالات التحليل غير الخطي والتحسين الرياضي.

كلمات مفتاحية: محلل، النقطة الثابتة، مؤثر رتيب أعظمي، احتواء تفاضلي، تقريب يوسفيد.

Abstract

Differential problems and differential inclusions are fundamental topics in applied mathematics, given their close connection to the theory of maximal monotone operators and their pivotal role in the analysis of numerous mathematical models in various fields. This framework includes the problem of finding the set of zeros for the sum of maximal monotone operators, which forms the basis for a lot of applications in non-linear analysis, optimization, and some theories related of fixed point.

This thesis aims to study the problem of finding a set of zeros for the sum of a family of maximal monotone operators, with a focus on developing efficient algorithms and achieving their convergence properties. First, we reviewed the most important algorithms known in the scientific literature for dealing with the case of a single maximal monotone operator, then the case of a sum of maximal monotone operators, presenting the most notable modifications and improvements proposed by researchers to accelerate convergence and enhance numerical stability. Next, we addressed the fundamental problem of finding the zeros of a finite family of maximal monotone operators, dividing it into several special cases. In the context of summing two maximal monotone operators, we proposed an improved method for finding solutions based on the Yosida approximation, with a theoretical analysis of the method's properties and a proof of its convergence. We also presented a new algorithm for dealing with the case of The sum of a pair of maximal monotone operators corresponding to a finite family of nonexpansive operators, Demonstrating strong convergence of their Sequences generated by iteration under appropriate conditions.

At a later stage, we generalized the study to the case of the sum of a finite family of maximal monotone operators, and proposed a general algorithm based on a radial function to compute the zero set of the problem, with a detailed theoretical analysis of its properties and proof of the appropriate convergence results. At the end of the thesis, we addressed the problem of the difference between two maximal periodic effects and proposed a new method for finding their solutions, studying their convergence properties and highlighting their contribution to expanding the range of possible applications of the theory of maximal monotone operators.

The results obtained in this thesis contribute to enriching the literature on algorithms for solving comprehensiveness problems associated with maximal monotone operators and open new horizons for their applications in the fields of nonlinear analysis and mathematical optimization.

Key words: *Resolvent, Fixed point, Maximal monotone operator, Yosida approximation, Differential inclusion.*

Résumé

Les problèmes différentiels et les inclusions différentielles sont des thèmes fondamentaux en mathématiques appliquées, compte tenu de leur lien étroit avec la théorie des opérateurs maximaux monotones et de leur rôle central dans l'analyse de nombreux modèles mathématiques dans divers domaines. Ce cadre comprend le problème de la recherche de l'ensemble des zéros pour la somme des opérateurs maximaux monotones, qui constitue la base d'un grand nombre d'applications en analyse spectrale, en optimisation et en théorie des points fixes. Cette thèse vise à étudier le problème de la recherche d'un ensemble de zéros pour la somme d'une famille finie d'opérateurs maximaux monotones, en mettant l'accent sur le développement d'algorithmes efficaces et l'obtention de leurs propriétés de convergence. Nous avons d'abord passé en revue les algorithmes les plus importants connus dans la littérature scientifique pour traiter le cas d'un seul opérateur maximal monotone, puis le cas d'une somme d'opérateurs maximaux monotones, en présentant les modifications et améliorations les plus notables proposées par les chercheurs pour accélérer la convergence et améliorer la stabilité numérique. Ensuite, nous avons abordé le problème fondamental de la recherche des zéros d'une famille finie d'opérateurs maximaux monotones, en le divisant en plusieurs cas particuliers. Dans le cas de la somme de deux opérateurs maximaux monotones, nous avons proposé une méthode améliorée pour trouver des solutions basée sur l'approximation de Yosida, avec une analyse théorique des propriétés de la méthode et une preuve de sa convergence. Nous avons également présenté un nouvel algorithme pour traiter le cas de la somme de deux opérateurs maximaux monotones associés à un ensemble fini d'opérateurs non expansifs, en prouvant la forte convergence de leurs suites itératives dans des conditions appropriées.

Dans un second temps, nous avons généralisé l'étude au cas de la somme d'une famille finie d'opérateurs maximaux monotones, et proposé un algorithme général basé sur une fonction radiale pour calculer l'ensemble zéro du problème, avec une analyse théorique détaillée de ses propriétés et une preuve des résultats de convergence appropriés. À la fin de la thèse, nous avons abordé le problème de la différence entre deux opérateurs maximaux monotones et proposé une nouvelle méthode pour trouver leurs solutions, en étudiant leurs propriétés de convergence et en soulignant leur contribution à l'élargissement du champ d'applications possibles de la théorie des effets périodiques maximaux.

Les résultats obtenus dans cette thèse contribuent à enrichir la recherche sur les algorithmes de résolution des problèmes d'exhaustivité associés aux opérateurs maximaux monotones et ouvrent de nouveaux horizons pour leurs

applications dans les domaines de l'analyse de formes et de l'optimisation mathématique. .

Mots-clés : *Résolvente, Point fixe, Opérateur maximal monotone, Inclusion différentielle, Yosida approximation.*

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List of Abbreviations

$:=$	Equal by definition
$a.e$	Almost every where
$i.e$	That is to say
$P(Y) := 2^Y$:	The set of all subset of Y
$F : X \rightrightarrows Y$:	A multifunction F of X with value in Y
$P(Y) := 2^Y$:	Predictive functional control
$\text{dom}F$:	The domain of F
$\text{Gr}(F)$:	The graph of F
$\text{Ran}(F)$:	The range of
$\text{zer } F$:	The set of zeros of F
$\mathbb{H}$:	A real Hilbert space
$\langle \cdot, \cdot \rangle$:	The inner product on $\mathbb{H}$
$\text{id}_{\mathbb{H}}$:	The identity function on $\mathbb{H}$
$N_K(x)$:	The normal cone to K at x
$\text{epi } f$:	The epigraph of the function f
$\partial f(x)$:	The subdifferential of the function f at x
$\text{prox}_f(\cdot)$:	The proximal operator of the function f
$\nabla f(x)$:	The gradient of the function f at x
$\Pi_C(\cdot)$:	The orthogonal projection onto the set C
$J^A(\cdot)$:	The resolvent of A
$A_\lambda^A(\cdot)$:	The Yosida approximation of A
(a, b) :	An open interval in $\mathbb{R}$
$\mathcal{R}(\alpha)$:	The real part of the complex number α
$[\mathcal{R}(\alpha)]$:	The integer part of $\mathcal{R}(\alpha)$
$L^1([a, b], \mathbb{R})$:	The space of all Lebesgue-integrable functions from $[a, b]$ into $\mathbb{R}$
$C([a, b], \mathbb{R})$:	The space of all continuous functions from $[a, b]$ into $\mathbb{R}$
$\Gamma_0(\mathbb{H})$:	f is proper, convex, and lower semicontinuous from $\mathbb{H}$ in $\mathbb{R}$

General introduction

Variational problems constitute one of the fundamental pillars of modern mathematical analysis and applied sciences. At their core, variational methods seek to determine extrema typically minima or maxima of functionals defined on infinite dimensional spaces. Originating from classical mechanics through the principle of least action, variational theory has evolved into a powerful framework that unifies diverse areas such as partial differential equations, optimization theory, control theory, and mathematical physics.

In many physical and natural phenomena, systems evolve toward configurations that minimize energy or satisfy equilibrium principles. This variational structure allows complex models to be formulated in terms of energy functionals whose critical points correspond to physically meaningful states. For example, equilibrium configurations in elasticity, stationary heat distributions, electrostatic potentials, and fluid flows can all be characterized as solutions of variational problems. The calculus of variations provides the theoretical foundation for deriving governing equations often expressed as Euler Lagrange equations that describe these systems.

A central mathematical tool in the modern study of variational problems is the theory of maximal monotone operators. Monotone operators arise naturally when dealing with convex energy functionals and nonlinear differential equations. In particular, the subdifferential of a proper, lower semicontinuous, convex functional is a maximal monotone operator. This deep connection between convex analysis and operator theory has led to a powerful framework for studying nonlinear evolution equations and variational inequalities.

Maximal monotone operators play a crucial role in addressing nonlinear and non-smooth problems that cannot be handled by classical differential methods. They provide a robust analytical structure for proving existence, uniqueness, and stability of solutions to partial differential equations, inclusions, and evolution systems. In physics, they appear in models of plasticity, phase transitions, diffusion processes, and thermodynamics, where constitutive laws may be nonlinear or multi-valued. In natural sciences, they are used to model phenomena such as population dynamics, chemical reactions, porous media flow, and biological systems

with constraints.

The interplay between variational principles and maximal monotone operator theory has significantly advanced the study of nonlinear systems. By combining energy methods with operator theoretic techniques, researchers can analyze complex models involving dissipation, constraints, and irreversible processes. This unified framework continues to be an essential tool in both theoretical investigations and real world applications across physics and natural sciences.

We attempt in this research to address the improvement of an algorithm for finding solutions to the sum of finite family of maximal monotone operators in a Hilbert space. This is achieved through the development of more efficient analytical and numerical approaches in terms of convergence and computational stability. This work is motivated by the growing interest in nonlinear problems that can be formulated as differential inclusions or variational problems associated with maximal monotone operators.

The objectives of this thesis are as follows:

- To study the theoretical properties of the sum of two maximal monotone operators in a Hilbert space, with particular emphasis on existence conditions of solutions.
- To analyze classical algorithms (such as splitting methods) and identify their strengths and limitations.
- To propose a modification or improvement of an existing algorithm in order to ensure faster convergence and enhanced numerical stability.
- To establish the convergence of the proposed algorithm under suitable assumptions using tools from functional analysis and monotone operator theory.
- we investigated a non-monotone variational problem by employing the difference of two maximal monotone operators. Furthermore, we proposed a new algorithm for computing the zeros of the problem under consideration.

This work aims to contribute to the development of numerical methods related to maximal monotone operator theory and to open new perspectives for their application in solving complex nonlinear problems across various scientific fields.

Research Structure

This thesis is organized into four chapters, preceded by a general introduction that presents an overview of the fundamental mathematical frameworks underlying maximal monotone

operators and differential inclusions.

- **Chapter 1:** This chapter introduces the basic concepts and foundational results concerning maximal monotone operators and differential inclusions. These notions constitute the theoretical groundwork necessary for the development and understanding of the subsequent chapters.
- **Chapter 2:** This chapter is devoted to the presentation of the principal algorithms proposed in the literature for computing the zeros of single and the sum of two maximal monotone operators. We review the most prominent splitting and iterative methods introduced by various researchers, emphasizing their core ideas and theoretical foundations. In addition, we discuss several fundamental convergence theorems that ensure the validity and effectiveness of these approaches under suitable assumptions. This overview provides the necessary scientific background upon which the methods developed in the subsequent chapters are built and evaluated.
- **Chapter 3:** In this part of the research, we present our original contribution, where we investigate several cases concerning the sum of a finite family of maximal monotone operators. We begin by studying the case of the sum of two maximal monotone operators combined with a finite family of nonexpansive operators. In this setting, we propose two new algorithms and analyze their convergence properties.

Subsequently, we extend this approach to the more general case of the sum of n maximal monotone operators. We provide a unified formulation of the proposed algorithm and examine its theoretical properties. The obtained results are supported by illustrative examples that demonstrate the effectiveness and potential applicability of the proposed methods.

- **Chapter 4:** This chapter is devoted to the study of a non-monotone problem defined as the difference of two maximal monotone operators. Such problems present significant analytical and numerical challenges due to the absence of monotonicity. In this context, we propose a new algorithm designed to address this class of problems and provide a detailed analysis of its theoretical properties, including convergence conditions.

The proposed approach is further supported by several illustrative examples that highlight its effectiveness and demonstrate its relevance in handling this type of nonlinear problem.

Chapter 1

Preliminaries

1.1 Preliminary concepts

This section is devoted to a systematic review of the fundamental concepts and analytical tools from operator theory and convex analysis that are essential for the development of the results presented in this work. In particular, we introduce the basic notions concerning operators in Hilbert spaces, with special emphasis on monotone and maximal monotone operators, which provide a central and unifying Approach for the study of nonlinear inclusion problems.

Furthermore, we present the principal elements of convex analysis, including convex functions and subdifferential operators, which arise naturally in optimization theory and variational formulations(see [1, 2]). These preliminary materials establish the theoretical foundation upon which the subsequent analysis and proposed algorithms are constructed, and they will be repeatedly employed throughout the remainder of this work. All definitions and properties presented in this section can be found in the existing in (see [3–21]).

1.1.1 Hilbert space

A linear space $\mathbb{H}$ equipped with an inner product $\langle \cdot, \cdot \rangle$, and which is complete with respect to the norm

$$\|x\| := \sqrt{\langle x, x \rangle}$$

induced by this inner product, is called a *Hilbert space*.

Throughout this work, we consider a real Hilbert space $\mathbb{H}$. Let C the nonempty closed

convex subset of $\mathbb{H}$, and by I a finite index set

$$I = \{1, 2, \dots, m\} \subset \mathbb{N}, \quad \text{with } m \in \mathbb{N}.$$

1.1.1.1 Properties of the Inner Product

In any Hilbert space $\mathbb{H}$, the following basic algebraic and geometric relations are satisfied for all vectors $u, v \in \mathbb{H}$.

Geometric identities

The relationship between the inner product and the induced norm is captured by the expanded square forms:

$$\begin{aligned} \|u + v\|^2 &= \|u\|^2 + \|v\|^2 + 2\operatorname{Re}\langle u, v \rangle, \\ \|u - v\|^2 &= \|u\|^2 + \|v\|^2 - 2\operatorname{Re}\langle u, v \rangle. \end{aligned}$$

A consequence of these relations is the *Parallelogram Law*, which characterizes Hilbertian norms:

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Furthermore, the difference of squares can be expressed via the inner product as:

$$\langle u + v, u - v \rangle = \|u\|^2 - \|v\|^2.$$

The cauchy–schwarz relation and linear dependence

The magnitude of the inner product is bounded by the product of the norms:

$$|\langle u, v \rangle| \leq \|u\| \|v\|.$$

Specifically, the boundaries of this inequality provide information regarding the alignment of the vectors:

1. $\langle u, v \rangle = \|u\| \|v\|$ if and only if $u = \alpha v$ for some scalar $\alpha \geq 0$.
2. $\langle u, v \rangle = -\|u\| \|v\|$ if and only if $u = -\alpha v$ for some scalar $\alpha \geq 0$.

Triangle inequalities and convexity

The norm satisfies the generalized triangle inequality. For any vectors $\{x_i\}_{i=1}^m$ and non-negative weights $\{\omega_i\}_{i=1}^m \subset \mathbb{R}_+$:

$$\left\| \sum_{i=1}^m \omega_i x_i \right\| \leq \sum_{i=1}^m \omega_i \|x_i\|.$$

If all weights ω_i are strictly positive, equality is achieved if and only if the vectors $\{x_i\}$ are positively linearly dependent.

The norm is also *strictly convex*. This can be demonstrated by the fact that for any $\lambda \in (0, 1)$ and vectors $u \neq v$ with $\|u\| = \|v\|$, we have:

$$\|(1 - \lambda)u + \lambda v\| < \|u\|.$$

Equivalently, if $\|(1 - \lambda)u + \lambda v\| = \|u\| = \|v\|$, then it must be that $u = v$.

Expanded norm of a weighted sum

For any finite set of vectors and real coefficients ω_i , the squared norm of the linear combination expands as follows:

$$\left\| \sum_{i=1}^m \omega_i x_i \right\|^2 = \sum_{i=1}^m \omega_i^2 \|x_i\|^2 + \sum_{i \neq j} \omega_i \omega_j \langle x_i, x_j \rangle.$$

Theorem 1.1.1 *A Banach space with a norm $\|\cdot\|$ is a Hilbert space if and only if there exists an inner product $\langle \cdot, \cdot \rangle$ such that the norm is induced by the inner product, that is,*

$$\|x\| = \sqrt{\langle x, x \rangle} \quad \text{for all } x \in \mathbb{H}.$$

Example 1.1.1 (Euclidean space) *The Euclidean space $\mathbb{R}^n$ is defined as*

$$\mathbb{R}^n := \{x = (\xi_1, \xi_2, \dots, \xi_n) : \xi_j \in \mathbb{R}, j = 1, 2, \dots, n\}.$$

Any element $x = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ can be identified with a matrix of type $n \times 1$ (a column vector), that is,

$$x = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}, \quad \text{or equivalently } x = (\xi_1, \xi_2, \dots, \xi_n)^\top.$$

The Euclidean space is equipped with the standard inner product defined by

$$\langle x, y \rangle := x^\top y = \sum_{j=1}^n \xi_j \eta_j,$$

where $x = (\xi_1, \dots, \xi_n)$ and $y = (\eta_1, \dots, \eta_n)$. The associated norm is given by

$$\|x\| := (x^\top x)^{1/2}.$$

Example 1.1.2 (Inner product induced by a matrix) Let $\mathbb{R}^m$ be endowed with an inner product induced by a positive definite matrix $G = [g_{ij}]_{m \times m}$, defined by

$$\langle w, v \rangle_G := w^\top G v = \sum_{i=1}^m \sum_{j=1}^m g_{ij} \omega_i v_j.$$

Here $w = (\omega_1, \dots, \omega_m)$ and $v = (v_1, \dots, v_m)$. The corresponding norm is

$$\|u\|_G := (u^\top G u)^{1/2}.$$

Example 1.1.3 (The space ℓ^2) The space ℓ^2 consists of all real sequences $x = (\xi_1, \xi_2, \dots)$ that are square summable, that is,

$$\ell^2 := \left\{ x : \xi_k \in \mathbb{R}, k = 1, 2, \dots, \sum_{k=1}^{\infty} \xi_k^2 < \infty \right\}.$$

The inner product on ℓ^2 is defined by

$$\langle x, y \rangle := \sum_{k=1}^{\infty} \xi_k \eta_k,$$

and the associated norm is

$$\|x\| := \left(\sum_{k=1}^{\infty} \xi_k^2 \right)^{1/2}.$$

1.1.1.2 Functions

Let $X \subseteq \mathbb{H}$ and let $f_i : X \rightarrow \mathbb{R}$, $i \in I$. The function

$$f := \max_{i \in I} f_i$$

is defined by

$$f(x) := \max\{f_i(x) : i \in I\}.$$

In particular, define

$$f_+ := \max\{f, 0\}, \quad f_- := \max\{-f, 0\}.$$

Let $f : X \rightarrow \mathbb{R}$. The subset

$$S(f, \alpha) := \{x \in X : f(x) \leq \alpha\}$$

is called a *sublevel set* of f at level $\alpha \in \mathbb{R}$.

The subset

$$\text{epi } f := \{(x, \rho) \in X \times \mathbb{R} : \rho \geq f(x)\} \subseteq X \times \mathbb{R}$$

is called the *epigraph* of f .

We say that f is *lower semicontinuous* at $x \in X$ if

$$\liminf_{y \rightarrow x} f(y) \geq f(x).$$

We say that f is *weakly lower semicontinuous* at $x \in X$ if

$$\liminf_{y \rightarrow x} f(y) \geq f(x).$$

We say that a function $f : \mathbb{H} \rightarrow \mathbb{R}$ is *coercive* if

$$\lim_{\|x\| \rightarrow +\infty} f(x) = +\infty.$$

We say that a function $f : X \rightarrow \mathbb{R}$ attains a *global minimum* at a point $x^* \in X$ if

$$f(x^*) \leq f(x) \quad \text{for all } x \in X.$$

We say that a function $f : X \rightarrow \mathbb{R}$ attains a *local minimum* at a point $x^* \in X$ if there exists $\delta > 0$ such that

$$f(x^*) \leq f(x) \quad \text{for all } x \in X \cap B(x^*, \delta).$$

Example 1.1.4 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x + \frac{1}{2}, & \text{if } x < 0, \\ x^2 + 1, & \text{if } x \geq 0. \end{cases}$$

We observe that

$$\lim_{x \rightarrow 0^-} f(x) = \frac{1}{2} \quad \text{and} \quad f(0) = 1.$$

Hence,

$$\liminf_{x \rightarrow 0} f(x) = \frac{1}{2} < f(0),$$

which shows that f is not lower semicontinuous at $x = 0$. However, since

$$\limsup_{x \rightarrow 0} f(x) = 1 = f(0),$$

the function f is lower semicontinuous from above at $x = 0$.

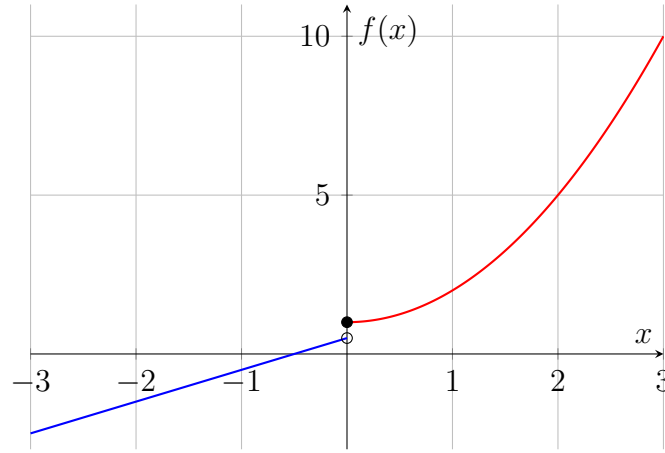


Figure 1.1: Graph of the function f .

Theorem 1.1.2 *A function $f : X \rightarrow \mathbb{R}$ is (weakly) lower semicontinuous if and only if $\text{epi} f$ is a (weakly) closed subset of $X \times \mathbb{R}$.*

Theorem 1.1.3 *If a function $f : X \rightarrow \mathbb{R}$ is weakly lower semicontinuous on a weakly closed and bounded subset $X \subseteq \mathbb{H}$, then f is bounded from below and attains its minimum.*

Theorem 1.1.4 *If a function $f : \mathbb{H} \rightarrow \mathbb{R}$ is weakly lower semicontinuous and coercive, and if $X \subseteq \mathbb{H}$ is weakly closed, then f attains its global minimum on X .*

1.1.2 Convex sets

Let $\mathbb{H}$ and $\mathbb{H}_i$, for $i = 1, 2, \dots, m$, be Hilbert spaces.

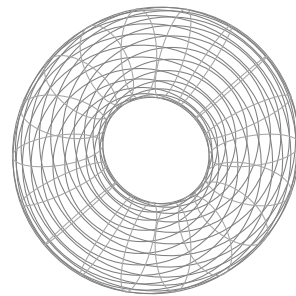
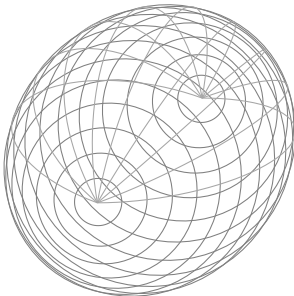
Definition 1.1.1 *A subset $C \subset \mathbb{H}$ is said to be convex if*

$$(1 - \lambda)x + \lambda y \in C, \quad \forall x, y \in C, \quad \forall \lambda \in [0, 1].$$

The sets depicted below illustrate examples of a convex set and a non-convex set.

Convex set : Sphere

non-convex set : torus



The *closure* of C is denoted by $\overline{C}$. A set C is said to be *sequentially weakly closed* if every weakly convergent sequence $(x_n)_{n \in \mathbb{N}} \subset C$ converges weakly to a point $x \in C$.

The *convex hull* of C , denoted by $\text{conv } C$, is the intersection of all convex sets containing C . Similarly, the *affine hull* of C , denoted by $\text{aff } C$, is the intersection of all affine subspaces of H containing C .

A subset $C \subset H$ is called a cone if

$$C = \mathbb{R}_+^* C,$$

that is, if $x \in C$ implies $\lambda x \in C$ for every $\lambda > 0$. A convex cone is a set that is both convex and a cone.

The *conical hull* of C , denoted by $\text{cone } C$, is the intersection of all cones in H containing C . The smallest linear subspace of H containing C is called the *span* of C and is denoted by $\text{span } C$.

The *interior* of C , denoted by $\text{int } C$, is the largest open set contained in C and is defined by

$$\text{int } C = \{x \in C : \exists \varepsilon > 0 \text{ such that } B(x, \varepsilon) \subset C\}.$$

The *relative interior* of C is defined by

$$\text{ri } C = \{x \in \text{aff } C : \exists \varepsilon > 0 \text{ such that } B(x, \varepsilon) \cap \text{aff } C \subset C\},$$

where $B(x, \varepsilon)$ denotes the open ball centered at x with radius ε .

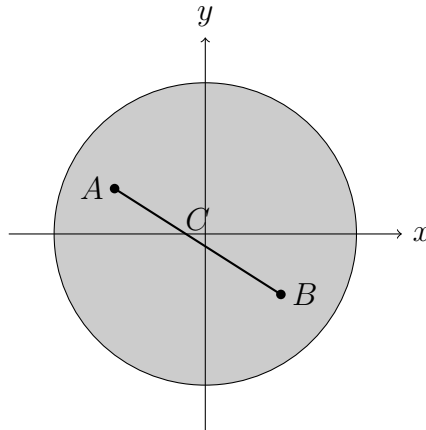
Example 1.1.5

$$C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

This set is convex: for any two points $u, v \in C$ and any $t \in [0, 1]$,

$$tu + (1 - t)v \in C,$$

that is, the line segment joining u and v lies entirely inside the disk.



Example 1.1.6 We now present several examples of convex and non-convex sets.

1. Any real interval is a convex set.
2. Circles in the plane and spheres in space are examples of non-convex sets.

1.1.2.1 Convex Functions

Let $\mathbb{H}$ denote a Hilbert space, and let X be a convex subset within it. We consider a functional $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$.

The function f is categorized as **convex** on X if for all pair of points x_1, x_2 and every weight $\lambda \in [0, 1]$, this inequality is satisfied:

$$f((1 - \lambda)x_1 + \lambda x_2) \leq (1 - \lambda)f(x_1) + \lambda f(x_2). \quad (1.1)$$

In instances where the above inequality holds strictly for all distinct $x_1, x_2 \in X$ and for $\lambda \in (0, 1)$, we classify f as **strictly convex**. Conversely, a mapping g is termed **concave** if its negation, $-g$, satisfies the convexity criteria.

A function $f : \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined as **proper** provided its effective domain, defined by the set

$$\text{dom } f = \{x \in \mathbb{H} : f(x) < +\infty\}, \quad (1.2)$$

is not empty. This ensures the function is not identically infinite.

For a parameter $\alpha > 0$, we say that f is **α -strongly convex** if it satisfies a strengthened version of the convexity inequality:

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y) - \frac{\alpha}{2}\lambda(1 - \lambda)\|x - y\|^2, \quad (1.3)$$

for all $x, y \in X$ and $\lambda \in [0, 1]$. This property implies a minimum degree of curvature related to the norm of the space.

Illustrative example: the indicator mapping

Let C be a subset of $\mathbb{H}$. The **indicator function** associated with C , represented by $\delta_C : \mathbb{H} \rightarrow \{0, +\infty\}$, is characterized by:

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C. \end{cases} \quad (1.4)$$

The analytical properties of δ_C are directly inherited from the geometry of C . Specifically:

- δ_C is proper if and only if C is non-empty.
- δ_C is a convex function if and only if C is a convex set.
- δ_C is lower semicontinuous if and only if C is a closed set in the topology of $\mathbb{H}$.

1.1.3 Subdifferential of a convex function and its properties

Definition 1.1.2 Let f represent a convex function on a real Hilbert space $\mathbb{H}$. For any given $x \in \mathbb{H}$, we define the **subdifferential**, denoted by $\partial f(x)$, as the collection of all vectors $g \in \mathbb{H}$ that satisfy the following variational inequality for every y in the space:

$$f(y) \geq f(x) + \langle g, y - x \rangle, \quad \forall y \in \mathbb{H}.$$

If the set $\partial f(x)$ is non-empty, the function f is considered **subdifferentiable** at the point x . We refer to any individual member g of the set $\partial f(x)$ as a **subgradient** of the function at x .

For each $g \in \partial f(x)$, the affine function

$$\tilde{f}_x : \mathbb{H} \rightarrow \mathbb{R}, \quad \tilde{f}_x(y) := \langle g, y - x \rangle + f(x),$$

is called a *linearization* of f at the point x .

Theorem 1.1.5 Suppose $f : \mathbb{H} \rightarrow \mathbb{R}$ is a convex function that is continuous on a real Hilbert space $\mathbb{H}$. For each $x \in \mathbb{H}$, the set of subgradients $\partial f(x)$ possesses the following properties: it is non-empty, convex, bounded, and exhibits closure in the weak topology of $\mathbb{H}$.

Theorem 1.1.6 Let $f : \mathbb{H} \rightarrow \mathbb{R}$ be a continuous convex function. Then f is Gâteaux differentiable at a point $x \in \mathbb{H}$ if and only if the subdifferential $\partial f(x)$ consists of a single element. In this case,

$$\partial f(x) = \{Df(x)\}.$$

Theorem 1.1.7 Let $f : \mathbb{H} \rightarrow \mathbb{R}$ be a continuous convex function, and let $x, s \in \mathbb{H}$. Then the directional derivative of f at x in the direction s satisfies

$$f'(x; s) = \sup \{ \langle y, s \rangle \mid y \in \partial f(x) \}.$$

In particular, if f is differentiable at x , then

$$f'(x; s) = \langle Df(x), s \rangle.$$

Moreover, if $\mathbb{H}$ is a Euclidean space, then

$$f'(x; s) = s^\top \nabla f(x).$$

Example 1.1.7 (Subdifferential of the norm) Let $f(x) = \|x\|$ on a Hilbert space $\mathbb{H}$. Then the subdifferential of f at x is given by

$$\partial\|x\| = \begin{cases} \left\{ \frac{x}{\|x\|} \right\}, & \text{if } x \neq 0, \\ \{u \in \mathbb{H} : \|u\| \leq 1\}, & \text{if } x = 0. \end{cases}$$

1.1.4 Metric projection

The metric projection on a Hilbert space $\mathbb{H}$ serves as a fundamental tool throughout the following sections of this study. In the current section, we introduce this operator and outline its primary characteristics. Furthermore, a more detailed analysis of its properties is reserved for discussion in Chapter 2.

Definition 1.1.3 (Metric projection) Given a point x in a Hilbert space $\mathbb{H}$ and a non-void subset $C \subseteq \mathbb{H}$, an element $y \in C$ is referred to as a **metric projection** of x onto C if it satisfies the following distance minimization:

$$\|x - y\| = \inf_{z \in C} \|x - z\|.$$

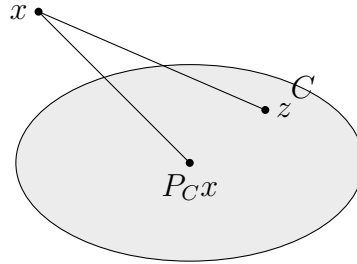


Figure 1.2: Metric projection

Let $C \subset \mathbb{H}$ be a nonempty subset and let $x \in \mathbb{H}$. If there exists $y \in C$ such that

$$\|y - x\| \leq \|z - x\| \quad \text{for all } z \in C,$$

then y is called the *metric projection* of x onto C and is denoted by $P_C x$ (see Fig. 1.2).

The vector

$$s := P_C x - x$$

is called the projection vector of x onto C .

If $P_C x$ exists and is uniquely determined for all $x \in \mathbb{H}$, then the operator

$$P_C : \mathbb{H} \rightarrow C$$

is called the *metric projection onto C* .

Remark 1.1.1 Let $C \subset \mathbb{H}$.

1. If $x \in C$, then it immediately from the definition that

$$P_C x = x.$$

2. If $x \notin C$ and the metric projection $P_C x$ exists, then $P_C x \in \partial C$. Indeed, assume that $P_C x \in \text{int}(C)$. Then there exists $\varepsilon > 0$ such that

$$B(P_C x, \varepsilon) \subset C.$$

Define

$$z := x + \left(1 - \frac{\varepsilon}{\|P_C x - x\|}\right) (P_C x - x) \in B(P_C x, \varepsilon) \subset C.$$

Since $\|z - P_C x\| = \varepsilon$, we obtain

$$\|z - x\| = \left(1 - \frac{\varepsilon}{\|P_C x - x\|}\right) \|P_C x - x\| < \|P_C x - x\|,$$

which contradicts the definition of $P_C x$.

Theorem 1.1.8 (Characterization Theorem) Let $x \in \mathbb{H}$, let $C \subset \mathbb{H}$ be a convex subset, and let $y \in C$. The following conditions are equivalent:

1. $y = P_C x$;
2. $\langle x - y, z - y \rangle \leq 0$ for all $z \in C$.

Proof 1.1.1 (i) $\Rightarrow$ (ii). Let $y = P_C x$, $z \in C$, and define

$$z_\lambda := y + \lambda(z - y),$$

for $\lambda \in (0, 1)$. Obviously, $z_\lambda \in C$ since C is convex. By (i) and by the properties of the inner product, we have

$$\begin{aligned}\|x - y\|^2 &\leq \|x - z_\lambda\|^2 \\ &= \|x - y - \lambda(z - y)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, z - y \rangle + \lambda^2\|z - y\|^2.\end{aligned}$$

Since $\lambda > 0$, the above inequality yields

$$\langle x - y, z - y \rangle \leq \frac{\lambda}{2} \|z - y\|^2,$$

and letting $\lambda \rightarrow 0$ we obtain (ii).

(ii) $\Rightarrow$ (i). By the properties of the inner product and by (ii), we obtain for any $z \in C$

$$\begin{aligned}\|z - x\|^2 &= \|z - y + y - x\|^2 \\ &= \|z - y\|^2 + \|y - x\|^2 + 2\langle z - y, y - x \rangle \\ &\geq \|y - x\|^2,\end{aligned}$$

which, by the definition of the metric projection, gives (i).

1.1.5 Single-valued and multifunction mapping

Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be real Hilbert spaces, and let $D \subset \mathbb{H}_1$. An operator (or mapping)

$$A : D \rightarrow \mathbb{H}_2$$

is called single-valued if to each point $x \in D$ it assigns a unique point $Ax \in \mathbb{H}_2$.

The range of the operator A is defined by

$$\text{ran } A := \{y \in \mathbb{H}_2 \mid \exists x \in D \text{ such that } A(x) = y\}.$$

Example 1.1.8 Consider a linear operator R that performs a cyclic right-shift on vectors in the space $\mathbb{H}^m$. This mapping is defined as:

$$R : \mathbb{H}^m \rightarrow \mathbb{H}^m, \quad (v_1, v_2, \dots, v_m) \mapsto (v_m, v_1, \dots, v_{m-1}).$$

Derivation of the adjoint operator

To find the adjoint R^* , we use the fundamental inner product identity $\langle Rv, w \rangle = \langle v, R^*w \rangle$ for any $v, w \in \mathbb{H}^m$. By expanding the left-hand side, we obtain:

$$\begin{aligned}\langle Rv, w \rangle &= \langle (v_m, v_1, \dots, v_{m-1}), (w_1, w_2, \dots, w_m) \rangle \\ &= v_m w_1 + v_1 w_2 + v_2 w_3 + \dots + v_{m-1} w_m.\end{aligned}$$

By rearranging the terms to align with the components of the original vector $v = (v_1, \dots, v_m)$, we can write:

$$\begin{aligned}\langle v, R^*w \rangle &= v_1 w_2 + v_2 w_3 + \dots + v_{m-1} w_m + v_m w_1 \\ &= \langle (v_1, v_2, \dots, v_m), (w_2, w_3, \dots, w_m, w_1) \rangle.\end{aligned}$$

Therefore, the adjoint operator R^* is the cyclic left-shift operator given by:

$$R^* : \mathbb{H}^m \rightarrow \mathbb{H}^m, \quad (w_1, w_2, \dots, w_m) \mapsto (w_2, w_3, \dots, w_m, w_1).$$

Definition of the kernel

The kernel (or null space) of a linear operator A acting on $\mathbb{H}$ is defined as the set of all elements that the operator maps to the zero vector:

$$\ker A := \{v \in \mathbb{H} \mid A(v) = 0\}.$$

Definition 1.1.4 A multivalued operator A from X into Y is an operator that assigns to each $x \in X$ a (possibly empty) subset $A(x)$ of Y . Such an operator is also called a multifunction, and we write:

$$A : X \rightrightarrows Y \quad \text{or} \quad A : X \rightarrow 2^Y.$$

The sets

$$\begin{aligned}\text{Gr}(A) &:= \{(x, y) \in X \times Y \mid y \in A(x)\}, \\ D(A) &:= \{x \in X \mid A(x) \neq \emptyset\}, \\ R(A) &:= \{y \mid \exists x : y \in A(x)\} = \bigcup_{x \in D(A)} A(x),\end{aligned}$$

are called respectively the *graph*, the *domain*, and the *range* of A .

The inverse of a multivalued operator is defined by

$$A^{-1}(y) := \{x \in X \mid y \in A(x)\}.$$

or equivalently

$$x \in A^{-1}(y) \iff (x, y) \in \text{Gr}(A).$$

We have the following relations:

$$D(A^{-1}) = R(A), \quad R(A^{-1}) = D(A), \quad (A^{-1})^{-1} = A.$$

1.1.5.1 Basic operations

We can define new operators using the following properties for addition, scalar multiplication, and composition of multivalued operators:

- For $A, B : X \rightarrow 2^Y$,

$$(A + B)(x) := A(x) + B(x);$$

- For $\lambda \in \mathbb{R}$ and $B : X \rightarrow 2^Y$,

$$(\lambda B)(x) := \lambda B(x);$$

- For $B : X \rightarrow 2^Y$ and $A : Z \rightarrow 2^X$, we define $B \circ A : Z \rightarrow 2^Y$ by

$$(B \circ A)(x) := B(A(x)) = \bigcup_{y \in A(x)} B(y).$$

1.1.6 Monotone and maximal monotone operators

Maximal monotone operators constitute a fundamental mathematical framework for addressing complex and nonlinear problems arising in diverse scientific and technological fields. They play a central role in modeling, analysis, and the resolution of numerous real-world applications [22–25]. In particular, maximal monotone operators in Hilbert spaces are essential in functional analysis and convex optimization, owing to their strong structural and geometric properties. Their importance is reflected in their capacity to treat intricate problems, including differential equations, through geometric techniques that ensure convergence and uniqueness of solutions. For a comprehensive treatment of maximal monotone operators and their properties, including detailed discussions, proofs, and applications, we refer the reader to [26–28]. All definitions and fundamental results used in this section can be found in [7, 8, 27, 29–36].

Definition 1.1.5 *A set-valued mapping $A : \mathbb{H} \rightrightarrows \mathbb{H}$ is said to be monotone if*

$$\langle x^* - y^*, x - y \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in \text{Gr}(A).$$

Definition 1.1.6 A monotone operator $A : \mathbb{H} \rightrightarrows \mathbb{H}$ is said to be maximal monotone if its graph is not strictly contained in the graph of any other monotone operator.

Proposition 1.1.1 Let $\mathbb{H}$ be a real Hilbert space and let $A : \mathbb{H} \rightrightarrows \mathbb{H}$ be an operator. Then A is maximal monotone if and only if

$$(x, u) \in \text{Graph}(A) \iff \langle x - y, u - v \rangle \geq 0, \quad \forall (y, v) \in \text{Graph}(A).$$

We now present some classical examples of monotone and maximal monotone operators.

Example 1.1.9 (Zero operator) Let $\mathbb{H}$ be a real Hilbert space. Define the operator

$$A : \mathbb{H} \rightrightarrows \mathbb{H}, \quad A(x) = \{0\}, \quad \forall x \in \mathbb{H}.$$

Then A is a monotone operator.

Example 1.1.10 (Identity operator) Let $\mathbb{H}$ be a real Hilbert space. Define the identity operator

$$\text{Id} : \mathbb{H} \rightrightarrows \mathbb{H}, \quad \text{Id}(x) = \{x\}, \quad \forall x \in \mathbb{H}.$$

Then Id is a monotone operator.

Example 1.1.11 (Normal cone operator) Let $\mathbb{H}$ be a real Hilbert space and let $C \subset \mathbb{H}$ be a nonempty closed convex set. The normal cone operator $N_C : \mathbb{H} \rightrightarrows \mathbb{H}$ is defined by

$$N_C(x) = \begin{cases} \{z \in \mathbb{H} : \langle z, y - x \rangle \leq 0, \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Then N_C is a maximal monotone operator.

Example 1.1.12 A famous example of a maximal monotone operator is the subgradient of the function $f(x) = |x|$ on $\mathbb{R}$:

$$\partial f(x) = \begin{cases} -1, & x < 0, \\ [-1, 1], & x = 0, \\ 1, & x > 0. \end{cases}$$

Proposition 1.1.2 A monotone operator A is maximal monotone if and only if

$$\forall (y, y^*) \in \text{Gr}(A), \langle x^* - y^*, x - y \rangle \geq 0 \implies (x, x^*) \in \text{Gr}(A).$$

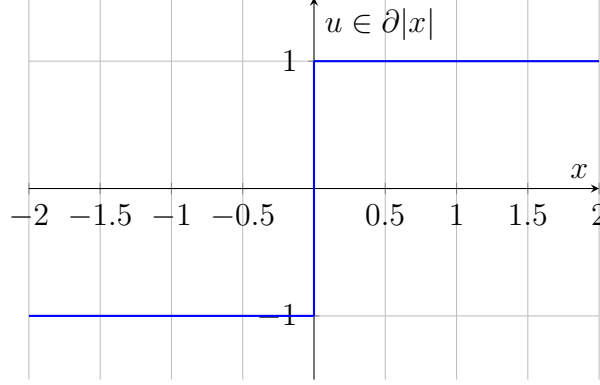


Figure 1.3: Set-valued monotone operator $\partial|x|$

Example 1.1.13 Let $C \subset \mathbb{H}$ be a nonempty closed convex set. The associated normal cone operator $N_C : \mathbb{H} \rightrightarrows \mathbb{H}$ is defined by

$$N_C(x) = \begin{cases} \{\tilde{u} \in \mathbb{H} \mid \langle \tilde{u}, y - x \rangle \leq 0, \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Since $N_C = \partial\delta_C$, where δ_C denotes the indicator function of C , and since δ_C is proper, convex, and lower semicontinuous, it follows that N_C is maximal monotone.

Moreover, if L is a linear subspace of $\mathbb{H}$, then

$$N_L(x) = L^\perp \quad \text{for all } x \in L, \quad N_L(x) = \emptyset \quad \text{otherwise.}$$

Proposition 1.1.3 (Maximal monotonicity and scalar multiplication) Let $A : \mathbb{H} \rightrightarrows \mathbb{H}$ be a maximal monotone operator and let $\lambda \in \mathbb{R}_{++}$. Let the operator λA by

$$\lambda A(x) := \{\lambda u \mid u \in A(x)\}.$$

Then λA is maximal monotone.

Proposition 1.1.4 Let $\mathbb{H}$ be a Hilbert space, and let $A : \mathbb{H} \rightrightarrows \mathbb{H}$ be a maximal monotone operator. Assume that there exists $\alpha > 0$ such that

$$\langle x^*, x \rangle \geq 0 \quad \text{whenever } \|x\| > \alpha, \quad x \in \text{dom}(A), \quad x^* \in A(x). \quad (1.5)$$

Then there exists an element $x \in H$ such that

$$0 \in A(x).$$

1.1.7 Some classes of monotone operators

Definition 1.1.7 A set-valued operator $A : \mathbb{H} \rightrightarrows \mathbb{H}$ is said to be strongly monotone if there exists a constant $\rho > 0$ such that

$$\langle x^* - y^*, x - y \rangle \geq \rho \|x - y\|^2, \quad \forall (x, x^*), (y, y^*) \in \text{Gr}(A).$$

Lemma 1.1.1 Let C be a nonempty, closed, and convex subset of a real Hilbert space $\mathbb{H}$. Let the mapping $A : C \rightarrow \mathbb{H}$ be α -inverse strongly monotone and $\lambda > 0$ be a constant. Then, we have

$$\|(I - \rho A)x - (I - \rho A)y\|^2 \leq \|x - y\|^2 + \rho(\rho - 2\alpha)\|Ax - Ay\|^2$$

for all $x, y \in C$. In particular, if $0 < \rho \leq 2\alpha$, then $I - \rho A$ is nonexpansive.

Example 1.1.14 Let $z \in \mathbb{R}_+$ and let $\beta \in \mathbb{R}_+^*$. Define $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$\phi(z) = \beta z^2.$$

Then it is clear that every operator that is strongly monotone with constant β is also uniformly monotone with modulus ϕ .

Remark 1.1.2 1. Strong monotonicity implies monotonicity.

2. Observe that an operator A is strongly monotone with modulus ρ if and only if the operator $A - \rho I$ is monotone, where I denotes the identity operator.

The strong monotonicity of the inverse operator A^{-1} is sometimes called the Dunn property or the co-coercivity of A . We say that A is γ -co-coercive if and only if A^{-1} is γ -strongly monotone. This property can be equivalently expressed as

$$\langle x^* - y^*, x - y \rangle \geq \gamma \|x^* - y^*\|^2, \quad \forall (x, x^*), (y, y^*) \in \text{Gr}(A).$$

An operator that is both strongly monotone and Lipschitz continuous is co-coercive. More precisely, the following result holds.

Proposition 1.1.5 If A is Lipschitz continuous with constant L and strongly monotone with modulus β , then A is co-coercive with modulus β/L^2 .

Every γ -co-coercive operator is monotone and $1/\gamma$ -Lipschitz continuous, but it is not necessarily strongly monotone (for example, a constant mapping).

Conversely, a monotone operator is not necessarily co-coercive, as illustrated by the following example. Consider the linear operator $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$A(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This operator is monotone, but it is not co-coercive on $\mathbb{R}^2$.

Definition 1.1.8 (Nonexpansive-type mappings) *Let D be a nonempty subset of a Hilbert space $\mathbb{H}$ and let $A : D \rightarrow \mathbb{H}$. The mapping A is said to be*

1. *nonexpansive, or Lipschitz continuous with constant 1, if*

$$(\forall x \in D)(\forall y \in D) \quad \|Ax - Ay\| \leq \|x - y\|; \quad (1.6)$$

2. *strictly nonexpansive if*

$$(\forall x \in D)(\forall y \in D) \quad x \neq y \Rightarrow \|Ax - Ay\| < \|x - y\|;$$

3. *firmly nonexpansive if*

$$(\forall x \in D)(\forall y \in D) \quad \|Ax - Ay\|^2 + \|(Id - A)x - (Id - A)y\|^2 \leq \|x - y\|^2;$$

4. *a Banach contraction, or Lipschitz continuous with constant β , if there exists $\beta \in [0, 1)$ such that*

$$(\forall x \in D)(\forall y \in D) \quad \|Ax - Ay\| \leq \beta \|x - y\|; \quad (1.16)$$

5. *strongly nonexpansive if A is nonexpansive and whenever $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are sequences in D such that $(x_n - y_n)_{n \in \mathbb{N}}$ is bounded and*

$$\|x_n - y_n\| - \|Ax_n - Ay_n\| \rightarrow 0,$$

it follows that

$$(x_n - y_n) - (Ax_n - Ay_n) \rightarrow 0.$$

Proposition 1.1.6 (Equivalent characterizations of firm nonexpansiveness) *Let D be a nonempty subset of a Hilbert space $\mathbb{H}$ and let $A : D \rightarrow \mathbb{H}$. Then the following statements are equivalent:*

1. *A is firmly nonexpansive.*

2. $Id - A$ is firmly nonexpansive.

3. $2A - Id$ is nonexpansive.

4. For all $x, y \in D$,

$$\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay \rangle.$$

5. For all $x, y \in D$,

$$0 \leq \langle Ax - Ay, (Id - A)x - (Id - A)y \rangle.$$

Example 1.1.15 Let $A : \mathbb{H} \rightarrow \mathbb{H}$ be a contraction and define $F = I - A$. Then F is $\frac{1}{2}$ -co-coercive.

Indeed, for all $x, y \in \mathbb{H}$, we have

$$\begin{aligned} \|F(x) - F(y)\|^2 &= \|(I - A)(x) - (I - A)(y)\|^2 \\ &= \|(x - y) - (A(x) - A(y))\|^2 \\ &= \|x - y\|^2 - 2\langle x - y, A(x) - A(y) \rangle + \|A(x) - A(y)\|^2 \\ &\leq 2(\|x - y\|^2 - \langle x - y, A(x) - A(y) \rangle) \\ &= 2\langle x - y, F(x) - F(y) \rangle. \end{aligned}$$

1.1.8 Some results on resolvents

Let $\mathbb{H}$ be a Hilbert space.

Definition 1.1.9 Let $\lambda > 0$ and let $A : \mathbb{H} \rightrightarrows \mathbb{H}$ be an operator defined on the Hilbert space $\mathbb{H}$. The mapping

$$J_\lambda^A = (I + \lambda A)^{-1}$$

is called the resolvent of A (with parameter λ).

Remark 1.1.3 The resolvent is a contraction (and therefore single-valued) for a monotone operator, and it is defined on the whole space if A is maximal monotone. This follows from the following result due to Minty (1962)(see [24]).

Example 1.1.16 Let $\mathbb{H}$ be a real Hilbert space, let $x_0 \in \mathbb{H}$, suppose that $\mathbb{H} = L^2([0, T]; \mathbb{H})$, and let A be the time-derivative operator

$$A : \mathbb{H} \rightarrow 2^{\mathbb{H}} : x \mapsto \begin{cases} \{x'\}, & \text{if } x \in W^{1,2}([0, T]; H) \text{ and } x(0) = x_0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then $\text{dom } J^A = \mathbb{H}$ and, for every $x \in \mathbb{H}$,

$$J^A x : [0, T] \rightarrow \mathbb{H} : t \mapsto e^{-t} x_0 + \int_0^t e^{s-t} x(s) ds.$$

Definition 1.1.10 Let A be a set-valued operator defined on a Hilbert space $\mathbb{H}$. The Yosida approximation (or Yosida regularization) of A with parameter $\lambda > 0$ is defined by

$$A_\lambda := \frac{I - J_\lambda^A}{\lambda} = (\lambda I + A^{-1})^{-1}.$$

For every $\lambda > 0$, if A is maximal monotone, then its Yosida approximation A_λ is a single-valued operator defined on the whole space $\mathbb{H}$. Moreover, A_λ is maximal monotone and Lipschitz continuous with Lipschitz constant $1/\lambda$.

In addition, the following property holds:

$$\forall \lambda > 0, \forall x \in \mathbb{H}, \quad A_\lambda(x) \in A(J_\lambda^A(x)).$$

Proposition 1.1.7 (Properties of the Yosida approximation) Let $A : \mathbb{H} \rightrightarrows \mathbb{H}$ be a maximal monotone operator on a real Hilbert space $\mathbb{H}$, and let $\lambda > 0$. Then the Yosida approximation A_λ satisfies the following properties:

1. **Single-valuedness.** The operator A_λ is single-valued and defined on the whole space $\mathbb{H}$.
2. **Monotonicity.** The operator A_λ is monotone, that is,

$$\langle A_\lambda(x) - A_\lambda(y), x - y \rangle \geq 0, \quad \forall x, y \in \mathbb{H}.$$

3. **Lipschitz continuity.** The operator A_λ is Lipschitz continuous with Lipschitz constant $\frac{1}{\lambda}$, i.e.,

$$\|A_\lambda(x) - A_\lambda(y)\| \leq \frac{1}{\lambda} \|x - y\|, \quad \forall x, y \in \mathbb{H}.$$

4. **Strong convergence.** As $\lambda \rightarrow 0^+$, the sequence $A_\lambda(x)$ converges strongly to an element of $A(x)$ for every $x \in \text{dom}(A)$. More precisely,

$$A_\lambda(x) \longrightarrow A^0(x) \in A(x),$$

where $A^0(x)$ denotes the element of minimal norm in $A(x)$.

Proposition 1.1.8 For all $\beta > 0$, $\lambda > 0$,

1. If A is maximally β -strongly monotone, then J_λ^A is $\frac{1}{\lambda\beta+1}$ -Lipschitz continuous.
2. If A is maximally β -inverse strongly monotone, then A_λ is $\frac{1}{\beta+\lambda}$ -Lipschitz continuous.

Corollary 1.1.1 *Let $\lambda > 0$ and let A be monotone on $\mathbb{H}$. Then every element z of $\mathbb{H}$ can be written in at most one way as*

$$z = x + \lambda y, \quad \text{where } y \in Ax.$$

Lemma 1.1.2 *For $\lambda > 0$, $\mu > 0$, and $x \in \mathbb{H}$, the following identity holds:*

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda} \right) J_\lambda x \right).$$

Lemma 1.1.3 *Let $\lambda > 0$ and let A be a maximal monotone operator on $\mathbb{H}$. Then, for every $z \in \mathbb{H}$, there exists a unique $u \in \mathbb{H}$ such that*

$$z - u \in \lambda A(u), \quad \text{i.e., } z \in (I + \lambda A)(u).$$

The following theorem establishes the connection between (maximal) monotonicity and (firm) contractions.

Theorem 1.1.9 *Let $\lambda > 0$. An operator A on $\mathbb{H}$ is monotone if and only if its resolvent J_λ^A is a firm contraction. Moreover, A is maximal monotone if and only if J_λ^A is a firm contraction and*

$$\text{dom}(J_\lambda^A) = \mathbb{H}.$$

The following results describe a remarkable connection between resolvents of maximal monotone operators and firm contractions.

Proposition 1.1.9 (Minty) *An operator $A : \mathbb{H} \rightarrow \mathbb{H}$ is a firm contraction if and only if it is the resolvent of a maximal monotone operator $S : \mathbb{H} \rightrightarrows \mathbb{H}$, that is,*

$$A = J^S.$$

Proposition 1.1.10 *Let $\mathbb{H}$ be a Hilbert space and let $C \subset \mathbb{H}$ be a nonempty closed convex set. Then the resolvent of the normal cone operator N_C coincides with the Euclidean projection onto C , namely*

$$J_\lambda^{N_C} = P_C.$$

1.1.9 Proximal operator of a convex function

The proximal operator is a fundamental tool in convex analysis and optimization, particularly in problems involving nonsmooth functions. All the facts presented in this section have been previously established in (see [37, 38])

Theorem 1.1.10 *Let $\mathbb{H}$ be a real Hilbert space, let $K \subset \mathbb{H}$ be a nonempty closed convex set, and let*

$$f : K \rightarrow \mathbb{R} \cup \{+\infty\}$$

be a proper, convex, and lower semicontinuous function. Assume that $\text{dom } f \cap K \neq \emptyset$.

1. *If f is coercive or if K is bounded, then there exists a point $x^* \in K$ such that*

$$f(x^*) = \inf_{x \in K} f(x).$$

2. *Moreover, if f is strictly convex, then this minimizer x^* is unique.*

Proposition 1.1.11 (The proximal point) *Let $\mathbb{H}$ be a real Hilbert space, let $f \in \Gamma_0(\mathbb{H})$, and let $\lambda > 0$. Then, for any $x \in \mathbb{H}$, there exists a unique point $x^* \in \mathbb{H}$ such that*

$$f(x^*) + \frac{1}{2\lambda} \|x^* - x\|^2 = \inf_{y \in \mathbb{H}} \left\{ f(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\}.$$

Definition 1.1.11 (Proximal operator) *Let $\mathbb{H}$ be a real Hilbert space and let $f \in \Gamma_0(\mathbb{H})$. For a parameter $\lambda > 0$, the proximal operator of f is defined by*

$$\text{prox}_{\lambda f}(x) := \arg \min_{y \in \mathbb{H}} \left\{ f(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\}, \quad x \in \mathbb{H}.$$

Example 1.1.17 *Set $f = \|\cdot\|$. Then*

$$\text{prox}_f x = \begin{cases} \left(1 - \frac{1}{\|x\|}\right) x, & \text{if } \|x\| > 1, \\ 0, & \text{if } \|x\| \leq 1. \end{cases}$$

Proof 1.1.2 *Set*

$$g(y) = \|y\| + \frac{1}{2} \|x - y\|^2.$$

If $x = 0$, then clearly $y = 0$ is the minimizer of g .

We now consider two cases.

Case 1: $\|x\| \leq 1$.

If $y = 0$, then

$$g(0) = \frac{1}{2}\|x\|^2.$$

If $\|y\| = \|x\|$, then

$$\begin{aligned} g(y) &= \|y\| + \frac{1}{2}\|x\|^2 - \langle x, y \rangle + \frac{1}{2}\|y\|^2 \\ &\geq \|y\| + \frac{1}{2}\|x\|^2 - \|x\| \|y\| + \frac{1}{2}\|y\|^2 \\ &= \|x\| + \frac{1}{2}\|x\|^2 - \|x\|^2 + \frac{1}{2}\|x\|^2 \\ &= \|x\| \\ &\geq \frac{1}{2}\|x\|^2. \end{aligned}$$

Hence, any y such that $\|y\| = \|x\|$ is not a minimizer. Clearly, if $\|y\| > \|x\|$, then $g(y) > g(0)$.

Finally, if $\|y\| = \lambda\|x\|$ for some $\lambda \in (0, 1)$, then

$$\begin{aligned} g(y) &\geq \|y\| + \frac{1}{2}\|x\|^2 - \|x\| \|y\| + \frac{1}{2}\|y\|^2 \\ &= \lambda\|x\| + \frac{1}{2}\|x\|^2 - \lambda\|x\|^2 + \frac{1}{2}\lambda^2\|x\|^2 \\ &= \lambda(\|x\| - \|x\|^2) + \frac{1}{2}(1 + \lambda^2)\|x\|^2. \end{aligned}$$

The first term is greater than or equal to zero, while the second term is greater than $\frac{1}{2}\|x\|^2$. Altogether, $y = 0$ is the minimizer if $\|x\| \leq 1$.

Case 2: $\|x\| > 1$.

When $\|x\| > 1$, we have $g(x) > g(0)$. Thus $y = 0$ is not the minimizer. Since g is convex and differentiable, differentiating with respect to y and setting the gradient equal to zero yields

$$\frac{y}{\|y\|} - (x - y) = 0. \tag{1.7}$$

Equivalently,

$$x = y \left(\frac{1}{\|y\|} + 1 \right).$$

Taking norms on both sides,

$$\|x\| = \|y\| \left(\frac{1}{\|y\|} + 1 \right),$$

which implies,

$$\|x\| = 1 + \|y\| \quad \Longleftrightarrow \quad \|y\| = \|x\| - 1.$$

Therefore, if $\|x\| > 1$, equation (1.7) becomes

$$y = x \left(1 - \frac{1}{\|x\|} \right),$$

and if $\|x\| \leq 1$, then $y = 0$ is the minimizer.

Example 1.1.18 (Projection onto a convex set) Let $C \subset \mathbb{H}$ be a nonempty closed convex set and let $f = \iota_C$ be the indicator function of C , defined by

$$\iota_C(y) = \begin{cases} 0, & y \in C, \\ +\infty, & y \notin C. \end{cases}$$

Then, for any $\lambda > 0$ and any $x \in \mathbb{H}$, the proximal operator satisfies

$$\text{prox}_{\lambda \iota_C}(x) = \arg \min_{y \in C} \frac{1}{2\lambda} \|y - x\|^2 = P_C(x),$$

where $P_C(x)$ denotes the orthogonal projection of x onto C .

Example 1.1.19 (Quadratic function) Let $f : \mathbb{H} \rightarrow \mathbb{R}$ be defined by

$$f(y) = \frac{1}{2} \|y\|^2.$$

Then, for any $\lambda > 0$ and any $x \in \mathbb{H}$, the proximal operator is given explicitly by

$$\text{prox}_{\lambda f}(x) = \frac{1}{1 + \lambda} x.$$

Example 1.1.20 (Absolute value and soft-thresholding) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(y) = |y|.$$

Then, for any $\lambda > 0$ and any $x \in \mathbb{R}$, the proximal operator is given by

$$\text{prox}_{\lambda f}(x) = \begin{cases} x - \lambda, & x > \lambda, \\ 0, & |x| \leq \lambda, \\ x + \lambda, & x < -\lambda. \end{cases}$$

This operator is known as the soft-thresholding operator.

Remark 1.1.4 1. Strong monotonicity implies monotonicity.

2. Observe that an operator A is strongly monotone with modulus ρ if and only if the operator $A - \rho I$ is monotone, where I denotes the identity operator.

The strong monotonicity of the inverse operator A^{-1} is sometimes called the Dunn property or the co-coercivity of A . We say that A is γ -co-coercive if and only if A^{-1} is γ -strongly monotone. This property can be equivalently expressed as

$$\langle x^* - y^*, x - y \rangle \geq \gamma \|x^* - y^*\|^2, \quad \forall (x, x^*), (y, y^*) \in \text{Gr}(A).$$

An operator that is both strongly monotone and Lipschitz continuous is co-coercive. More precisely, the following result holds.

Proposition 1.1.12 *If A is Lipschitz continuous with constant L and strongly monotone with modulus β , then A is co-coercive with modulus β/L^2 .*

Every γ -co-coercive operator is monotone and $1/\gamma$ -Lipschitz continuous, but it is not necessarily strongly monotone (for example, a constant mapping).

Conversely, a monotone operator is not necessarily co-coercive.

1.1.10 Fundamental convergence results

Lemma 1.1.4 *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X , and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1.$$

Suppose that

$$x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n \quad \text{for all } n \geq 0,$$

and

$$\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0.$$

Lemma 1.1.5 *Let C be a nonempty, closed, and convex subset of a Hilbert space $\mathbb{H}$, and let $S : C \rightarrow C$ be a nonexpansive mapping. Then the mapping $I - S$ is demiclosed at zero; that is, if $\{x_n\}$ is a sequence in C such that*

$$x_n \rightharpoonup x \quad \text{and} \quad \|x_n - Sx_n\| \rightarrow 0,$$

then $x \in F(S)$.

Lemma 1.1.6 Suppose that $\mathbb{H}$ is a real Hilbert space and that $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathbb{H}$ such that

$$\limsup_{n \rightarrow \infty} \|x_n\| \leq r, \quad \limsup_{n \rightarrow \infty} \|y_n\| \leq r, \quad \text{and} \quad \lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n) y_n\| = r$$

hold for some $r > 0$. Then

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0.$$

Lemma 1.1.7 Assume that $\{s_n\}$ is a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - \gamma_n) s_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

1. $\sum_{n=1}^{\infty} \gamma_n = \infty$;
2. $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then

$$\lim_{n \rightarrow \infty} s_n = 0.$$

Chapter 2

A brief presentation of algorithms proposed for solving the zero-finding problem for maximal monotone operators.

2.1 Historical Background and Introduction

The problem of solving equations and inclusions involving maximal monotone operators plays a fundamental role in modern nonlinear analysis, convex optimization, and variational theory. Its importance arises both from its rich theoretical structure and from its wide range of applications in areas such as partial differential equations, mechanics, signal processing, and machine learning (see [3, 22, 25, 27, 39]).

The foundations of monotone operator theory were laid in the early 1960s through the seminal works of Minty and Browder (see [24, 40]). Their contributions established deep connections between monotonicity, maximality, and the solvability of nonlinear operator equations in Hilbert spaces. In particular, Minty's characterization of maximal monotone operators provided a crucial existence and uniqueness framework for inclusions of the form

$$0 \in Ax,$$

which later became the basis for the development of iterative algorithms.

2.1.1 Algorithm for a Single Maximal Monotone Operator

The first systematic algorithm for solving inclusions involving a single maximal monotone operator was the proximal point algorithm, introduced independently by Martinet in 1970 and Rockafellar in 1976. This method consists of iteratively applying the resolvent of the operator, which is firmly nonexpansive under maximal monotonicity. The proximal point algorithm exhibits strong theoretical convergence properties and serves as a unifying framework for many classical optimization methods.

Theorem 2.1.1 *Let $\mathbb{H}$ be a real Hilbert space and $A : \mathbb{H} \rightrightarrows \mathbb{H}$ be a maximal monotone operator.*

Let $\{z_n\}_{n \geq 0} \subset \mathbb{H}$ be a sequence generated by

$$z_{n+1} = (1 - \rho_n)z_n + \rho_n w_n, \quad n \geq 0,$$

where for each $n \geq 0$,

$$\|w_n - J_{\lambda_n}^A(z_n)\| \leq \varepsilon_n,$$

Assume that the sequences $\varepsilon_n \in [0, \infty)$, $\rho_n \in]0, 2[$, and $\lambda_n \in]0, \infty[$ satisfy

$$E_1 = \sum_{n=0}^{\infty} \varepsilon_n < \infty, \quad \Delta_1 = \inf_{n \geq 0} \rho_n > 0, \quad \Delta_2 = \sup_{n \geq 0} \rho_n < 2, \quad \tilde{\lambda} = \inf_{n \geq 0} \lambda_n > 0.$$

If the set of zeros of A ,

$$\text{zer}(A) := \{x \in \mathbb{H} \mid 0 \in Ax\},$$

is nonempty, then the sequence $\{z_n\}$ converges weakly to a point in $\text{zer}(A)$.

Proof 2.1.1 *Assume that $\text{zer}(A) \neq \emptyset$ and define, for each n ,*

$$S_n := I - J_{\lambda_n}^A = I - (I + \lambda_n A)^{-1}.$$

Since every zero of A is a fixed point of $J_{\lambda_n}^A$, it follows that each zero of A belongs to $\text{zer}(S_n)$ for all n .

Set

$$\tilde{z}_{n+1} = (1 - \rho_n)z_n + \rho_n J_{\lambda_n}^A(z_n) = (I - \rho_n S_n)(z_n).$$

Let $\bar{z} \in \text{zer}(A)$. Then

$$\begin{aligned} \|\tilde{z}_{n+1} - \bar{z}\|^2 &= \|z_n - \rho_n S_n(z_n) - \bar{z}\|^2 \\ &= \|z_n - \bar{z}\|^2 - 2\rho_n \langle z_n - \bar{z}, S_n(z_n) \rangle + \rho_n^2 \|S_n(z_n)\|^2. \end{aligned}$$

Because S_n is firmly nonexpansive and $S_n(\bar{z}) = 0$, we obtain

$$\|\tilde{z}_{n+1} - \bar{z}\|^2 \leq \|z_n - \bar{z}\|^2 - \rho_n(2 - \rho_n)\|S_n(z_n)\|^2.$$

Using $\Delta_1 \leq \rho_n \leq \Delta_2 < 2$, it follows that

$$\|\tilde{z}_{n+1} - \bar{z}\|^2 \leq \|z_n - \bar{z}\|^2 - \Delta_1(2 - \Delta_2)\|S_n(z_n)\|^2,$$

and hence

$$\|\tilde{z}_{n+1} - \bar{z}\| \leq \|z_n - \bar{z}\|.$$

Since $\|w_n - J_{\lambda_n A}(z_n)\| \leq \varepsilon_n$ and $z_{n+1} = (1 - \rho_n)z_n + \rho_n w_n$, we have

$$\|z_{n+1} - \tilde{z}_{n+1}\| \leq \rho_n \varepsilon_n.$$

Therefore,

$$\|z_{n+1} - \bar{z}\| \leq \|\tilde{z}_{n+1} - \bar{z}\| + \|z_{n+1} - \tilde{z}_{n+1}\| \leq \|z_n - \bar{z}\| + \rho_n \varepsilon_n.$$

Iterating this inequality gives

$$\|z_{n+1} - \bar{z}\| \leq \|z_0 - \bar{z}\| + \sum_{i=0}^n \rho_i \varepsilon_i \leq \|z_0 - \bar{z}\| + 2E_1,$$

so $\{z_n\}$ is bounded.

Expanding again,

$$\begin{aligned} \|z_{n+1} - \bar{z}\|^2 &= \|\tilde{z}_{n+1} - \bar{z} + (z_{n+1} - \tilde{z}_{n+1})\|^2 \\ &\leq \|\tilde{z}_{n+1} - \bar{z}\|^2 + 2\|\tilde{z}_{n+1} - \bar{z}\|\|z_{n+1} - \tilde{z}_{n+1}\| + \|z_{n+1} - \tilde{z}_{n+1}\|^2 \\ &\leq \|z_n - \bar{z}\|^2 - \Delta_1(2 - \Delta_2)\|S_n(z_n)\|^2 \\ &\quad + 2\rho_n \varepsilon_n (\|z_0 - \bar{z}\| + 2E_1) + \rho_n^2 \varepsilon_n^2. \end{aligned}$$

Since $\sum \varepsilon_n < \infty$, also $\sum \varepsilon_n^2 < \infty$. Summing the above inequality yields

$$\sum_{n=0}^{\infty} \|S_n(z_n)\|^2 < \infty, \quad \text{hence} \quad S_n(z_n) \rightarrow 0.$$

For each n , let $(x_n, y_n) \in A$ satisfy

$$x_n + \lambda_n y_n = z_n.$$

Then $S_n(z_n) = z_n - x_n$, so $z_n - x_n \rightarrow 0$. Because $\inf_n \lambda_n > 0$, we also have

$$y_n = \lambda_n^{-1} S_n(z_n) \rightarrow 0.$$

Since $\{z_n\}$ is bounded, it admits weak cluster points. Let z^∞ be one such cluster point and $z_{n(j)} \rightharpoonup z^\infty$. Then $x_{n(j)} \rightharpoonup z^\infty$ and $y_{n(j)} \rightarrow 0$. For any $(x, y) \in A$, monotonicity gives

$$\langle x - x_n, y - y_n \rangle \geq 0.$$

Passing to the limit along $n(j)$ yields

$$\langle x - z^\infty, y \rangle \geq 0.$$

By maximality of A , this implies $(z^\infty, 0) \in A$, so $z^\infty \in \text{zer}(A)$.

To prove uniqueness of the weak cluster point, let z_1^∞ and z_2^∞ be two such points. For any zero $\bar{z}$ of A , the limit

$$\alpha^* = \lim_{n \rightarrow \infty} \|z_n - \bar{z}\|$$

exists. In particular,

$$\alpha_i = \lim_{n \rightarrow \infty} \|z_n - z_i^\infty\|, \quad i = 1, 2,$$

exist and are finite. Using

$$\|z_n - z_2^\infty\|^2 = \|z_n - z_1^\infty\|^2 + 2\langle z_n - z_1^\infty, z_1^\infty - z_2^\infty \rangle + \|z_1^\infty - z_2^\infty\|^2,$$

and passing to the limit gives

$$\lim_{n \rightarrow \infty} \langle z_n - z_1^\infty, z_1^\infty - z_2^\infty \rangle = \frac{1}{2}(\alpha_2^2 - \alpha_1^2 - \|z_1^\infty - z_2^\infty\|^2).$$

Since z_1^∞ is a weak cluster point, the left-hand side is zero, hence

$$\alpha_2^2 = \alpha_1^2 + \|z_1^\infty - z_2^\infty\|^2.$$

Interchanging the roles of z_1^∞ and z_2^∞ gives

$$\alpha_1^2 = \alpha_2^2 + \|z_1^\infty - z_2^\infty\|^2,$$

which implies

$$\|z_1^\infty - z_2^\infty\| = 0.$$

Thus $z_1^\infty = z_2^\infty$, and the sequence $\{z_n\}$ has a unique weak cluster point. Therefore $z_n \rightharpoonup z^\infty \in \text{zer}(A)$.

2.2 Algorithms for the Sum of Two Maximal Monotone Operators

In this section, we employ the forward-backward splitting algorithm to derive strong convergence results for approximating a zero of the sum of an α -inverse strongly monotone operator and a maximal monotone operator.

To establish our first main result, we adopt the analytical approach introduced by Yao and Shahzad. [41, 42].

Theorem 2.2.1 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $A : C \rightarrow H$ be an α -inverse strongly monotone mapping and let B be a maximal monotone operator from H into 2^H such that $\text{dom}(B) \subset C$ and $(A + B)^{-1}(0) \neq \emptyset$. For $\lambda > 0$, denote by J_λ^B the resolvent of B , that is,*

$$J_\lambda^B = (I + \lambda B)^{-1}.$$

Let the sequence $\{z_n\}$ be generated by

$$\begin{cases} w_n = J_{\lambda_n}(z_n - \lambda_n A z_n), \\ z_{n+1} = \alpha_n z_n + \beta_n w_n + \gamma_n e_n, \end{cases} \quad (2.1)$$

where $\{e_n\}$ is an error sequence in H , $z_1 \in H$, $\alpha_n, \beta_n, \gamma_n \in (0, 1)$, and

$$\alpha_n + \beta_n + \gamma_n = 1.$$

Assume that the control sequences satisfy the following conditions:

1. $\lim_{n \rightarrow \infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$;
2. $e_n \rightarrow 0$;
3. there exist constants $0 < a \leq \alpha_n \leq b < 1$ and $0 < c \leq \beta_n \leq d < 1$;
4. $0 < \varepsilon \leq \lambda_n \leq 2\alpha$ and $\lim_{n \rightarrow \infty} (\lambda_n - \lambda_{n+1}) = 0$.

Then the sequence $\{z_n\}$ converges strongly to a point $q \in (A + B)^{-1}(0)$, where

$$q = P_{(A+B)^{-1}(0)}(0).$$

Proof 2.2.1 First, we show that the sequence $\{z_n\}$ generated by algorithm (2.1) is bounded. Fix $p \in (A + B)^{-1}(0)$. Since J_{λ_n} is nonexpansive, we obtain

$$\begin{aligned}\|w_n - p\| &= \|J_{\lambda_n}(z_n - \lambda_n A z_n) - J_{\lambda_n}(p - \lambda_n A p)\| \\ &\leq \|(I - \lambda_n A)z_n - (I - \lambda_n A)p\| \\ &\leq \|z_n - p\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|z_{n+1} - p\| &\leq \alpha_n \|z_n - p\| + \beta_n \|w_n - p\| + \gamma_n \|e_n - p\| \\ &\leq (1 - \gamma_n) \|z_n - p\| + \gamma_n (\|e_n\| + \|p\|) \\ &\leq (1 - \gamma_n) \|z_n - p\| + \gamma_n M_1 \\ &\leq \max\{\|z_n - p\|, M_1\},\end{aligned}$$

where $\sup_n \|e_n\| + \|p\| \leq M_1$ for some $M_1 > 0$. By induction on n , we obtain

$$\|z_{n+1} - p\| \leq \max\{\|z_1 - p\|, M_1\}.$$

Hence, the sequence $\{z_n\}$ is bounded, and so is $\{w_n\}$.

Define

$$z_{n+1} = \alpha_n z_n + (1 - \alpha_n) v_n \quad \text{for all } n \geq 0.$$

Then, from (2.1), we have

$$\begin{aligned}v_{n+1} - v_n &= \frac{z_{n+2} - \alpha_{n+1} z_{n+1}}{1 - \alpha_{n+1}} - \frac{z_{n+1} - \alpha_n z_n}{1 - \alpha_n} \\ &= \frac{\beta_{n+1} w_{n+1} + \gamma_{n+1} e_{n+1}}{1 - \alpha_{n+1}} - \frac{\beta_n w_n + \gamma_n e_n}{1 - \alpha_n} \\ &= \frac{\beta_{n+1}}{1 - \alpha_{n+1}} (w_{n+1} - w_n) + \left(\frac{\beta_{n+1}}{1 - \alpha_{n+1}} - \frac{\beta_n}{1 - \alpha_n} \right) w_n \\ &\quad + \frac{\gamma_{n+1} e_{n+1}}{1 - \alpha_{n+1}} - \frac{\gamma_n e_n}{1 - \alpha_n}.\end{aligned}$$

Set $u_n = (I - \lambda_n A)z_n$. It follows from Lemma 1.1 that ...

$$\|u_{n+1} - u_n\| \leq \|z_{n+1} - z_n\|.$$

If $\lambda_n \leq \lambda_{n+1}$, then by Lemma 1.2 we have

$$w_{n+1} = J_{\lambda_{n+1}}(u_{n+1}) = J_{\lambda_n} \left(\frac{\lambda_n}{\lambda_{n+1}} u_{n+1} + \left(1 - \frac{\lambda_n}{\lambda_{n+1}} \right) J_{\lambda_{n+1}}(u_{n+1}) \right).$$

Hence, we obtain

$$\begin{aligned}
\|w_{n+1} - w_n\| &= \|J_{\lambda_n}(u_{n+1}) - J_{\lambda_n}(u_n)\| \\
&\leq \frac{\lambda_n}{\lambda_{n+1}} \|u_{n+1} - u_n\| + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right) \|w_{n+1} - u_n\| \\
&\leq \|z_{n+1} - z_n\| + \frac{1}{\varepsilon} |\lambda_{n+1} - \lambda_n| \|w_{n+1} - u_n\|.
\end{aligned} \tag{2}$$

If $\lambda_n > \lambda_{n+1}$, again using Lemma 1.2, we obtain

$$\begin{aligned}
\|w_n - w_{n+1}\| &= \|J_{\lambda_n}(u_n) - J_{\lambda_{n+1}}(u_{n+1})\| \\
&\leq \frac{\lambda_{n+1}}{\lambda_n} \|u_n - u_{n+1}\| + \left(1 - \frac{\lambda_{n+1}}{\lambda_n}\right) \|w_n - u_{n+1}\| \\
&\leq \|z_{n+1} - z_n\| + \frac{1}{\varepsilon} |\lambda_{n+1} - \lambda_n| \|w_n - u_{n+1}\|.
\end{aligned} \tag{3}$$

Therefore, from (2) and (3), we infer that

$$\|w_{n+1} - w_n\| \leq \|z_{n+1} - z_n\| + \frac{M_2}{\varepsilon} |\lambda_{n+1} - \lambda_n|,$$

where M_2 satisfies

$$\sup\{\|w_{n+1} - u_n\|, \|w_n\| + \|u_n\| : n \geq 0\} \leq M_2.$$

Hence, we have

$$\begin{aligned}
\|v_{n+1} - v_n\| &\leq \frac{\beta_{n+1}}{1 - \alpha_{n+1}} \|w_{n+1} - w_n\| + \left| \frac{\beta_{n+1}}{1 - \alpha_{n+1}} - \frac{\beta_n}{1 - \alpha_n} \right| \|w_n\| \\
&\quad + \frac{\gamma_{n+1}}{1 - \alpha_{n+1}} \|e_{n+1}\| + \frac{\gamma_n}{1 - \alpha_n} \|e_n\| \\
&\leq \frac{\beta_{n+1}}{1 - \alpha_{n+1}} \|z_{n+1} - z_n\| + \frac{\beta_{n+1}}{1 - \alpha_{n+1}} \frac{M_2}{\varepsilon} |\lambda_{n+1} - \lambda_n| \\
&\quad + \left| \frac{\beta_{n+1}}{1 - \alpha_{n+1}} - \frac{\beta_n}{1 - \alpha_n} \right| M_2 + \left(\frac{\gamma_{n+1}}{1 - \alpha_{n+1}} + \frac{\gamma_n}{1 - \alpha_n} \right) M_1.
\end{aligned}$$

which implies that

$$\begin{aligned}
\limsup_{n \rightarrow \infty} (\|v_{n+1} - v_n\| - \|z_{n+1} - z_n\|) &\leq \limsup_{n \rightarrow \infty} \left[\frac{\beta_{n+1}}{1 - \alpha_{n+1}} \frac{M_2}{\varepsilon} |\lambda_{n+1} - \lambda_n| \right. \\
&\quad \left. + \left| \frac{\beta_{n+1}}{1 - \alpha_{n+1}} - \frac{\beta_n}{1 - \alpha_n} \right| M_2 + \frac{\gamma_{n+1}}{1 - \alpha_{n+1}} M_1 + \frac{\gamma_n}{1 - \alpha_n} M_1 \right] \\
&= 0.
\end{aligned}$$

By Lemma 1.4, we have

$$\lim_{n \rightarrow \infty} \|z_n - v_n\| = 0,$$

and hence,

$$\lim_{n \rightarrow \infty} \|z_{n+1} - z_n\| = \lim_{n \rightarrow \infty} (1 - \gamma_n) \|v_n - z_n\| = 0.$$

Moreover, we observe that

$$\begin{aligned} \|z_n - w_n\| &\leq \|z_n - z_{n+1}\| + \|z_{n+1} - w_n\| \\ &\leq \|z_n - z_{n+1}\| + \alpha_n \|z_n - w_n\| + \gamma_n \|e_n - w_n\|. \end{aligned}$$

Therefore,

$$\|z_n - w_n\| \leq \frac{1}{1 - \alpha_n} \|z_n - z_{n+1}\| + \frac{\gamma_n}{1 - \alpha_n} \|e_n - w_n\|,$$

which implies that

$$\lim_{n \rightarrow \infty} \|z_n - w_n\| = 0.$$

Next, we show that $w_\omega(z_n) \subset (A + B)^{-1}(0)$. Let $p \in w_\omega(z_n)$. Then there exists a subsequence $\{z_{n_j}\}$ which converges weakly to p . Since J_{λ_n} is the resolvent of B , by the definition of w_n we have

$$\frac{z_{n_j} - w_{n_j}}{\lambda_{n_j}} - Az_{n_j} \in Bw_{n_j}.$$

By the monotonicity of B , it follows that

$$0 \leq \left\langle \frac{z_{n_j} - w_{n_j}}{\lambda_{n_j}} - Az_{n_j} - w, w_{n_j} - u \right\rangle \quad (2.2)$$

for each $(u, w) \in B$. Taking the limit in (2.2) as $j \rightarrow \infty$, we obtain

$$\langle -Ap - w, p - u \rangle \geq 0.$$

The maximal monotonicity of B implies that $-Ap \in Bp$, and hence $p \in (A + B)^{-1}(0)$.

Set $q = P_{(A+B)^{-1}(0)}(0)$ and take a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $z_{n_j} \rightharpoonup z$ and

$$\limsup_{n \rightarrow \infty} \langle q, q - z_n \rangle = \lim_{j \rightarrow \infty} \langle q, q - z_{n_j} \rangle.$$

Thus, $z \in (A + B)^{-1}(0)$ and hence

$$\limsup_{n \rightarrow \infty} \langle q, q - z_n \rangle = \langle q, q - z \rangle \leq 0.$$

Finally, we show that the sequence $\{z_n\}$ converges strongly to $q = P_{(A+B)^{-1}(0)}(0)$. By (2.1), we have

$$\begin{aligned}
\|z_{n+1} - q\|^2 &= \|\alpha_n(z_n - q) + \beta_n(w_n - q) - \gamma_n q + \gamma_n e_n\|^2 \\
&\leq \|\alpha_n(z_n - q) + \beta_n(w_n - q) - \gamma_n q\|^2 + \gamma_n \|e_n\|^2 \\
&= \|\alpha_n(z_n - q) + \beta_n(w_n - q)\|^2 - 2\gamma_n \langle z_n - q + w_n - q, q \rangle + \gamma_n^2 \|q\|^2 + \gamma_n \|e_n\|^2 \\
&\leq \|\alpha_n(z_n - q) + \beta_n(w_n - q)\|^2 - 2\gamma_n \langle z_{n+1} - q - \gamma_n e_n, q \rangle + M_3 \gamma_n \|e_n\| \\
&\leq (\alpha_n \|z_n - q\| + \beta_n \|w_n - q\|)^2 - 2\gamma_n \langle z_{n+1} - q, q \rangle + 2\gamma_n^2 \|e_n\| \|q\| + M_3 \gamma_n \|e_n\| \\
&\leq (1 - \gamma_n) \|z_n - q\|^2 + \gamma_n [2\langle q, q - z_{n+1} \rangle + (2\|q\| + M_3) \|e_n\|] \\
&= (1 - \gamma_n) \|z_n - q\|^2 + \delta_n,
\end{aligned}$$

where M_3 satisfies

$$\sup \left\{ \|e_n\| + 2\|\alpha_n(z_n - q) + \beta_n(w_n - q) - \gamma_n q\| : n \geq 0 \right\} \leq M_3.$$

Then Lemma 2.7 implies that $z_n \rightarrow q$ as $n \rightarrow \infty$, and this completes the proof.

The problem of finding solutions to inclusions of the form

$$0 \in (A + B)x, \tag{2.3}$$

where A and B are maximal monotone operators, naturally arises in constrained optimization, equilibrium problems, and saddle-point formulations. This setting led to the development of *operator splitting methods*, which decompose the original problem into simpler subproblems associated with each operator (see [43–48]).

One of the most influential methods in this class is the *Douglas–Rachford splitting algorithm*, which was originally introduced in the 1950s for the numerical solution of partial differential equations and later reformulated within the framework of monotone operator theory by Lions and Mercier in 1979 (see [49]).

Within this framework, the problem under consideration reduces to a classical case that has been extensively studied in the literature. In this setting, the solution is obtained via the *Douglas–Rachford iterative algorithm (DRIA)*, defined by the recursion

$$Z_{n+1} = J_\lambda^B(2J_\lambda^A - I)(Z_n) + (I - J_\lambda^B)(Z_n), \quad n \geq 0.$$

Closely related is the *Peaceman–Rachford splitting method*, which, although more aggressive, requires stronger assumptions to ensure convergence.

Another important class of methods is the *forward-backward splitting algorithm*, applicable when one operator is maximal monotone and the other is cocoercive or Lipschitz continuous. This approach, further developed by Passty, Lions, and Mercier, has become a cornerstone of modern convex optimization. In 2020 A. Beddani present a new algorithm for solving the problem which defined by two maximal monotone operators(see [50]), he propose a vector function which has relation with the set of zeroes of the problem (2.3), it defined as:

$$f_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad f_\lambda(x, y) = \begin{pmatrix} J_\lambda^A(x) - \frac{x+y}{2} \\ J_\lambda^B(y) - \frac{x+y}{2} \end{pmatrix}.$$

For all $\lambda > 0$, we consider the following problem:

$$\min_{(x,y) \in \mathbb{R} \times \mathbb{R}} \|f_\lambda(x, y)\| = 0. \quad (2.4)$$

It is well known that problem(2.4) admits several algorithms for computing one of its solutions (see [51]).

Theorem 2.2.2 *For all $\lambda > 0$, problem(2.3) has a solution if and only if problem(2.4) has a solution.*

Proof 2.2.2 *Let $\lambda > 0$ and assume that (x, y) is a solution of problem(2.4). Then*

$$\begin{cases} J_\lambda^A(x) = \frac{x+y}{2}, \\ J_\lambda^B(y) = \frac{x+y}{2}. \end{cases}$$

Therefore,

$$\begin{cases} y = 2J_\lambda^A(x) - x, \\ 2J_\lambda^B(y) = x + y. \end{cases}$$

Replacing y by $2J_\lambda^A(x) - x$ in the identity $2J_\lambda^B(y) = x + y$, we obtain

$$J_\lambda^B(2J_\lambda^A(x) - x) = J_\lambda^A(x).$$

So,

$$2J_\lambda^A(x) - x \in J_\lambda^A(x) + \lambda B(J_\lambda^A(x)).$$

That means

$$A_\lambda(x) \in B(J_\lambda^A(x)).$$

Since $A_\lambda(x) \in A(J_\lambda^A(x))$, we obtain

$$0 \in A(J_\lambda^A(x)) + B(J_\lambda^A(x)).$$

Consequently,

$$J_\lambda^A(x) \in (A + B)^{-1}(0).$$

Conversely, if $z \in (A + B)^{-1}(0)$, then $z \in \text{dom } A$. This implies that there exists an element $z' \in X$ such that

$$z' + z \in z + A(z).$$

By the definition of the resolvent of A , we get

$$z = J_\lambda^A(z' + z).$$

We conclude that, for all $z \in (A + B)^{-1}(0)$, there exists $s = z' + z \in X$ such that

$$z = J_\lambda^A(s).$$

Thus,

$$J_\lambda^A(s) \in (A + B)^{-1}(0).$$

Then

$$0 \in A(J_\lambda^A(s)) + B(J_\lambda^A(s)).$$

Therefore, there exists $a \in X$ such that

$$\begin{cases} a \in A(J_\lambda^A(s)), \\ -a \in B(J_\lambda^A(s)). \end{cases}$$

Hence,

$$\begin{cases} a + J_\lambda^A(s) \in J_\lambda^A(s) + A(J_\lambda^A(s)), \\ -a + J_\lambda^A(s) \in J_\lambda^A(s) + B(J_\lambda^A(s)). \end{cases}$$

By the definition of the resolvent of A , we have ...

$$\begin{cases} J_\lambda^A(s) = J_\lambda^A(J_\lambda^A(s) + a), \\ J_\lambda^A(s) = J_\lambda^B(J_\lambda^A(s) - a). \end{cases}$$

We can write $J_\lambda^A(s)$ as

$$J_\lambda^A(s) = \frac{x + y}{2},$$

where

$$\begin{cases} x = J_\lambda^A(s) + a, \\ y = J_\lambda^A(s) - a. \end{cases}$$

Then

$$\begin{cases} J_\lambda^A(x) = \frac{x+y}{2}, \\ J_\lambda^B(y) = \frac{x+y}{2}. \end{cases}$$

Consequently, problem (1.2) has a solution.

Remark 2.2.1 Let $\lambda > 0$ and define

$$S_\lambda = \{(x, y) \in X \times X : \|f_\lambda(x, y)\| = 0\} \quad \text{and} \quad S_\mu = \{(x, y) \in X \times X : \|f_\mu(x, y)\| = 0\}.$$

Then

$$\{J_\lambda^A(x) : (x, y) \in S_\lambda\} = \{J_\mu^A(x) : (x, y) \in S_\mu\}.$$

That is,

$$(A + B)^{-1}(0) = \{J_\lambda^A(x) : (x, y) \in S_\lambda\}, \quad \forall \lambda > 0.$$

Theorem 2.2.3 For all $\lambda > 0$, if

$$\theta_\lambda(a) = J_\lambda^B(2J_\lambda^A(a) - a) - J_\lambda^A(a) + a,$$

then

$$J_\lambda^A(S(\theta_\lambda)) = (A + B)^{-1}(0),$$

where $S(\theta_\lambda)$ denotes the set of all fixed points of θ_λ .

Proof 2.2.3 If a is a fixed point of the function θ_λ , then

$$a = J_\lambda^B(2J_\lambda^A(a) - a) - J_\lambda^A(a) + a \iff J_\lambda^B(2J_\lambda^A(a) - a) = J_\lambda^A(a).$$

This is equivalent to

$$2J_\lambda^A(a) - a \in J_\lambda^A(a) + \lambda B(J_\lambda^A(a)) \iff \frac{1}{\lambda}(J_\lambda^A(a) - a) \in B(J_\lambda^A(a)).$$

Since

$$A_\lambda(a) = \frac{1}{\lambda}(a - J_\lambda^A(a)) \in A(J_\lambda^A(a)),$$

we obtain

$$0 \in A(J_\lambda^A(a)) + B(J_\lambda^A(a)).$$

Example 2.2.1 Consider $A(x) = \{3x\}$, $B(x) = \{2x\}$, and $X = \mathbb{R}$. We seek a solution of the following problem:

$$0 \in A(x) + B(x). \quad (3.1)$$

Clearly, for all $\lambda > 0$,

$$J_\lambda^A(x) = \frac{x}{3\lambda + 1} \quad \text{and} \quad J_\lambda^B(x) = \frac{x}{2\lambda + 1}.$$

We define the function $f_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$f_\lambda(x, y) = \begin{pmatrix} J_\lambda^A(x) - \frac{x+y}{2} \\ J_\lambda^B(y) - \frac{x+y}{2} \end{pmatrix} = \begin{pmatrix} \frac{x}{3\lambda+1} - \frac{x+y}{2} \\ \frac{y}{2\lambda+1} - \frac{x+y}{2} \end{pmatrix}.$$

If $\|f_\lambda(x, y)\| = 0$, then

$$\begin{cases} \frac{x}{2\lambda+1} - \frac{x+y}{2} = 0, \\ \frac{y}{4\lambda+1} - \frac{x+y}{2} = 0. \end{cases}$$

Hence, $(x, y) = (0, 0)$. Applying Theorem (2.3), we obtain $J_\lambda^A(0) = 0$ as a solution of (2.3).

Numerical Example

We have:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(x, y) \mapsto f(x, y) = \begin{pmatrix} J_A^\lambda(x) - \frac{x+y}{2} \\ J_B^\lambda(y) - \frac{x+y}{2} \end{pmatrix}$$

The Jacobian matrix of f is:

$$J_f(x, y) = \begin{pmatrix} \frac{\partial J_A^\lambda}{\partial x}(x) - \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\partial J_B^\lambda}{\partial y}(y) - \frac{1}{2} \end{pmatrix}$$

We calculate the determinant of $J_f(x, y)$:

$$\begin{aligned} \det J_f(x, y) &= \left(\frac{\partial J_A^\lambda}{\partial x}(x) - \frac{1}{2} \right) \left(\frac{\partial J_B^\lambda}{\partial y}(y) - \frac{1}{2} \right) - \frac{1}{4} \\ &= \frac{\partial J_A^\lambda}{\partial x}(x) \frac{\partial J_B^\lambda}{\partial y}(y) - \frac{1}{2} \frac{\partial J_A^\lambda}{\partial x}(x) - \frac{1}{2} \frac{\partial J_B^\lambda}{\partial y}(y) \end{aligned}$$

Newton–Raphson method, or Newton method, is a powerful technique for solving the equations numerically:

$$(x_{k+1}, y_{k+1}) = (x_k, y_k) - J_f^{-1}(x_k, y_k) f(x_k, y_k), \quad k = 0, 1, 2, \dots$$

2.3 Conclusion

In this chapter, we reviewed a collection of previous works devoted to the study of maximal monotone operators, which constitute one of the fundamental pillars of functional analysis and optimization theory. Particular attention was given to the Douglas–Rachford splitting method as a prominent iterative scheme for computing the proximal point, together with a rigorous presentation of its convergence properties within appropriate frameworks.

The Douglas–Rachford algorithm is regarded as one of the most influential tools in this area, serving as a foundation for the development of more efficient iterative methods, either in terms of accelerating convergence or extending applicability to broader classes of problems. For this reason, we emphasized the main theoretical features that make this method a central reference in constructing modern algorithms for finding the zeros of the sum of two maximal monotone operators.

We also presented the approach proposed by A. Beddani, which is based on the introduction of a vector-valued function whose minimal values lead to the determination of the zeros of the sum of two maximal monotone operators. This perspective establishes a meaningful connection between convex analytical techniques and zero-finding methods, thereby enriching the theoretical structure of the problem and opening new avenues for algorithmic development.

Consequently, this chapter provides the theoretical and methodological foundation upon which our subsequent work is built. The approaches discussed herein will serve as essential tools in the following chapters, where we develop our own contributions and establish new results in this framework.

Chapter 3

Study of various cases for locating the zeroes of the sum of a finite family of maximal monotone operators

3.1 Problem statement

In this part we consider the real Hilbert space $\mathbb{H}$ and let $\{A_i\}_{i=1}^n$ be family of maximal monotone operators on H . We are focused on the monotone inclusion formula which interested of finding $x \in \mathbb{H}$ satisfying

$$0 \in A_1(x) + \cdots + A_n(x). \quad (3.1)$$

The solution set of the problem (3.1) is denoted by

$$(A_1 + \cdots + A_n)^{-1}(0).$$

The problem under consideration plays a significant role in various fields such as physics, chemistry, and electronics. Important contributions to this area were made by M. Thera and A. Moudafi (see, e.g., [44, 52]), whose work established foundational principles, especially for the case $n = 1, 2$.

The inclusion problem (3.1), in particular for $n = 2$, has attracted considerable attention from researchers whenever the solution set is nonempty. Representative references include [53–58]. It is worth noting that many of these works develop algorithms under the hypothesis that the solution set is nonempty.

In 2009, as discussed in [59], a new splitting methodology was introduced for algorithmic decomposition. The approach reformulates problem (3.1) by identifying an element in an

extended solution set denoted by

$$S(\{A_i\}_{i=1}^n),$$

which is embedded in the product space $\mathbb{H} \times \mathbb{H}^n$. This extended solution set is characterized as follows:

$$S(A_1, \dots, A_n) = \{(z, v_1, \dots, v_n) \in V : z \in \mathbb{H}, v_k \in A_k(z), k = 1, 2, \dots, n\},$$

where

$$V = \{(z, v_1, \dots, v_n) \in \mathbb{H} \times \mathbb{H}^n : v_1 + \dots + v_n = 0\}.$$

In 2017, M. Alimohammady introduced several algorithm aimed at obtaining common elements in S . This last also interest on finding the solutions of variational inequalities associated with inverse-strongly monotone operators, and finding elements in the set of fixed points of two infinite families of relatively nonexpansive mappings, In addition, he was interested in studying the zero set of two finite families of maximal monotone operators. These contributions provided a unified iterative framework for treating a broad class of nonlinear problems within Hilbert spaces.

Subsequently, in 2020, D.B. Wega and Z. Habtup proposed a novel and impactful algorithm motivated by the foundational principles established in earlier studies. Their approach was designed to identify a point in the extended solution set

$$S(A_1, \dots, A_n), \quad \text{for } n \geq 2,$$

where $A_1, \dots, A_n$ denote a family of maximal monotone mappings, under the assumption that

$$S(A_1, \dots, A_n) \neq \emptyset.$$

The conclusions of their study were strongly influenced by the theoretical framework previously developed in the literature. Beyond its theoretical significance, this line of research has stimulated further investigations into new directions and specialized applications. Recent advances have not only deepened the theoretical understanding of monotone operator theory and equilibrium problems but have also facilitated practical implementations, thereby highlighting the applicability of these foundational principles in related areas of optimization and nonlinear analysis.

In this context, The primary objective of this work is to prove the existence of solutions and to propose specific algorithmic procedures for effectively addressing the problem under consideration.

First, we focused on studying several cases of the problem:

$$0 \in A(x) + B(x) + \sum_{i=1}^n C_i(x), \quad (3.2)$$

where

- $A : \mathbb{H} \rightarrow 2^{\mathbb{H}}$ denote an α -inverse strongly monotone mapping.
- $B : \mathbb{H} \rightarrow 2^{\mathbb{H}}$ represents a β -strongly monotone operator.
- $\{C_i\}_{i=1}^n$ consisting of n nonexpansive operators, where each $C_i : \mathbb{H} \rightarrow \mathbb{H}$.

Let us introduce the operator $F(S_e)$ the set of all fixed points of the operator S_e , and this last defined as:

$$S_e = \left\{ x \in \mathbb{H} \mid 0 \in A(x) + B(x) + \sum_{i=1}^n C_i(x) \right\}.$$

Consider first the problem on two maximal monotone operators over a Hilbert space $\mathbb{H}$. It is stated as follows: find $x \in \mathbb{H}$ such that

$$0 \in A(x) + B(x). \quad (3.3)$$

So, let us present a simple algorithm using the Yosida approach, which can be used to solve (3.3).

Proposition 3.1.1 *For all $\iota > 0$, $\lambda > 0$, we have $A_\lambda(x) = A_\iota(x + \iota A_\lambda(x))$.*

Proof 3.1.1 *let $\iota > 0$, $\lambda > 0$, we have*

$$J_\lambda^A(x) = J_\iota^A \left(\frac{\iota}{\lambda} x + \left(1 - \frac{\iota}{\lambda} \right) J_\lambda^A(x) \right),$$

this equivalent to,

$$x - \lambda A_\lambda(x) = \frac{\iota}{\lambda} x + \left(1 - \frac{\iota}{\lambda} \right) (x - \lambda A_\lambda(x)) - \iota A_\iota \left(\frac{\iota}{\lambda} x + \left(1 - \frac{\iota}{\lambda} \right) (x - \lambda A_\lambda(x)) \right),$$

after simplifying,

$$x - \lambda A_\lambda(x) = \frac{\iota}{\lambda} x + x - \frac{\iota}{\lambda} x - \lambda A_\lambda(x) + \iota A_\lambda(x) - \iota A_\iota \left(\frac{\iota}{\lambda} x + \left(1 - \frac{\iota}{\lambda} \right) (x - \lambda A_\lambda(x)) \right),$$

therefore,

$$-\lambda A_\lambda(x) = -\lambda A_\lambda(x) + \iota A_\lambda(x) - \iota A_\iota \left(\frac{\iota}{\lambda} x + \left(1 - \frac{\iota}{\lambda} \right) (x - \lambda A_\lambda(x)) \right),$$

so we conclude,

$$A_\lambda(x) = A_\iota((x + \iota A_\lambda(x))).$$

This complete the proof.

Proposition 3.1.2 *for all $\lambda > 0$ we consider the function $\theta_\lambda : \mathbb{H} \rightarrow \mathbb{H}$ by*

$$\theta_\lambda(x) = x + A_\lambda(x) + B_\lambda(x - 2\lambda A_\lambda(x)).$$

If a is a fixed point of θ_λ , then $a - \lambda A_\lambda(a)$ is a solution of problem (3.3).

Proof 3.1.2 *Suppose that a is a fixed point of θ_λ . By definition, this implies:*

$$a + A_\lambda(a) + B_\lambda(a - 2\lambda A_\lambda(a)) = a.$$

Is equivalent to:

$$A_\lambda(a) + B_\lambda(a - 2\lambda A_\lambda(a)) = 0.$$

We set $t = a - \lambda A_\lambda(a)$. Then:

$$a = t + \lambda A_\lambda(a), \quad \text{and} \quad a - 2\lambda A_\lambda(a) = t - \lambda A_\lambda(a).$$

The substitution of these into the previous equation yields:

$$A_\lambda(a) + B_\lambda(t - \lambda A_\lambda(a)) = 0.$$

Thus, we have:

$$0 \in A(t) + B(t),$$

consequently, as asserted, t is a solution of the monotone inclusion (3.3)

Theorem 3.1.1 *For a fixed $\lambda > 0$, let the sequence $\{x_k\}$ be generated by the following iterative process:*

$$\begin{cases} x_0, x_1 \in \mathbb{H}, \\ x_{k+1} = (1 - \alpha)x_k + \alpha\theta_\lambda(x_k) + \epsilon_k(x_k - x_{k-1}), \quad \forall k \geq 1. \end{cases} \quad (3.4)$$

Assume that the relaxation parameter satisfies $0 < \alpha < 1$ and that the sequence of inertial effects $\{\epsilon_k\} \subset (0, 1)$ is summable, such that $\sum_{k=1}^{\infty} \epsilon_k < \infty$.

If $\{x_k\}$ strongly converges to a limit $l \in \mathbb{H}$, then the point $l - \lambda A_\lambda(l)$ is a solution to the problem described in (3.3).

We now turn to the study of the principal problem (3.2).

Theorem 3.1.2 *Let $\lambda > 0$, if $T_\lambda(x) = J_\lambda^B(-\lambda \sum_{i=1}^n C_i(J_\lambda^A(x)) - x + 2J_\lambda^A(x)) + \lambda A_\lambda(x)$ if $S \neq \emptyset$, then $\{J_\lambda^A(F(T_\lambda))\} \subseteq S$.*

Proof 3.1.3 Let $\bar{x}$ be a fixed point of T_λ , so:

$$\begin{aligned}
 \bar{x} \in F(T_\lambda) &\implies T_\lambda(\bar{x}) = \bar{x} \\
 &\implies J_\lambda^B \left(-\lambda \sum_{i=1}^n C_i(J_\lambda^A(\bar{x})) - \bar{x} + 2J_\lambda^A(\bar{x}) \right) + \lambda A_\lambda(\bar{x}) = \bar{x} \\
 &\implies \bar{x} - \lambda A_\lambda(\bar{x}) = J_\lambda^B \left(-\lambda \sum_{i=1}^n C_i(J_\lambda^A(\bar{x})) - \bar{x} + 2J_\lambda^A(\bar{x}) \right) \\
 &\implies J_\lambda^A(\bar{x}) = J_\lambda^B \left(-\lambda \sum_{i=1}^n C_i(J_\lambda^A(\bar{x})) - \bar{x} + 2J_\lambda^A(\bar{x}) \right) \\
 &\implies -\lambda \sum_{i=1}^n C_i(J_\lambda^A(\bar{x})) - \bar{x} + 2J_\lambda^A(\bar{x}) \in \lambda B(J_\lambda^A(\bar{x})) + J_\lambda^A(\bar{x}) \\
 &\implies -\bar{x} + J_\lambda^A(\bar{x}) \in \lambda \sum_{i=1}^n C_i(J_\lambda^A(\bar{x})) + \lambda B(J_\lambda^A(\bar{x})) \\
 &\quad \text{and } \bar{x} - J_\lambda^A(\bar{x}) \in \lambda A(J_\lambda^A(\bar{x})) \\
 &\implies 0 \in \lambda \sum_{i=1}^n C_i(J_\lambda^A(\bar{x})) + \lambda B(J_\lambda^A(\bar{x})) + \lambda A(J_\lambda^A(\bar{x})) \\
 &\implies J_\lambda^A(\bar{x}) \in \text{zer} \left(\sum_{i=1}^n C_i + A + B \right).
 \end{aligned}$$

3.1.1 Algorithm 1

In order to correctly construct this algorithms, we need the following condition The condition concerning the family $\{C_i\}_{1 \leq i \leq n}$ is precisely that each operator $I + \lambda C_i$ be bijective.

Proposition 3.1.3 for all $\lambda > 0$, if $F(T_\lambda) \neq \emptyset$, then the system defined as follow:

$$\left\{ \begin{array}{l} J_\lambda^A(x) = \frac{x + y + \sum_{i=1}^{i=n} z_i}{n + 2} \\ J_\lambda^B(y) = \frac{x + y + \sum_{i=1}^{i=n} z_i}{n + 2} \\ J_\lambda^{C_1}(z_1) = \frac{x + y + \sum_{i=1}^{i=n} z_i}{n + 2} \\ \cdot \\ \cdot \\ \cdot \\ J_\lambda^{C_n}(z_n) = \frac{x + y + \sum_{i=1}^{i=n} z_i}{n + 2}. \end{array} \right. \quad (3.5)$$

has a solution $(x, y, z_1, \dots, z_n) \in \mathbb{H}^{n+2}$.

Proof 3.1.4 let $F(T_\lambda) \neq \emptyset$, and we denote $\bar{x} \in \mathbb{H}$ fixed point of T_λ , this implies,

$$J_\lambda^A(\bar{x}) = J_\lambda^B(-\lambda \sum_{i=1}^n C_i(J_\lambda^A(\bar{x})) - \bar{x} + 2J_\lambda^A(\bar{x})).$$

let we pose,

$$\begin{cases} \bar{x} = x, \\ -\lambda \sum_{i=1}^n C_i(J_\lambda^A(\bar{x})) - \bar{x} + 2J_\lambda^A(\bar{x}) = y, \\ J_\lambda^A(\bar{x}) + \lambda C_i(J_\lambda^A(\bar{x})) = z_i. \end{cases}$$

Therefore,

$$\begin{cases} J_\lambda^A(x) = J_\lambda^B(y), \\ J_\lambda^A(x) = J_\lambda^{C_i}(z_i), \\ x + y + \sum_{i=1}^n z_i = (n+2)J_\lambda^A(x). \end{cases}$$

Consequently,

$$\begin{cases} J_\lambda^A(x) = \frac{x + y + \sum_{i=1}^n z_i}{n+2}, \\ J_\lambda^B(y) = \frac{x + y + \sum_{i=1}^n z_i}{n+2}, \\ J_\lambda^{C_1}(z_1) = \frac{x + y + \sum_{i=1}^n z_i}{n+2}, \\ \cdot \\ \cdot \\ \cdot \\ J_\lambda^{C_n}(z_n) = \frac{x + y + \sum_{i=1}^n z_i}{n+2}. \end{cases}$$

This implies that $(x, y, z_1, \dots, z_n)$ is solution of (3.5).

Theorem 3.1.3 for all $\lambda > 0$, if the system of sequences defined as:

$$\begin{cases} (x_0, y_0, z_{10}, \dots, z_{n0}) \in \mathbb{H}^{n+2}, \\ x_{k+1} = (n+2)J_\lambda^A(x_k) - y_k - \sum_{i=1}^n z_{ik}, \\ y_{k+1} = (n+2)J_\lambda^A(y_k) - x_k - \sum_{i=1}^n z_{ik}, \\ z_{ik+1} = J_\lambda^A(x_k) + \lambda C_i(J_\lambda^A(x_k)), \end{cases} \quad k \geq 0.$$

Converge to $(\bar{x}, \bar{y}, \bar{z}_1, \dots, \bar{z}_n)$ then $J_\lambda^A(\bar{x})$ is solution of (3.2).

Proof 3.1.5 let assume that the last system converge to $(\bar{x}, \bar{y}, \bar{z}_1, \dots, \bar{z}_n)$ in $\mathbb{H}^{n+2}$ so, we have

$$\begin{cases} \bar{x} = (n+2)J_\lambda^A(\bar{x}) - y_k - \sum_{i=1}^{i=n} \bar{z}_i, \\ \bar{y} = (n+2)J_\lambda^A(\bar{y}) - \bar{x} - \sum_{i=1}^{i=n} \bar{z}_i, \\ \bar{z}_i = J_\lambda^A(\bar{x}) + \lambda C_i(J_\lambda^A(\bar{x})). \end{cases}$$

Therefore,

$$\begin{cases} \bar{x} + \bar{y} + \sum_{i=1}^{i=n} \bar{z}_i = (n+2)J_\lambda^A(\bar{x}), \\ \bar{y} + \bar{x} + \sum_{i=1}^{i=n} \bar{z}_i = (n+2)J_\lambda^A(\bar{y}), \\ \bar{z}_i = J_\lambda^A \bar{x} + \lambda C_i(J_\lambda^A(\bar{x})). \end{cases}$$

Then,

$$\bar{x} + \bar{y} + nJ_\lambda^A \bar{x} + \lambda \sum_{i=1}^{i=n} C_i(J_\lambda^A \bar{x}) = (n+2)J_\lambda^A(\bar{x}),$$

After simplifying,

$$\bar{y} = 2J_\lambda^A \bar{x} - \lambda \sum_{i=1}^{i=n} C_i(J_\lambda^A \bar{x}) - \bar{x}$$

We have also, $J_\lambda^A(\bar{x}) = J_\lambda^B(\bar{y})$.

Consequently,

$$J_\lambda^A(\bar{x}) = J_\lambda^B(-\lambda \sum_{i=1}^n C_i(J_\lambda^A(\bar{x})) - \bar{x} + 2J_\lambda^A \bar{x}).$$

So we conclude that $\bar{x}$ is fixed point of T_λ , which prove that $J_\lambda^A(\bar{x})$ is solution of (3.2).

Below, we examine the case where $n = 1$. In this setting, problem (3.2) reduces to the following: find an element $x \in \mathbb{H}$ such that

$$0 \in A(x) + B(x) + C(x). \quad (3.6)$$

Here, A and B are maximally monotone operators defined on the Hilbert space $\mathbb{H}$, while C is a single-valued nonexpansive mapping on $\mathbb{H}$.

The corresponding algorithm is then defined as follows:

$$\begin{cases} (x_0, y_0, z_0) \in \mathbb{H}^3, \\ x_{k+1} = 3J_\lambda^A(x_k) - y_k - z_k, \\ y_{k+1} = 3J_\lambda^A(y_k) - x_k - z_k, \\ z_{k+1} = J_\lambda^A(x_k) + \lambda C(J_\lambda^A(x_k)), \end{cases} \quad k \geq 0.$$

3.1.2 Algorithm 2

Proposition 3.1.4 *Let $\{C_i\}_{1 \leq i \leq n}$ be a finite family of nonexpansive operators defined on a real Hilbert space $\mathbb{H}$. Let $A : \mathbb{H} \rightarrow \mathbb{H}$ be an α -inverse strongly monotone operator and $B : \mathbb{B} \rightarrow \mathbb{H}$ be a β -strongly monotone operator.*

For all $\alpha > 0$, $\beta > 0$, and $\lambda > 0$, let T_λ be an L -Lipschitz operator, where L is defined as:

$$L = \frac{n\lambda^2 + (\alpha n + 2)\lambda + \alpha}{(\beta\lambda + 1)(\lambda + \alpha)} + \frac{\lambda}{\lambda + \alpha}.$$

Proof 3.1.6

$$\begin{aligned} \|T_\lambda(x) - T_\lambda(y)\| &= \left\| J_\lambda^B \left(-\lambda \sum_{i=1}^n C_i(J_\lambda^A x) - x + 2J_\lambda^A x \right) \right. \\ &\quad \left. - J_\lambda^B \left(-\lambda \sum_{i=1}^n C_i(J_\lambda^A y) - y + 2J_\lambda^A y \right) - \lambda A_\lambda(x) + \lambda A_\lambda(y) \right\| \\ &< \left\| J_\lambda^B \left(-\lambda \sum_{i=1}^n C_i(J_\lambda^A x) - (x - J_\lambda^A x) \right) \right. \\ &\quad \left. - J_\lambda^B \left(-\lambda \sum_{i=1}^n C_i(J_\lambda^A y) - (y - J_\lambda^A y) \right) \right\| + \frac{\lambda}{\lambda + \alpha} \|x - y\| \\ &< \frac{1}{\beta\lambda + 1} \left\| -\lambda \sum_{i=1}^n C_i(J_\lambda^A x) - \lambda A_\lambda(x) - J_\lambda^A x \right. \\ &\quad \left. + \lambda \sum_{i=1}^n C_i(J_\lambda^A y) + \lambda A_\lambda(y) + J_\lambda^A y \right\| + \frac{\lambda}{\lambda + \alpha} \|x - y\| \\ &< \frac{1}{\beta\lambda + 1} \left[\lambda \sum_{i=1}^n \|C_i(J_\lambda^A x) - C_i(J_\lambda^A y)\| + \left(\frac{\lambda}{\lambda + \alpha} + 1 \right) \|x - y\| \right] \\ &\quad + \frac{\lambda}{\lambda + \alpha} \|x - y\| \\ &< \frac{1}{\beta\lambda + 1} \left[\lambda \sum_{i=1}^n \|J_\lambda^A x - J_\lambda^A y\| + \left(\frac{\lambda}{\lambda + \alpha} + 1 \right) \|x - y\| \right] \\ &\quad + \frac{\lambda}{\lambda + \alpha} \|x - y\| \\ &< \frac{1}{\beta\lambda + 1} \left(\lambda n + \frac{2\lambda + \alpha}{\lambda + \alpha} \right) \|x - y\| + \frac{\lambda}{\lambda + \alpha} \|x - y\| \\ &< \left[\frac{n\lambda^2 + (\alpha n + 2)\lambda + \alpha}{(\beta\lambda + 1)(\lambda + \alpha)} + \frac{\lambda}{\lambda + \alpha} \right] \|x - y\|. \end{aligned}$$

Theorem 3.1.4 *For all λ , α and β positive real numbers:*

- $\beta > n$ for a given $n \in \mathbb{N}^*$;

- $\alpha > \frac{2}{\beta - n}$;
- $0 < \lambda < \frac{(\beta - n)\alpha - 2}{n}$.

Then T_λ is a contractive mapping.

Proof 3.1.7 Assume that the operator T_λ is contractive. Then the contractivity condition is equivalent to the inequality

$$\begin{aligned} \frac{n\lambda^2 + (\alpha n + 2)\lambda + \alpha}{(\beta\lambda + 1)(\lambda + \alpha)} + \frac{\lambda}{\lambda + \alpha} < 1 &\implies \frac{(n + \beta)\lambda^2 + (3 + \alpha n)\lambda + \alpha}{(\beta\lambda + 1)(\alpha + \lambda)} - 1 < 0 \\ &\implies \frac{n\lambda^2 + (2 + \alpha n - \alpha\beta)\lambda}{(\beta\lambda + 1)(\alpha + \lambda)} < 0. \end{aligned}$$

Since $(\beta\lambda + 1)(\alpha + \lambda) > 0$ for $\lambda > 0$, the above inequality reduces to the polynomial inequality

$$n\lambda^2 + (2 + \alpha n - \alpha\beta)\lambda < 0.$$

Factoring out λ , we obtain

$$\lambda(n\lambda + 2 + n\alpha - \alpha\beta) < 0.$$

Because $\lambda > 0$ and $n > 0$, this inequality holds if and only if

$$n\lambda + 2 + n\alpha - \alpha\beta < 0.$$

Consequently, the contractivity of T_λ is ensured under these conditions:

$$\begin{cases} \beta > n, \\ \alpha > \frac{2}{\beta - n}, \\ 0 < \lambda < \frac{\alpha\beta - 2 - n\alpha}{n}, \end{cases} \quad n > 0.$$

In this section, we introduce a modified version of the Ishikawa algorithm designed to accelerate the convergence of the sequence $\{T_\lambda x_k\}$. The following theorem establishes the framework and properties of this enhanced algorithm:

Theorem 3.1.5 Consider a real Hilbert space $\mathbb{H}$ and a closed convex subset $\mathbb{C} \subseteq \mathbb{H}$. Let $T_\lambda : \mathbb{C} \rightarrow \mathbb{C}$ be a contraction mapping. We define the sequence $\{x_k\}$ generated iteratively, for any initial point $x_0 \in \mathbb{C}$ and for each integer $k \geq 0$, as follows:

$$\begin{cases} u_0 \in \mathbb{H}, \\ u_{k+1} = a_k u_k + b_k T_\lambda v_k, \\ v_k = c_k u_k + d_k T_\lambda u_k, \end{cases} \quad k \geq 0,$$

where the sequences $\{a_k\}$, $\{b_k\}$, $\{c_k\}$, and $\{d_k\}$ satisfy the following conditions:

$$1. \ 0 \leq d_k \leq b_k < 1,$$

$$2. \ b_k + a_k = 1,$$

$$3. \ c_k + d_k = 1.$$

Assuming the sequence $\{u_k\}$ converges, its limit corresponds to the unique fixed point of T_λ .

3.1.3 Convergence of algorithm 2

Ishikawa showed that for any $\tilde{x}, \tilde{y}, \tilde{z} \in \mathbb{H}$ and any $\lambda \in \mathbb{R}$,

$$\|\lambda\tilde{x} + (1-\lambda)\tilde{y} - \tilde{z}\|^2 = \lambda\|\tilde{x} - \tilde{z}\|^2 + (1-\lambda)\|\tilde{y} - \tilde{z}\|^2 - \lambda(1-\lambda)\|\tilde{x} - \tilde{y}\|^2.$$

Let $\bar{x}$ be a fixed point of T_λ , then we have

$$\begin{aligned} \|u_{k+1} - \bar{x}\|^2 &= \|a_k u_k + b_k T_\lambda v_k - \bar{x}\|^2 \\ &= b_k \|T_\lambda v_k - \bar{x}\|^2 + a_k \|u_k - \bar{x}\|^2 - b_k a_k \|u_k - T_\lambda v_k\|^2 \end{aligned} \quad (3.7)$$

From contraction condition we have:

$$\|T_\lambda v_k - \bar{u}\|^2 = \|T_\lambda v_k - T_\lambda \bar{u}\|^2 \leq \|v_k - \bar{u}\|^2 + h \|v_k - T_\lambda v_k\|^2, \quad \text{where } h = L^2. \quad (3.8)$$

On the other hand,

$$\|v_k - \bar{u}\|^2 = \|c_k u_k + d_k T_\lambda u_k - \bar{u}\|^2$$

which expands to:

$$\|v_k - \bar{u}\|^2 = d_k \|T_\lambda u_k - \bar{u}\|^2 + c_k \|u_k - \bar{u}\|^2 - d_k c_k \|u_k - T_\lambda u_k\|^2 \quad (3.9)$$

Similarly, we can express:

$$\|v_k - T_\lambda v_k\|^2 = d_k \|T_\lambda u_k - T_\lambda v_k\|^2 + c_k \|u_k - T_\lambda v_k\|^2 - d_k c_k \|u_k - T_\lambda u_k\|^2 \quad (3.10)$$

Moreover, we have the following inequality:

$$\|T_\lambda u_k - \bar{u}\|^2 \leq \|u_k - \bar{u}\|^2 + h \|u_k - T_\lambda u_k\|^2 \quad (3.11)$$

By introducing equations (3.11), (3.10), and (3.9) into (3.8), we obtain:

$$\begin{aligned} \|T_\lambda v_k - \bar{u}\|^2 &\leq c_k \|u_k - \bar{u}\|^2 + hc_k \|u_k - T_\lambda u_k\|^2 + d_k \|u_k - \bar{u}\|^2 \\ &\quad - d_k c_k \|u_k - T_\lambda u_k\|^2 + hd_k \|T_\lambda u_k - T_\lambda v_k\|^2 + hc_k \|u_k - T_\lambda v_k\|^2 \end{aligned}$$

Thus,

$$\begin{aligned} \|T_\lambda v_k - \bar{u}\|^2 &\leq \|u_k - \bar{u}\|^2 - d_k(c_k - hd_k) \|u_k - T_\lambda u_k\|^2 \\ &\quad + hd_k \|T_\lambda u_k - T_\lambda v_k\|^2 + hc_k \|u_k - T_\lambda v_k\|^2 \end{aligned} \quad (3.12)$$

Substituting equation (3.12) into equation (3.7), we get:

$$\begin{aligned} \|u_{k+1} - \bar{u}\|^2 &\leq \|u_k - \bar{u}\|^2 + hb_k \|T_\lambda u_k - T_\lambda v_k\|^2 \\ &\quad - b_k d_k (c_k - hd_k) \|u_k - T_\lambda u_k\|^2 - b_k (a_k - h + hd_k) \|u_k - T_\lambda v_k\|^2 \end{aligned}$$

This implies that the sequence $\{\|u_k - \bar{u}\|^2\}$ is monotonically decreasing for sufficiently large k . Consequently, there exists a subsequence $\{u_{k_m}\}$ of $\{u_k\}$ such that:

$$\lim_{m \rightarrow \infty} \|u_{k_m} - T_\lambda u_{k_m}\| = 0$$

Now, we show that $\{T_\lambda u_k\}$ is a Cauchy sequence. Indeed,

$$\|T_\lambda u_{k_m} - T_\lambda u_{k_\ell}\| \leq \|u_{k_\ell} - T_\lambda u_{k_m}\| + \|u_{k_\ell} - T_\lambda u_{k_\ell}\|$$

Taking the limit as $m, \ell \rightarrow \infty$, we have:

$$\|T_\lambda u_{k_m} - T_\lambda u_{k_\ell}\| \rightarrow 0.$$

Thus, $\{T_\lambda u_k\}$ is a Cauchy sequence, hence convergent.

We call the limit $\bar{u}$. Then:

$$\lim_{m \rightarrow \infty} T_\lambda u_{k_m} = \lim_{m \rightarrow \infty} u_{k_m} = \bar{u}.$$

Using the contraction of T_λ , we have:

$$\|T_\lambda \bar{u} - T_\lambda u_{k_m}\| \leq \|\bar{u} - u_{k_m}\| + \|u_{k_m} - T_\lambda u_{k_m}\| + \|T_\lambda \bar{u} - T_\lambda u_{k_m}\|.$$

Taking the limit as $m \rightarrow \infty$, we obtain:

$$\lim_{m \rightarrow \infty} \|T_\lambda \bar{u} - T_\lambda u_{k_m}\| = 0.$$

Hence, we conclude that:

$$\|\bar{u} - T_\lambda \bar{u}\| \leq \|\bar{u} - u_{k_m}\| + \|u_{k_m} - T_\lambda u_{k_m}\| + \|T_\lambda u_{k_m} - T_\lambda \bar{u}\|$$

By passing to the limit as $m \rightarrow \infty$, we obtain $\|\bar{u} - T_\lambda \bar{u}\| = 0$, which implies that $\bar{u} = T_\lambda \bar{u}$. Consequently, $\bar{u}$ is a fixed point of T_λ . We now proceed to demonstrate that the sequence $\{u_k\}$ converges strongly to this unique fixed point

$$\begin{aligned} \|u_{k+1} - \bar{u}\|^2 &= \|a_k u_k + b_k T_\lambda v_k - \bar{u}\|^2 \\ &= b_k \|T_\lambda v_k - \bar{u}\|^2 + a_k \|u_k - \bar{u}\|^2 - b_k a_k \|T_\lambda v_k - u_k\|^2 \end{aligned} \quad (3.13)$$

we know that:

$$\|T_\lambda v_k - \bar{u}\|^2 \leq L^2 \|v_k - \bar{u}\|^2 + L^2 \|v_k - T_\lambda v_k\|^2$$

We set $L^2 = \varrho$, then:

$$\|T_\lambda v_k - \bar{u}\|^2 \leq \varrho \|v_k - \bar{u}\|^2 + h \|v_k - T_\lambda v_k\|^2.$$

Moreover:

$$\begin{aligned} \|v_k - \bar{u}\|^2 &= \|d_k T_\lambda u_k + c_k u_k - \bar{u}\|^2 \\ &= d_k \|T_\lambda u_k - \bar{u}\|^2 + c_k \|u_k - \bar{u}\|^2 - d_k c_k \|T_\lambda u_k - u_k\|^2 \end{aligned} \quad (3.14)$$

And similarly:

$$\begin{aligned} \|v_k - T_\lambda v_k\|^2 &= \|d_k T_\lambda u_k + c_k u_k - T_\lambda v_k\|^2 \\ &= d_k \|T_\lambda u_k - T_\lambda v_k\|^2 + c_k \|u_k - T_\lambda v_k\|^2 - d_k c_k \|T_\lambda u_k - u_k\|^2 \end{aligned} \quad (3.15)$$

Hence (3.15) can be rewritten as follows:

$$\begin{aligned} \|T_\lambda v_k - \bar{u}\|^2 &\leq \varrho d_k \|T_\lambda u_k - \bar{u}\|^2 + \varrho c_k \|u_k - \bar{u}\|^2 \\ &\quad - \varrho d_k c_k \|T_\lambda u_k - u_k\|^2 + \varrho d_k \|T_\lambda u_k - T_\lambda v_k\|^2 \\ &\quad + \varrho c_k \|u_k - T_\lambda v_k\|^2 - \varrho d_k c_k \|T_\lambda u_k - u_k\|^2 \end{aligned} \quad (3.16)$$

However, we also have

$$\|T_\lambda u_k - \bar{u}\|^2 \leq \varrho \|u_k - \bar{u}\|^2 + \varrho \|T_\lambda u_k - u_k\|^2 \quad (3.17)$$

By substituting (3.17) into (3.16), we obtain:

$$\begin{aligned} \|T_\lambda v_k - \bar{u}\|^2 &\leq \varrho^2 d_k \|x_k - \bar{u}\|^2 + \varrho^2 \|T_\lambda u_k - u_k\|^2 + \varrho c_k \|u_k - \bar{u}\|^2 \\ &\quad - \varrho d_k c_k \|T_\lambda u_k - u_k\|^2 + \varrho d_k c_k \|T_\lambda u_k - T_\lambda v_k\|^2 \\ &\quad + \varrho c_k \|u_k - T_\lambda v_k\|^2 - \varrho d_k c_k \|T_\lambda u_k - u_k\|^2 \\ &\leq (\varrho c_k + \varrho d_k) \|u_k - \bar{u}\|^2 - \varrho d_k (2 - 2d_k) \|T_\lambda u_k - u_k\|^2 \\ &\quad + \varrho c_k \|u_k - T_\lambda v_k\|^2 + \varrho d_k \|T_\lambda u_k - T_\lambda v_k\|^2. \end{aligned} \quad (3.18)$$

Incorporating (3.18) into (3.13) yields:

$$\begin{aligned} \|u_{k+1} - \bar{u}\|^2 &\leq b_k \varrho \|u_k - \bar{u}\|^2 - b_k d_k \varrho (2 - \varrho - 2d_k) \|T_\lambda u_k - u_k\|^2 \\ &\quad + b_k c_k \varrho \|u_k - T_\lambda v_k\|^2 + b_k d_k \varrho \|T_\lambda u_k - T_\lambda v_k\|^2 \\ &\quad + a_k \|u_k - \bar{u}\|^2 - b_k a_k \varrho \|u_k - T_\lambda v_k\|^2 \\ &\leq a_k (1 - \varrho) \|u_k - \bar{u}\|^2 - b_k d_k \varrho (2 - \varrho - 2d_k) \|T_\lambda u_k - u_k\|^2 \\ &\quad + b_k d_k \varrho \|T_\lambda u_k - T_\lambda v_k\|^2 - b_k a_k (1 - \varrho + \varrho d_k) \|u_k - T_\lambda v_k\|^2 \end{aligned} \quad (3.19)$$

Let

- $\frac{1-\varrho}{2} \leq b_k \leq 1 - \varrho$;
- $\varrho \in]0; 1[$;
- $d_k \geq 0$;
- $\lim_{k \rightarrow \infty} d_k = 0$; There exists a natural number N such that for all $k > N$:

$$2 - \varrho - 2d_k \geq 0 \quad \text{and} \quad a_k - \varrho + \varrho d_k \geq 0.$$

Thus, for $k \geq N$, we have

$$\|u_{k+1} - \bar{u}\|^2 \leq \tilde{\varrho} \|u_k - \bar{u}\|^2 + b_k d_k \varrho \|T_\lambda u_k - T_\lambda v_k\|^2$$

where $0 < \tilde{\varrho} = 1 - \frac{(1-\varrho)^2}{2}$. Given that the set C is bounded, it follows that the sequence of squared norms $\{\|T_\lambda u_k - T_\lambda v_k\|^2\}$ remains within finite bounds. Consequently, we can deduce that

$$\lim_{k \rightarrow \infty} b_k d_k \varrho \|T_\lambda u_k - T_\lambda v_k\|^2 = 0.$$

From Lemma (1.1.7) we conclude that $\lim_{k \rightarrow \infty} u_k = \bar{x}$.

3.1.4 Applications of maximal monotone operators to minimization problems

We define the optimization problem:

$$\min_{x \in \mathbb{R}^n} [F_1(x) + F_2(x) + F_3(x)], \quad (3.20)$$

where:

- The functions F_2 and F_3 are assumed to be proper, closed, and convex.
- $F_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable, where the gradient ∇F_1 satisfies the 1-Lipschitz continuity condition.

Proposition 3.1.5 *Let $A = \partial F_1$, $B = \partial F_2$ and $C = \nabla F_3$. It follows that the minimization problem (3.20) can be reformulated as the problem of finding a zero of the sum of maximally monotone operators, namely,*

$$\text{Find } x \in \mathbb{R}^n \text{ such that } 0 \in A(x) + B(x) + C(x). \quad (3.21)$$

3.1.5 Example

Consider the functions $F_1, F_2, F_3 : \mathbb{R} \rightarrow \mathbb{R}$ defined by:

$$F_1(x) = \frac{1}{2}x^2, \quad F_2(x) = \frac{1}{10}x^2 + 2, \quad F_3(x) = 2x^2 + 1.$$

Our objective is to find a solution for the following minimization problem:

$$\min_{x \in \mathbb{R}} \sum_{i=1}^3 F_i(x). \quad (3.22)$$

Let A, B , and C be the monotone operators corresponding to the gradients of F_2, F_3 , and F_1 , respectively, such that:

$$A(x) = \{\nabla F_2(x)\} = \left\{\frac{1}{5}x\right\}, \quad B(x) = \{\nabla F_3(x)\} = \{4x\}, \quad C(x) = \nabla F_1(x) = x.$$

(3.22) is equivalent to finding $x \in \mathbb{R}$ that satisfies the inclusion:

$$0 \in (A + B + C)(x).$$

We have:

$$J_\lambda^A(x) = \frac{5}{5 + \lambda}x \quad \text{and} \quad J_\lambda^B(x) = \frac{1}{1 + 4\lambda}x.$$

Consequently, the operator T_λ is defined as:

$$T_\lambda(x) = \frac{4\lambda^2 - 5\lambda + 10}{(4\lambda + 1)(\lambda + 5)}x.$$

In accordance with Theorem 3.1.5, we select the following parameters for our numerical implementation:

$$\lambda = 2, \quad b_n = \frac{1}{(n+1)^2}, \quad \text{and} \quad d_n = \frac{1}{n+1}.$$

3.1.5.1 Numerical Validations for Algorithm 2

We show sigle-valued mapping:

$$T_\lambda(x) = \frac{4\lambda^2 - 5\lambda + 10}{(4\lambda + 1)(\lambda + 5)}x.$$

Letting $\lambda = 2$, so:

$$T_2(x) = \frac{16}{63}x.$$

We take:

$$x_0 = y_0 = 1,$$

and define:

$$\begin{cases} a_n = 1 - b_n \\ b_n = \frac{1}{n+1} \\ c_n = 1 - d_n \\ d_n = \frac{1}{(n+1)^2} \end{cases}$$

Iterative scheme and convergence analysis

For each iteration $n \geq 0$, the sequences are updated as follows:

$$\begin{aligned} y_n &= c_n x_n + d_n T_2(x_n) \\ &= \left(1 - \frac{1}{(n+1)^2}\right) x_n + \frac{16}{63(n+1)^2} x_n, \\ x_{n+1} &= a_n x_n + b_n T_2(y_n) \\ &= \left(1 - \frac{1}{n+1}\right) x_n + \frac{16}{63(n+1)} y_n. \end{aligned}$$

These parameter selections align with the hypotheses of Theorem 3.1.5, which ensures the **strong convergence** of the sequence $\{x_n\}$ to the unique fixed point of the operator T_2 . The fixed point $\bar{x}$ is determined by:

$$\bar{x} = T_2(\bar{x}) \implies \bar{x} = 0.$$

3.2 On the finite family sum of maximally monotone operators

In this part, we present the method employed to study the topic under consideration. We begin by defining the adjoint operator, which allows us to derive a connection between its set of zeros and the solution set of the given problem.

For all $\lambda > 0$, we define

$$\xi_\lambda(x_1, \dots, x_n) = \begin{pmatrix} J_\lambda^{A_1}(x_1) - \frac{1}{n} \sum_{j=1}^n x_j \\ J_\lambda^{A_2}(x_2) - \frac{1}{n} \sum_{j=1}^n x_j \\ \vdots \\ J_\lambda^{A_n}(x_n) - \frac{1}{n} \sum_{j=1}^n x_j \end{pmatrix}$$

The issue here is considered for all λ strictly positive, to establish the link between the zeros of the proposed operator and the underlying investigation:

$$\xi_\lambda(x_1, \dots, x_n) = 0. \quad (3.23)$$

Let an auxiliary function be constructed from ξ_λ such that its fixed points coincide with the solutions to the problem at hand. For all $\lambda > 0$, this function is defined as

$$\chi_\lambda(x_1, \dots, x_n) = \begin{pmatrix} \lambda(A_1)_\lambda(x_1) + \frac{1}{n} \sum_{j=1}^n x_j \\ \lambda(A_2)_\lambda(x_2) + \frac{1}{n} \sum_{j=1}^n x_j \\ \vdots \\ \lambda(A_n)_\lambda(x_n) + \frac{1}{n} \sum_{j=1}^n x_j \end{pmatrix}$$

Theorem 3.2.1 *For any $\lambda > 0$, we have the assertions are equivalent:*

1. The problem (3.1) hold for x^∞ .
2. There exists a vector $(a_1, \dots, a_n) \in \mathbb{H}^n$ that solves (3.23), such that the identity $J_\lambda^{A_{i_0}}(a_{i_0}) = x^\infty$ holds for each $i_0 \in \{1, 2, \dots, n\}$.

Proof 3.2.1 We begin by proving the first implication.

Let $\lambda > 0$ and let x^∞ be a solution of problem (3.1), that is,

$$0 \in A_1(x^\infty) + \cdots + A_n(x^\infty).$$

Then, there exists $(y_1^\infty, \dots, y_n^\infty) \in \mathbb{X}^n$ such that

$$y_i^\infty \in \lambda A_i(x^\infty), \quad i \in \{1, \dots, n\}, \quad \text{and} \quad \sum_{i=1}^n y_i^\infty = 0.$$

By adding $\bar{x}$ to both sides of each inclusion, we obtain:

$$y_i^\infty + x^\infty \in x^\infty + \lambda A_i(x^\infty), \quad i \in \{1, \dots, n\},$$

while still satisfying $\sum_{i=1}^n \bar{y}_i = 0$. From resolvent definition, this can be rewritten as

$$J_\lambda^{A_i}(y_i^\infty + x^\infty) = x^\infty, \quad i = 1, \dots, n.$$

Define $a_i := y_i^\infty + x^\infty$ for $i = 1, \dots, n$. Then,

$$x^\infty = \frac{1}{n} \sum_{j=1}^n a_j \quad \text{and} \quad J_\lambda^{A_i}(a_i) - \frac{1}{n} \sum_{j=1}^n a_j = 0, \quad i = 1, \dots, n.$$

Hence, $(a_1, \dots, a_n)$ is a solution of problem (3.23), with

$$J_\lambda^{A_{i_0}}(a_{i_0}) = x^\infty, \quad \text{for some } i_0 \in \{1, \dots, n\}.$$

Next, we prove the converse implication.

Let $\lambda > 0$ and let $(a_1, \dots, a_n)$ be a solution of problem (3.23). By definition of (3.23), we have

$$J_\lambda^{A_i}(a_i) - \frac{1}{n} \sum_{j=1}^n a_j = 0, \quad i = 1, \dots, n.$$

Equivalently,

$$n J_\lambda^{A_i}(a_i) = \sum_{j=1}^n a_j, \quad i = 1, \dots, n,$$

which implies

$$J_\lambda^{A_1}(a_1) = \cdots = J_\lambda^{A_n}(a_n).$$

Thus, we obtain

$$\sum_{i=1}^n (A_i)_\lambda(a_i) = 0$$

and

$$J_\lambda^{A_1}(a_1) = \cdots = J_\lambda^{A_n}(a_n).$$

Moreover, since

$$(A_i)_\lambda(a_i) \in A_i(J_\lambda^{A_i}(a_i)), \quad i = 1, \dots, n,$$

it follows that

$$0 \in \sum_{j=1}^n A_j(J_\lambda^{A_{i_0}}(a_{i_0}))$$

Consequently,

$$\bar{x} := J_\lambda^{A_{i_0}}(a_{i_0})$$

is a solution to this problem (3.1).

It follows from the preceding discussion that solving Problem (3.1) is the same as solving the system below:

$$\begin{cases} J_{\lambda_1}^A(x_1) - \frac{1}{n} \sum_{j=1}^n x_j = 0, \\ J_{\lambda_2}^A(x_2) - \frac{1}{n} \sum_{j=1}^n x_j = 0, \\ \vdots \\ J_{\lambda_n}^A(x_n) - \frac{1}{n} \sum_{j=1}^n x_j = 0. \end{cases} \quad (3.24)$$

To illustrate, consider the following example

Example 3.2.1 Consider three convex functions $f_1, f_2, f_3 : \mathbb{R} \rightarrow \mathbb{R}$ defined by the following cases:

$$f_i(x) = \begin{cases} (3x + 2)^2 & \text{if } i = 1 \\ (3x + 1)^2 & \text{if } i = 2 \\ x^2 & \text{if } i = 3 \end{cases}$$

The corresponding subdifferentials $A_1 = \partial f_1$, $A_2 = \partial f_2$, and $A_3 = \partial f_3$ are maximal monotone operators on $\mathbb{R}$, given by:

$$A_i(x) = \begin{cases} \{18x + 12\} & \text{if } i = 1 \\ \{18x + 6\} & \text{if } i = 2 \\ \{2x\} & \text{if } i = 3 \end{cases}$$

The objective is to solve the inclusion problem:

$$0 \in A_1(x) + A_2(x) + A_3(x) \quad (3.25)$$

For all $\lambda > 0$, the resolvents $J_\lambda^{A_i}(x) = (I + \lambda A_i)^{-1}(x)$ are defined as:

$$J_{\lambda}^{A_i}(x) = \begin{cases} \frac{x - 12\lambda}{18\lambda + 1} & \text{if } i = 1 \\ \frac{x - 6\lambda}{18\lambda + 1} & \text{if } i = 2 \\ \frac{x}{2\lambda + 1} & \text{if } i = 3 \end{cases}$$

Our system,

$$\begin{cases} \frac{x - 12\lambda}{18\lambda + 1} = \frac{x + y + z}{3}, \\ \frac{y - 6\lambda}{18\lambda + 1} = \frac{x + y + z}{3}, \\ \frac{z}{2\lambda + 1} = \frac{x + y + z}{3}. \end{cases}$$

If $\lambda = 1$, we find

$$\begin{cases} \frac{x - 12}{19} = \frac{x + y + z}{3}, \\ \frac{y - 6}{19} = \frac{x + y + z}{3}, \\ \frac{z}{3} = \frac{x + y + z}{3}. \end{cases}$$

Thus $(x, y, z) = (3, -3, -\frac{27}{19})$. Apply Theorem (3.2.1), we obtain $J_1^{A_1}(3) = -\frac{9}{19}$ is a solution of (3.25).

We introduce an alternative approach for obtaining candidate solutions to problem 3.1 by defining the two functions below.

$$\begin{aligned} h_{\lambda} : \mathbb{H}^n &\rightarrow \mathbb{H}^n, h_{\lambda}(x_1, \dots, x_n) = (J_{\lambda}^{A_1}(x_1), \dots, J_{\lambda}^{A_n}(x_n)), \\ k : \mathbb{H}^n &\rightarrow \mathbb{H}^n, k(x_1, \dots, x_n) = \left(\frac{1}{n} \sum_{j=1}^n x_j, \dots, \frac{1}{n} \sum_{j=1}^n x_j \right). \end{aligned}$$

Proposition 3.2.1 For each $\lambda \in \mathbb{R}_{++}$. If $((x_1^{\infty}, \dots, x_n^{\infty}), (y_1^{\infty}, \dots, y_n^{\infty})) \in \text{Gr}(h_{\lambda}) \cap \text{Gr}(k)$, then $J_{\lambda}^{A_{i_0}}(x_{i_0}^{\infty})$ is a solution of (3.1), $i_0 = 1, 2, \dots, n$.

Proof 3.2.2 For all $\lambda > 0$. Let $((x_1^{\infty}, \dots, x_n^{\infty}), (y_1^{\infty}, \dots, y_n^{\infty})) \in \text{Gr}(h_{\lambda}) \cap \text{Gr}(k)$, then,

$$\begin{cases} J_{\lambda_1}^A(x_1^{\infty}) = \frac{1}{n} \sum_{j=1}^n x_j^{\infty}, \\ J_{\lambda_2}^A(x_2^{\infty}) = \frac{1}{n} \sum_{j=1}^n x_j^{\infty}, \\ \vdots \\ J_{\lambda_n}^A(x_n^{\infty}) = \frac{1}{n} \sum_{j=1}^n x_j^{\infty}. \end{cases}$$

So, $(x_1^\infty, \dots, x_n^\infty)$ is a solution to the problem (3.23). From Theorem 3.2.1, we find that $J_\lambda^{A_{i_0}}(x_{i_0}^\infty)$ is a solution of (3.1).

Proposition 3.2.2 For each $\lambda \in \mathbb{R}_{++}$. If $J_\lambda^{A_{i_0}}(x_{i_0}^\infty)$ is a solution of (3.1) then there is $\{x_i^\infty\}_{i=1}^n \subset \mathbb{H}^n$ such that $\left((x_1^\infty, \dots, x_n^\infty), \left(\frac{1}{n} \sum_{j=1}^n x_j^\infty, \dots, \frac{1}{n} \sum_{j=1}^n x_j^\infty\right)\right) \in \text{Gr}(h_\lambda) \cap \text{Gr}(k)$.

Proof 3.2.3 Assume that $J_\lambda^{A_{i_0}}(x_{i_0}^\infty)$ is a solution of problem (3.1) in Theorem 3.2.1. Then there exists a vector $(x_1^\infty, \dots, x_n^\infty)$ which is a solution of problem (3.23). In particular, this vector satisfies the following system of equations:

$$\begin{cases} J_{\lambda_1}^A(x_1^\infty) = \frac{1}{n} \sum_{j=1}^n x_j^\infty, \\ J_{\lambda_2}^A(x_2^\infty) = \frac{1}{n} \sum_{j=1}^n x_j^\infty, \\ \vdots \\ J_{\lambda_n}^A(x_n^\infty) = \frac{1}{n} \sum_{j=1}^n x_j^\infty. \end{cases}$$

Consequently, we obtain

$$\left((x_1^\infty, \dots, x_n^\infty), \left(\frac{1}{n} \sum_{j=1}^n \bar{x}_j, \dots, \frac{1}{n} \sum_{j=1}^n x_j^\infty\right)\right) \in \text{Gr}(h_\lambda) \cap \text{Gr}(k).$$

The convergence of the two aforementioned propositions suggests that

$$(\sum_{i=1}^n A_i)^{-1}(0) = \left\{ J_\lambda^{A_{i_0}}(x_{i_0}^\infty) \mid ((x_i^\infty)_{i=1}^n, (\bar{y}_i)_{i=1}^n) \in \text{Gr}(h_\lambda) \cap \text{Gr}(k) \right\}$$

Given the established relationship between the current problem and the roots of the aforementioned function, the subsequent stage involves the implementation of various numerical techniques.

3.2.1 On the computation of zeros for composite vector functions

The determination of roots for mappings within the Euclidean space $\mathbb{R}^n$ represents a cornerstone of computational mathematics and optimization theory, underpinned by its extensive utility in the applied sciences. While various algorithmic frameworks exist—each contingent upon the mapping's regularity and the problem's constraints—the present discussion is confined to the initial framework. Specifically, we investigate the problem through an operator-theoretic lens. Let $\{A_i\}_{i=1}^n$ denote a collection of maximal monotone operators on

$\mathbb{R}$. Our primary objective is to characterize an element $x \in \mathbb{R}$ that satisfies the equilibrium conditions governed by these operators. Consider the inclusion problem:

$$0 \in \sum_{i=1}^n A_i(x).$$

Provided that the resolvents $J_\lambda^{A_i}$ for $i \in \{1, \dots, n\}$ are differentiable, the solution to this problem may be approximated using the following iterative procedure.

Numerical iteration

Consider the function $\xi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, defined by:

$$\xi(x) = \begin{pmatrix} J_\lambda^{A_1}(x_1) - \frac{1}{n} \sum_{j=1}^n x_j \\ J_\lambda^{A_2}(x_2) - \frac{1}{n} \sum_{j=1}^n x_j \\ \vdots \\ J_\lambda^{A_n}(x_n) - \frac{1}{n} \sum_{j=1}^n x_j \end{pmatrix}$$

The Jacobian matrix of ξ is:

$$J_\xi(x) = \begin{pmatrix} \frac{\partial J_\lambda^{A_1}}{\partial x_1}(x_1) - \frac{1}{n} & -\frac{1}{n} & \dots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{\partial J_\lambda^{A_2}}{\partial x_2}(x_2) - \frac{1}{n} & \dots & -\frac{1}{n} \\ \vdots & \vdots & & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \dots & \frac{\partial J_\lambda^{A_n}}{\partial x_n}(x_n) - \frac{1}{n} \end{pmatrix}$$

In instances where the Jacobian matrix $J_\xi(x)$ is non-singular—specifically, when $\det J_\xi(x) \neq 0$ the problem of identifying a root $x \in \mathbb{R}^n$ such that $\xi(x) = 0$ is conventionally addressed via the Newton-Raphson method. This approach represents a robust iterative framework for achieving local convergence in multi-dimensional systems:

$$\begin{cases} x_{1,k+1} = x_{1,k} - [J_\xi^{-1}(x_{1,k}, \dots, x_{n,k})\xi(x_{1,k}, \dots, x_{n,k})]_1 \\ x_{2,k+1} = x_{2,k} - [J_\xi^{-1}(x_{1,k}, \dots, x_{n,k})\xi(x_{1,k}, \dots, x_{n,k})]_2 \\ \vdots \\ x_{n,k+1} = x_{n,k} - [J_\xi^{-1}(x_{1,k}, \dots, x_{n,k})\xi(x_{1,k}, \dots, x_{n,k})]_n \end{cases} \quad k = 0, 1, 2, \dots$$

3.2.2 Example

Let f_1 , f_2 , and f_3 be functions from $\mathbb{R}$ into $\mathbb{R}$ defined by

$$\begin{cases} f_1(x) = \frac{1}{2}x^2, \\ f_2(x) = x^2, \\ f_3(x) = \frac{3}{2}x^2. \end{cases}$$

It is readily verified that the functions under consideration are proper, convex, and lower semicontinuous. Moreover, their subdifferentials are given by

$$\partial f_1(x) = \{x\}, \quad \partial f_2(x) = \{2x\}, \quad \partial f_3(x) = \{3x\},$$

which are maximal monotone.

For simplicity, we define

$$\begin{cases} A_1(x) = x, \\ A_2(x) = 2x, \\ A_3(x) = 3x. \end{cases}$$

For $\lambda = 1$, the corresponding resolvent operators satisfy

$$J_\lambda^{A_i}(x) = \begin{cases} \frac{x}{2}, & \text{if } i = 1, \\ \frac{x}{3}, & \text{if } i = 2, \\ \frac{x}{4}, & \text{if } i = 3. \end{cases}$$

Let us consider the following inclusion problem: find $x \in \mathbb{R}$ such that

$$0 \in A_1(x) + A_2(x) + A_3(x). \quad (3.26)$$

To this end, define the mapping $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$F(x_1, x_2, x_3) = \begin{pmatrix} J_\lambda^{A_1}(x_1) - \frac{1}{3} \sum_{j=1}^3 x_j \\ J_\lambda^{A_2}(x_2) - \frac{1}{3} \sum_{j=1}^3 x_j \\ J_\lambda^{A_3}(x_3) - \frac{1}{3} \sum_{j=1}^3 x_j \end{pmatrix}.$$

J_F is given by

$$J_F(x_1, x_2, x_3) = \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{12} \end{pmatrix},$$

Table 3.1: Numerical verification of the Newton–Raphson method

Iteration k	$x_{1,k}$	$x_{2,k}$	$x_{3,k}$
0	1.000000	1.000000	1.000000
1	0.333333	0.333333	0.333333
2	0.111111	0.111111	0.111111
3	0.037037	0.037037	0.037037
4	0.012346	0.012346	0.012346
5	0.004115	0.004115	0.004115
6	0.001372	0.001372	0.001372
7	0.000457	0.000457	0.000457
8	0.000152	0.000152	0.000152
9	0.000051	0.000051	0.000051

where J_F is the jacobian matrix.

Its determinant satisfies

$$\det J_F(x_1, x_2, x_3) = -\frac{1}{12} \neq 0,$$

which ensures the local applicability of Newton’s method.

To compute a solution of $F(x_1, x_2, x_3) = 0$, we apply the Newton–Raphson method

$$x^{k+1} = x^k - J_F(x^k)^{-1}F(x^k), \quad k = 0, 1, 2, \dots$$

Starting from an initial guess x^0 , the sequence $\{x^k\}$ converges to

$$(x_1, x_2, x_3) = (0, 0, 0).$$

Consequently,

$$J_\lambda^{A_1}(0) = 0$$

is a solution of the inclusion problem (3.26).

3.2.3 Fixed points and null sets for the aggregate of finite maximal monotone mappings

The objective of this section is to construct a mapping such that the set of its fixed points is in direct correspondence with the zero-set of the sum of a finite collection of maximal monotone operators.

For any $\lambda > 0$, we define the mapping $\chi_\lambda : \mathbb{H}^n \rightarrow \mathbb{H}^n$ as follows:

$$\chi_\lambda(x_1, \dots, x_n) = \begin{pmatrix} \lambda(A_1)_\lambda(x_1) + \frac{1}{n} \sum_{j=1}^n x_j \\ \lambda(A_2)_\lambda(x_2) + \frac{1}{n} \sum_{j=1}^n x_j \\ \vdots \\ \lambda(A_n)_\lambda(x_n) + \frac{1}{n} \sum_{j=1}^n x_j \end{pmatrix}.$$

Corollary 3.2.1 *The following properties are equivalent for every $\lambda > 0$:*

1. *Problem (3.1) admits at least one solution.*
2. *The mapping χ_λ possesses a fixed point.*

It is evident that for each $\lambda > 0$, $(a_1, \dots, a_n)$ solves (3.23) if and only if $(a_1, \dots, a_n) \in \text{Fix}(g_\lambda)$. Utilizing the identity $\lambda(A_i)_\lambda(a_i) = a_i - J_\lambda^{A_i}(a_i)$ for $i = 1, \dots, n$, and invoking Theorem 3.2.1, we arrive at the concluded result.

Remark: The preceding analysis establishes that the problem of determining the zeros of a finite sum of maximal monotone operators can be reduced to the study of the fixed points of the mapping g . Within the context of Hilbert spaces, this investigation leverages specific structural properties inherent to complete inner product spaces. Such environments facilitate the application of robust fixed-point theorems, as detailed in the literature (see [19, 21]). The following section outlines the fundamental theoretical frameworks and computational strategies employed for identifying fixed points in this setting.

3.2.4 Zeros of sums comprising lipschitz mappings and maximal monotone operators

We first examine the properties of an operator defined as the sum of a maximal monotone mapping and a Lipschitz perturbation.

We aim to find an element $x \in \mathbb{H}$ that solves a multi-operator inclusion problem within a real Hilbert space $\mathbb{H}$. Specifically, we consider a k -Lipschitz continuous function $f : \mathbb{H} \rightarrow \mathbb{H}$ and a finite collection of maximal monotone operators denoted by $A_1, \dots, A_n$. The problem involves identifying a point x that fulfills the following relation:

$$0 \in f(x) + \sum_{i=1}^n A_i(x), \quad (3.27)$$

where f is a Lipschitzian mapping and $\{A_i\}_{i=1}^n$ denotes a collection of monotone operators. The set of all solutions fulfilling (3.27) is represented by the zero-set of the aggregate operator:

$$\Omega = \left(f + \sum_{i=1}^n A_i \right)^{-1} (0).$$

For any regularization parameter $\lambda > 0$, we introduce the mapping $S_\lambda : \mathbb{H}^n \rightarrow \mathbb{H}^n$ characterized by the following structural form:

$$S_\lambda(x_1, \dots, x_n) = \begin{pmatrix} \Psi_{\lambda,1}(x_1) - \bar{x} \\ \Psi_{\lambda,2}(x_2) - \bar{x} \\ \vdots \\ \Psi_{\lambda,n}(x_n) - \bar{x} \end{pmatrix}, \quad (3.28)$$

where the components $\Psi_{\lambda,i}$ are defined using the resolvent operators $J_\lambda^{A_i}$ as:

$$\Psi_{\lambda,i}(x_i) = J_\lambda^{A_i}(x_i) + \frac{\lambda}{n} f \left(J_\lambda^{A_i}(x_i) \right), \quad \forall i \in \{1, \dots, n\},$$

and $\bar{x} = \frac{1}{n} \sum_{j=1}^n x_j$ denotes the arithmetic mean (barycenter) of the vector components in $\mathbb{H}^n$.

We consider the problem below: For all $\lambda > 0$,

$$S_\lambda(x) = 0 \quad (3.29)$$

where $x = (x_1, \dots, x_n)$.

Lemma 3.2.1 *For each $\lambda > 0$, every solution $\bar{x}$ of (3.27) corresponds to a solution $(a_1, \dots, a_n) \in \mathbb{H}^n$ of the system (3.29), linked by the condition $J_\lambda^{A_{i_0}}(a_{i_0}) = \bar{x}$ for all $i_0 \in \{1, \dots, n\}$.*

Proof 3.2.4 *Let $\lambda > 0$ be given. Consider $\bar{x}$ to be a solution to the inclusion problem:*

$$0 \in f(\bar{x}) + \sum_{i=1}^n A_i(\bar{x}).$$

By the definition of the summation of sets, there exist elements $(\bar{y}_1, \dots, \bar{y}_n) \in \mathbb{X}^n$ such that:

$$\bar{y}_i \in \lambda A_i(\bar{x}), \quad \forall i \in \{1, \dots, n\},$$

under the constraint that their sum balances the function f :

$$\sum_{i=1}^n \bar{y}_i = -\lambda f(\bar{x}).$$

By adding the identity $\bar{x}$ to the inclusion, we observe:

$$\bar{x} + \bar{y}_i \in \bar{x} + \lambda A_i(\bar{x}) = (I + \lambda A_i)(\bar{x}).$$

Applying the resolvent operator $J_\lambda^{A_i} = (I + \lambda A_i)^{-1}$, this relation is equivalent to:

$$\bar{x} = J_\lambda^{A_i}(\bar{x} + \bar{y}_i), \quad i = 1, \dots, n.$$

We now introduce the change of variables $a_i = \bar{x} + \bar{y}_i$ for all i . It follows from the summation constraint that:

$$\sum_{j=1}^n a_j = \sum_{j=1}^n (\bar{x} + \bar{y}_j) = n\bar{x} - \lambda f(\bar{x}).$$

Isolating $\bar{x}$ yields:

$$\bar{x} = \frac{1}{n} \sum_{j=1}^n a_j + \frac{\lambda}{n} f(\bar{x}).$$

Substituting $\bar{x} = J_\lambda^{A_i}(a_i)$ into the expression above, we obtain the coupled system:

$$J_\lambda^{A_i}(a_i) = \frac{1}{n} \sum_{j=1}^n a_j + \frac{\lambda}{n} f(J_\lambda^{A_i}(a_i)).$$

Rearranging the terms, we conclude that for every $i \in \{1, \dots, n\}$:

$$J_\lambda^{A_i}(a_i) - \frac{\lambda}{n} f(J_\lambda^{A_i}(a_i)) = \frac{1}{n} \sum_{j=1}^n a_j.$$

This confirms that the vector $(a_1, \dots, a_n)$ solves the problem associated with (3.29), and the original solution is recovered as $\bar{x} = J_\lambda^{A_i}(a_i)$.

Let, $\chi_\lambda : \mathbb{H}^n \rightarrow \mathbb{H}^n$, defined as:

$$\chi_\lambda(x_1, \dots, x_n) = \begin{pmatrix} \lambda(A_1)_\lambda(x_1) - \frac{\lambda}{n} f(J_\lambda^{A_1}(x_1)) + \frac{1}{n} \sum_{j=1}^n x_j \\ \lambda(A_2)_\lambda(x_2) - \frac{\lambda}{n} f(J_\lambda^{A_2}(x_2)) + \frac{1}{n} \sum_{j=1}^n x_j \\ \vdots \\ \lambda(A_n)_\lambda(x_n) - \frac{\lambda}{n} f(J_\lambda^{A_n}(x_n)) + \frac{1}{n} \sum_{j=1}^n x_j \end{pmatrix}$$

It is observed that the null set of the mapping S_λ is precisely the set of fixed points for the operator χ_λ . This equivalence yields the following result:

Corollary 3.2.2 *For any $\lambda > 0$, the solvability of the inclusion problem (3.27) implies that the operator χ_λ admits at least one fixed point.*

In the following, we explore the specific instances where n equals 1 and 2. Let us first investigate the single-operator inclusion problem:

Find a point $x \in \mathbb{H}$ that satisfies the relation:

$$0 \in f(x) + A(x), \quad (3.30)$$

where $\mathbb{H}$ denotes a Hilbert space. In this context, we assume the mapping $f : \mathbb{H} \rightarrow \mathbb{H}$ is k -Lipschitz continuous, while $A : \mathbb{H} \rightrightarrows \mathbb{H}$ represents a maximal monotone set-valued operator.

Proposition 3.2.3 *The problem (3.30) is guaranteed to possess a single, unique solution provided that $\beta > k$ and the operator A is maximally β -strongly monotone.*

Proof 3.2.5 *We begin by characterizing the solutions to the inclusion problem as fixed points of a resolvent-based operator. For any $\lambda > 0$, the following equivalences hold:*

$$\begin{aligned} 0 \in f(x) + A(x) &\iff -f(x) \in A(x) \\ &\iff -\lambda f(x) \in \lambda A(x) \quad (\text{for } \lambda > 0) \\ &\iff x - \lambda f(x) \in x + \lambda A(x) \\ &\iff x - \lambda f(x) \in (I + \lambda A)(x) \\ &\iff x = (I + \lambda A)^{-1}(x - \lambda f(x)) \\ &\iff x = J_\lambda^A(x - \lambda f(x)). \end{aligned}$$

To analyze the existence and uniqueness of such a point, we define the operator $\tilde{\mathfrak{S}}_\lambda : \mathbb{H} \rightarrow \mathbb{H}$ by:

$$\tilde{\mathfrak{S}}_\lambda(x) = J_\lambda^A(x - \lambda f(x)).$$

Given that A is maximally β -strongly monotone, its resolvent J_λ^A is Lipschitz continuous with a constant of $(1 + \lambda\beta)^{-1}$. Using this property along with the k -Lipschitz continuity of f , let $x, y \in \mathbb{H}$:

$$\begin{aligned} \|\tilde{\mathfrak{S}}_\lambda(x) - \tilde{\mathfrak{S}}_\lambda(y)\| &= \|J_\lambda^A(x - \lambda f(x)) - J_\lambda^A(y - \lambda f(y))\| \\ &\leq \frac{1}{1 + \lambda\beta} \|(x - y) - \lambda(f(x) - f(y))\| \\ &\leq \frac{1}{1 + \lambda\beta} (\|x - y\| + \lambda\|f(x) - f(y)\|) \\ &\leq \frac{1 + \lambda k}{1 + \lambda\beta} \|x - y\|. \end{aligned}$$

The mapping $\mathfrak{S}_\lambda$ constitutes a contraction if its Lipschitz constant is strictly less than unity:

$$\frac{1 + \lambda k}{1 + \lambda \beta} < 1 \iff 1 + \lambda k < 1 + \lambda \beta \iff k < \beta.$$

Consequently, under the condition $k < \beta$, the Banach Fixed Point Theorem ensures that $\mathfrak{S}_\lambda$ possesses a unique fixed point in $\mathbb{H}$. This last is the unique solution to the original problem (3.30).

Algorithm: we present the iterative sequence $(y_n)_{n \in \mathbb{N}}$ defined by recursive scheme:

$$y_0 = a \in \mathbb{H}, \quad y_{n+1} = J_\lambda^A(y_n - \lambda f(y_n)). \quad (3.31)$$

Under the assumptions that $\lambda > 0$ and $k < \beta$, if the sequence (y_n) converges to a limit l , it follows from the continuity of the resolvent and the function f that l is a zero of the operator sum, i.e., $l \in (f + A)^{-1}(0)$.

The Case of Multiple Operators

Building upon the previous results, we now explore a methodology to identify the zeros of a functional sum involving two maximal monotone operators and a Lipschitzian perturbation. Specifically, we investigate the existence of a vector $x \in \mathbb{H}$ such that:

$$0 \in f(x) + A(x) + B(x). \quad (3.32)$$

In this formulation, $A : \mathbb{H} \rightrightarrows \mathbb{H}$ and $B : \mathbb{H} \rightrightarrows \mathbb{H}$ represent two maximal monotone mappings, while $f : \mathbb{H} \rightarrow \mathbb{H}$ denotes a k -Lipschitz continuous perturbation affecting the equilibrium of the system.

Theorem 3.2.2 For $\lambda > 0$ and $\theta_\lambda(x) = J_\lambda^B(-\lambda f(J_\lambda^A x) - x + 2J_\lambda^A x) + \lambda A_\lambda(x)$. $J_\lambda^A(x)$ solves (3.32) for any fixed point x of θ_λ .

Proof 3.2.6 Let $y \in \mathbb{H}$, and λ strictly positive, let x^∞ be a fixed point of the function θ_λ so we have,

$$x^\infty = J_\lambda^B(-\lambda f(J_\lambda^A x^\infty) - x^\infty + 2J_\lambda^A x^\infty) + \lambda A_\lambda(x^\infty), \quad (3.33)$$

and we have

$$\begin{aligned}
 (3.33) &\implies J_\lambda^A x^\infty = J_\lambda^B \left(-\lambda f \left(J_\lambda^A x^\infty \right) - x^\infty + 2J_\lambda^A x^\infty \right) \\
 &\implies -\lambda f \left(J_\lambda^A x^\infty \right) - x^\infty + 2J_\lambda^A x^\infty \in J_\lambda^A x^\infty + \lambda B \left(J_\lambda^A x^\infty \right) \\
 &\implies -\lambda f \left(J_\lambda^A x^\infty \right) \in x^\infty - J_\lambda^A x^\infty + \lambda B \left(J_\lambda^A x^\infty \right) \\
 &\implies -f \left(J_\lambda^A x^\infty \right) \in A_\lambda \left(x^\infty \right) + B \left(J_\lambda^A x^\infty \right) \\
 &\implies 0 \in f \left(J_\lambda^A x^\infty \right) + A_\lambda \left(J_\lambda^A x^\infty \right) + B \left(J_\lambda^A x^\infty \right)
 \end{aligned}$$

Since the Yosida approximation satisfies the inclusion $A_\lambda(\bar{x}) \in A(J_\lambda^A(\bar{x}))$, it follows directly that the point $J_\lambda^A(\bar{x})$ hold the inclusion problem:

$$0 \in f(J_\lambda^A(\bar{x})) + A(J_\lambda^A(\bar{x})) + B(J_\lambda^A(\bar{x})).$$

Algorithm: We examine the iterative sequence generated by the following expression:
 $(x_k)_k \in \mathbb{N}$ by:

$$x_0 = a, x_{k+1} = J_\lambda^B \left(-\lambda f \left(J_\lambda^A x_k \right) - x_k + 2J_\lambda^A x_k \right) + \lambda A_\lambda(x_k).$$

If $\lim_{n \rightarrow \infty} x_k = l$, then $0 \in (f + A + B)(J_\lambda^A(l))$.

3.3 Conclusion

In the concluding segment of this chapter, we introduced a novel algorithmic architecture designed to resolve inclusion problems involving the collective sum of multiple maximal monotone operators. The investigation commenced with a detailed examination of the interplay between a finite collection of nonexpansive mappings and the dual sum of two maximal monotone operators. Building upon these foundations, the analysis was expanded to encompass the more general configuration of a finite family of such operators. Within this extended paradigm, we formulated a computational scheme and established its convergence to a singular zero of the operator sum. To substantiate the theoretical framework, we integrated several practical, domain-specific applications. These examples serve to demonstrate the functional efficacy and the broader significance of the developed algorithms within the field of monotone operator theory.

Chapter 4

On the difference of two maximal monotone operators

The problem of finding the set zeroes of non-monotone operator, particularly when formulated as the difference of two maximal monotone operators in a Hilbert space, remains a relatively unexplored yet important topic in convex and nonconvex analysis (see [60–63]). Such problems arise in various real-world applications, including image processing, discrete tomography, portfolio optimization, and inverse problems, where traditional convex approaches often fall short.

Let $\mathbb{H}$ be a real Hilbert space and $A : \mathbb{H} \rightarrow 2^{\mathbb{H}}$, $B : \mathbb{H} \rightarrow 2^{\mathbb{H}}$ be two maximal monotone operators on $\mathbb{H}$. Our focus is on the principal problem given below: find an element x in $\mathbb{H}$ such that

$$0 \in A(x) - B(x) - x. \quad (4.1)$$

We denote the set of solutions of (4.1) by $(A - B - \text{Id})^{-1}(0)$.

In 2008 Moudafi abdellatif presented a semellar problem (see [64–66]) which defined as:

$$\text{Find } x \text{ such that } 0 \in A(x) - B(x).$$

In his paper establish theoretical links between the solutions of the original problem, its regularized form, and the associated dual formulation. These connections are particularly useful in understanding the behavior and structure of nonconvex variational problems. Moreover he proved that the last problem is equivalent to:

$$\text{Find } x \text{ such that } 0 = A_{\lambda}(x) - B_{\lambda}(x)$$

.

Moudafi introduces a splitting-type fixed-point algorithm that involves the resolvents of both

operators, which formulated as:

$$X_{n+1} = J_\lambda^B(X_n) + (Id - J_\lambda^A)(X_n)$$

He demonstrates convergence properties under reasonable assumptions, including strong monotonicity or cocoercivity. In the special case of DC (Difference of Convex) programming, the proposed algorithm reduces to a proximal-like method, which has been shown to generate weakly convergent sequences toward critical points.

In this work, we aim to solve the problem stated at the beginning of the paper.

4.1 A method for finding where the difference of two maximal monotone operators vanishes

In this section, we present our main result. We begin by defining the function $\Psi_\lambda : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H}$ as follows:

$$\Psi_\lambda(x, y) = \begin{pmatrix} J_\lambda^A(x) - \frac{x - y}{\lambda} \\ J_\lambda^B(y) - \frac{x - y}{\lambda} \end{pmatrix},$$

To advance our analysis, we introduce the following auxiliary problem:

$$\min_{(x, y) \in \mathbb{H} \times \mathbb{H}} \|\Psi_\lambda(x, y)\| = 0, \quad (4.2)$$

with the aim of proving the equivalence between the original problem given in (4.1) and the auxiliary problem (4.2).

Proposition 4.1.1 *If $(\mathfrak{x}, \eta_1, \eta_2)$ is solution of the system $\begin{cases} J_\lambda^A(\lambda y_1 + x) = x, \\ J_\lambda^B(\lambda y_2 + x) = x, \\ y_1 - y_2 = x \end{cases}$ Then $\mathfrak{x}$ is solution of (4.1).*

Proof 4.1.1 *Let $(\mathfrak{x}, \eta_1, \eta_2) \in \mathbb{H}^3$ be a solution of the following system:*

$$\begin{cases} J_\lambda^A(\lambda y_1 + x) = x, \\ J_\lambda^B(\lambda y_2 + x) = x, \\ y_1 - y_2 = x. \end{cases}$$

Then, in particular,

$$\begin{cases} J_\lambda^A(\lambda y_1 + x) = x, \\ J_\lambda^B(\lambda y_2 + x) = x, \\ y_1 - y_2 = x. \end{cases}$$

By definition, we have:

$$\bar{y}_1 \in A(\bar{x}), \quad \bar{y}_2 \in B(\bar{x}).$$

Substituting into the last equation, we get:

$$\bar{x} = \bar{y}_1 - \bar{y}_2 \quad \Rightarrow \quad 0 \in A(\bar{x}) - B(\bar{x}) - \bar{x}.$$

that what we want to show.

Theorem 4.1.1 For each $\lambda > 0$, the issue (4.2) has a solution if and only if (4.1) has a solution.

Proof 4.1.2 Let $\lambda > 0$, and assume that (x, y) is a solution of problem (4.2). So

$$\begin{cases} J_\lambda^A(x) - \frac{x - y}{\lambda} = 0, \\ J_\lambda^B(y) - \frac{x - y}{\lambda} = 0. \end{cases}$$

Therefore,

$$\begin{cases} J_\lambda^A(x) = \frac{x - y}{\lambda}, \\ J_\lambda^B(y) = \frac{x - y}{\lambda}. \end{cases}$$

From the first equation, we deduce:

$$y = x - \lambda J_\lambda^A(x).$$

Substituting this expression for y into the second equation:

$$J_\lambda^B(x - \lambda J_\lambda^A(x)) = \frac{x - (x - \lambda J_\lambda^A(x))}{\lambda} = J_\lambda^A(x).$$

Hence,

$$x - \lambda J_\lambda^A(x) \in J_\lambda^A(x) + \lambda B(J_\lambda^A(x)).$$

This implies:

$$x - J_\lambda^A(x) \in \lambda J_\lambda^A(x) + \lambda B(J_\lambda^A(x)).$$

But by the definition of the resolvent again, we also have:

$$x - J_\lambda^A(x) \in \lambda A(J_\lambda^A(x)).$$

Combining the two inclusions, we conclude:

$$0 \in A(J_\lambda^A(x)) - B(J_\lambda^A(x)) - J_\lambda^A(x).$$

Conversely, suppose $0 \in A(s) - B(s) - s$. Then there exist $t_1 \in A(s)$ and $t_2 \in B(s)$ such that:

$$t_1 - t_2 = s.$$

Let us now consider the values:

$$\begin{cases} x = \lambda t_1 + s, \\ y = \lambda t_2 + s. \end{cases}$$

Then we have:

$$s = \frac{x - y}{\lambda}.$$

Moreover, since $x = \lambda t_1 + s$, we obtain:

$$J_\lambda^A(x) = (I + \lambda A)^{-1}(x) = s.$$

Similarly, $J_\lambda^B(y) = s$. Hence:

$$\begin{cases} J_\lambda^A(x) = \frac{x - y}{\lambda}, \\ J_\lambda^B(y) = \frac{x - y}{\lambda}. \end{cases}$$

This confirms that (x, y) satisfies system (4.2), which completes the proof.

Example 4.1.1 Consider $A(x) = \{2x\}$, $B(x) = \{3x\}$ and $\mathbb{H} = \mathbb{R}$. We will find solution of the problem below:

$$0 \in A(x) - B(x) - Id(x)$$

Clearly, for all $\lambda > 0$,

$$J_\lambda^A(x) = \frac{x}{2\lambda + 1} \quad \text{and} \quad J_\lambda^B(x) = \frac{x}{3\lambda + 1}.$$

We can define the function: $\theta_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\theta_\lambda(x, y) = \begin{pmatrix} J_\lambda^A(x) - \frac{x - y}{\lambda} \\ J_\lambda^B(y) - \frac{x - y}{\lambda} \end{pmatrix} = \begin{pmatrix} \frac{x}{2\lambda + 1} - \frac{x - y}{\lambda} \\ \frac{y}{3\lambda + 1} - \frac{x - y}{\lambda} \end{pmatrix}.$$

If $\|\theta_\lambda(x, y)\| = 0$, then

$$\begin{cases} \frac{x}{2\lambda + 1} - \frac{x - y}{\lambda} = 0, \\ \frac{y}{3\lambda + 1} - \frac{x - y}{\lambda} = 0. \end{cases}$$

Hence $(x, y) = (0, 0)$. Apply theorem (2.2), we get $J_\lambda^A(0) = 0$

4.2 The Case of differentiable functions

Consider the function $\theta_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by:

$$(x, y) \mapsto \theta_\lambda(x, y) = \begin{pmatrix} J_\lambda^A(x) - \frac{x-y}{\lambda} \\ J_\lambda^B(y) - \frac{x-y}{\lambda} \end{pmatrix}.$$

We have J_{θ_λ} is:

$$J_{\theta_\lambda}(x, y) = \begin{pmatrix} \frac{\partial J_\lambda^A}{\partial x}(x) - \frac{1}{\lambda} & \frac{1}{\lambda} \\ -\frac{1}{\lambda} & \frac{\partial J_\lambda^B}{\partial y}(y) - \frac{1}{\lambda} \end{pmatrix}.$$

We compute the determinant of $J_{\theta_\lambda}(x, y)$ as follows:

$$\begin{aligned} \det J_{\theta}(x, y) &= \left(\frac{\partial J_\lambda^A}{\partial x}(x) - \frac{1}{\lambda} \right) \left(\frac{\partial J_\lambda^B}{\partial y}(y) - \frac{1}{\lambda} \right) - \left(\frac{1}{\lambda} \right)^2 \\ &= \frac{\partial J_\lambda^A}{\partial x}(x) \frac{\partial J_\lambda^B}{\partial y}(y) - \frac{1}{\lambda} \frac{\partial J_\lambda^A}{\partial x}(x) - \frac{1}{\lambda} \frac{\partial J_\lambda^B}{\partial y}(y). \end{aligned}$$

To numerically solve the system $\theta_\lambda(x, y) = 0$, we employ the Newton–Raphson method. The iterative formula is given by:

$$(x_{k+1}, y_{k+1}) = (x_k, y_k) - J_{\theta_\lambda}^{-1}(x_k, y_k) \theta_\lambda(x_k, y_k), \quad k = 0, 1, 2, \dots$$

Theorem 4.2.1 For all $\lambda > 0$, if $\Psi_\lambda(x) = J_\lambda^B(x - \lambda J_\lambda^A(x)) - J_\lambda^A(x) + x$ has a fixed point $\bar{x}$. Then $J_\lambda^A(\bar{x})$ is solution of the principal problem.

Proof 4.2.1 Let $\lambda > 0$, and suppose that a is a fixed point of the operator Ψ_λ . By definition, we have:

$$\Psi_\lambda(\bar{x}) = \bar{x}.$$

This implies:

$$\Psi_\lambda(\bar{x}) = J_\lambda^B(\bar{x} - \lambda J_\lambda^A(\bar{x})) - J_\lambda^A(\bar{x}) + \bar{x} = \bar{x}.$$

Subtracting a from both sides yields:

$$J_\lambda^B(\bar{x} - \lambda J_\lambda^A(\bar{x})) = J_\lambda^A(\bar{x}).$$

By the definition of the resolvent operator J_λ^B , this equality implies:

$$a - \lambda J_\lambda^A(\bar{x}) \in J_\lambda^A(\bar{x}) + \lambda B(J_\lambda^A(\bar{x})).$$

We get:

$$\bar{x} - J_\lambda^A(\bar{x}) \in \lambda J_\lambda^A(\bar{x}) + \lambda B(J_\lambda^A(\bar{x})).$$

On the other hand, from the definition of $J_\lambda^A(\bar{x})$, we also have:

$$\bar{x} - J_\lambda^A(\bar{x}) \in \lambda A(J_\lambda^A(\bar{x})).$$

Therefore, combining both inclusions:

$$\bar{x} - J_\lambda^A(\bar{x}) \in \lambda A(J_\lambda^A(\bar{x})) \cap (\lambda J_\lambda^A(\bar{x}) + \lambda B(J_\lambda^A(\bar{x}))).$$

Dividing by $\lambda > 0$, we obtain:

$$0 \in A(J_\lambda^A(\bar{x})) - B(J_\lambda^A(\bar{x})) - J_\lambda^A(\bar{x}).$$

Proposition 4.2.1 For all $\lambda > 0$, if A is a maximal monotone operator and B is a β -strongly monotone operator, then Ψ_λ is a $\frac{\lambda+1}{\beta\lambda+1} + \frac{1}{\lambda}$ -Lipschitz continuous mapping.

Proof 4.2.2

$$\begin{aligned} \|\Psi_\lambda(x) - \Psi_\lambda(y)\| &= \|J_\lambda^B(x - \lambda J_\lambda^A(x)) - J_\lambda^A(x) + x - J_\lambda^B(y - \lambda J_\lambda^A(y)) + J_\lambda^A(y) - y\| \\ &\leq \frac{1}{\beta\lambda+1}(\|x - \lambda J_\lambda^A(x) - y + \lambda J_\lambda^A(y)\|) + \frac{1}{\lambda}\|x - y\| \\ &\leq \frac{1}{\beta\lambda+1}(\|x - y\| + \lambda\|x - y\|) + \frac{1}{\lambda}\|x - y\| \\ &\leq (\frac{\lambda+1}{\beta\lambda+1} + \frac{1}{\lambda})\|x - y\|. \end{aligned}$$

Theorem 4.2.2 For all $\lambda > 0, \beta > 0$. If $\beta > 1$ and $\lambda > \frac{-\beta - \sqrt{\beta^2 + 4\beta - 4}}{2(1 - \beta)}$, then Ψ_λ is contractive operator.

Proof 4.2.3 If Ψ_λ is contractive, then we have:

$$\begin{aligned} \frac{\lambda+1}{\beta\lambda+1} + \frac{1}{\lambda} < 1 &\iff \frac{(1-\beta)\lambda^2 + \beta\lambda + 1}{\lambda(\beta\lambda+1)} < 0 \\ &\iff (1-\beta)\lambda^2 + \beta\lambda + 1 < 0. \end{aligned}$$

After studying the sign of this polynomial inequation we deduce the following condition to be Ψ_λ contractive,

$$\begin{cases} \beta > 1, \\ \lambda > \frac{-\beta - \sqrt{\beta^2 + 4\beta - 4}}{2(1 - \beta)}. \end{cases}$$

this complete the proof.

Corollary 4.2.1 *If A is a maximal monotone operator and B is a β -strongly monotone operator, then the sequence*

$$X_{n+1} = J_{\lambda}^B(X_n - \lambda J_{\lambda}^A(X_n)) - J_{\lambda}^A(X_n) + X_n, \quad X_0 \in \mathbb{H},$$

converges strongly to the unique fixed point of Ψ_{λ} .

Example 4.2.1 *We consider the operators:*

$$A(x) = \{2x\}, \quad B(x) = \{3x\}, \quad \mathbb{H} = \mathbb{R}$$

We aim to solve the following inclusion:

$$0 \in A(x) - B(x) - Id(x)$$

Substituting the expressions of the operators, we obtain:

$$0 \in 2x - 3x - x = -2x \quad \Rightarrow \quad x = 0$$

Thus, the exact solution is $x = 0$.

Iterative algorithm

According to Corollary (4.2.1), the sequence (x_n) is defined by:

$$x_{n+1} = J_{\lambda}^B(x_n - \lambda J_{\lambda}^A(x_n)) - J_{\lambda}^A(x_n) + x_n$$

The resolvents are:

$$J_{\lambda}^A(x) = \frac{1}{1+2\lambda}x, \quad J_{\lambda}^B(x) = \frac{1}{1+3\lambda}x$$

Let us now apply the algorithm with $\lambda = 1$ and initial value $x_0 = 1$:

- $J_1^A(x_0) = \frac{1}{1+2} \cdot 1 = \frac{1}{3}$
- $x_0 - \lambda J_1^A(x_0) = 1 - \frac{1}{3} = \frac{2}{3}$
- $J_1^B(x_0 - \lambda J_1^A(x_0)) = \frac{1}{1+3} \cdot \frac{2}{3} = \frac{1}{6}$
- $x_1 = \frac{1}{6} - \frac{1}{3} + 1 = \frac{5}{6}$

Next iteration:

- $J_1^A(x_1) = \frac{1}{3} \cdot \frac{5}{6} = \frac{5}{18}$

-
- $x_1 - J_1^A(x_1) = \frac{5}{6} - \frac{5}{18} = \frac{25}{36}$
 - $J_1^B(x_1 - J_1^A(x_1)) = \frac{1}{4} \cdot \frac{25}{36} = \frac{25}{144}$
 - $x_2 = \frac{25}{144} - \frac{5}{18} + \frac{5}{6} = \frac{25-40+120}{144} = \frac{105}{144} \approx 0.7292$

The sequence (x_n) generated by the algorithm converges strongly to the unique solution $x = 0$, as predicted by Corollary (4.2.1) This example illustrates the convergence behavior in a simple one-dimensional Hilbert space $\mathbb{R}$ with linear monotone operators.

General Conclusion

In this thesis, we investigated several fundamental problems related to maximal monotone operators and their applications in nonlinear analysis and optimization. The main focus was on the problem of finding zeros of maximal monotone operators under different structural settings, together with the development of appropriate solution methods and convergence analysis.

We first addressed the classical problem of finding the zeros of the sum of a finite family of maximal monotone operators. By introducing a suitable function whose zeros correspond to the solutions of the original problem, we established an equivalent reformulation that allowed a deeper analytical treatment. This approach enabled us to derive a system of equations whose solutions coincide with the solutions of the initial problem. Furthermore, we extended the framework by incorporating a Lipschitz continuous perturbation into the sum of maximal monotone operators and presented several results concerning strong monotonicity and inverse strong monotonicity. These properties played a crucial role in ensuring well-posedness and convergence of the proposed methods.

We also studied the problem under various scenarios and proposed several algorithms contributing to its resolution. The theoretical results were supported by illustrative numerical examples, which demonstrated the convergence behavior of the generated sequences toward the desired solutions.

In addition, a dedicated chapter was devoted to the development of an algorithm for finding the zeros of two maximal monotone operators. This case, which can be interpreted as solving problems involving the difference or structured combination of two maximal monotone operators, represents a significant extension of the classical sum framework. We introduced an iterative scheme tailored to this setting and established its convergence under appropriate assumptions. This contribution provides a constructive and practical approach for treating problems that fall outside the standard sum formulation.

Overall, this work contributes to the ongoing development of algorithmic and theoretical tools for problems involving maximal monotone operators. Nevertheless, the design of alter-

native algorithms under weaker assumptions, as well as the extension of these methods to more general operator classes and applied models, remains an open and promising direction for future research.

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