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THESIS

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SPECIALTY :Partial Differential Equations

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Subject

**Existence and Uniqueness of Solutions for Impulsive Fractional
Partial Differential Equation Boundary Value Problems with
the ψ -Caputo Derivative under the Weak Topology.**

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Dedication

I dedicate this work to :

My Father and Mother

No words can fully express my deep love and gratitude for the affection you have given me, and for your endless sacrifices, encouragement, love, support, kindness, guidance, and more.

May this thesis reflect my profound tenderness and appreciation. I hope to be worthy of your trust and expectations. I pray to Allah to grant you lifelong good health and the fulfilment of your wishes.

My Brothers and Sisters

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All those I love and who love me.

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Published Articles

- **Solutions to Variable-Order Fractional BVPs with Multipoint Data in $W^{s,p}$ Spaces**, (July 2025).
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- **Existence of Solutions to the Variable Order Caputo Fractional Thermistor Problem**, (February 2025).
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Articles Under Review

- **Weak Solution for a Boundary Value Problem Involving Kernel Function Power Law Caputo Impulsive Fractional Differential Equations**; Zineb BELLABES^{1,2}, Naima BOUSSEKKINE^{1,4}, Saad ABDELKEBIR^{3*} and Melouka REMIL^{1,4} (Accepted).
- **On the Study of the Global Stability of Volterra-Type Hadamard Fractional Partial Integral Equations with Variable Order**; Hussien ALBALA; Kadda MAAZOUZ; Zineb BELLABES; Naima BOUSSEKKINE; Abdelkader MOU-MEN; Hicham SABER (Accepted).
- **On Initial Value Problems for Sequential Fractional Differential Equations in Weak Topology**; Zineb BELLABES, Said BAGHDAD and Naima BOUSSEKKINE (Submitted).
- **Existence Results for Nonlinear Implicit Fractional Differential Equations Involving the Katugampola Derivative with Variable Order**; John R. GRAEF; Zineb BELLABES; Naima BOUSSEKKINE; Kadda MAAZOUZ (Submitted).

Notations

$\Gamma(z)$	The Gamma function
$B(z, w)$	The Beta function
$L^p([\epsilon_1, \epsilon_2])$	The space of Lebesgue p -integrable functions
$\mathcal{C}(J, \mathbb{R})$	The space of continuously differentiable functions on J
$\mathcal{C}^n(J, \mathbb{R})$	n -times continuously differentiable functions on I , where $n \in \mathbb{N}$
$\mathcal{AC}([\epsilon_1, \epsilon_2])$	The class of absolutely continuous functions on $[\epsilon_1, \epsilon_2]$
${}^\psi I_{\epsilon^+}^\gamma v(z)$	The left-sided ψ -Riemann-Liouville fractional-order integral
${}^{\psi, C} \mathcal{D}_{\epsilon^+}^\gamma v(z)$	The left-sided ψ -Caputo fractional-order derivative
${}^C \mathcal{D}_{\epsilon^+}^\gamma v(z)$	The left-sided Caputo fractional-order derivative
$I_{\epsilon^+}^{\gamma(z)} v(z)$	The left-sided Riemann-Liouville variable-order fractional integral
${}^C \mathcal{D}_{\epsilon^+}^{\gamma(z)} v(z)$	The left-sided Caputo variable-order fractional derivative
$W_{\mathcal{RL}, \epsilon^+}^{s, 1}$	Sobolev space of fractional order s defined via the Riemann–Liouville derivative starting at ϵ^+
\mathcal{M}_E	The family of all nonempty, bounded subsets of a space E
\mathcal{N}_E	The subfamily of relatively weakly compact subsets of E
E, \mathcal{H}	Banach space

Acronyms

FC	Fractional Calculus
BVP	Boundary Value Problem
IVP	Initial Value Problem
FBVP	Fractional Boundary Value Problem
FIVP	Fractional Initial Value Problem
FDE	Fractional Differential Equation
UHR	Ulam-Hyers-Rassias Stability
UH	Ulam-Hyers Stability
GUHR stability	(Generalized Ulam-Hyers-Rassias stability)
ODEs	Ordinary Differential Equations
LRLFI	the left Riemann-Liouville fractional integral
CFD	Caputo Fractional Derivative

Abstract

In this thesis, we study existence, uniqueness, and stability results for several classes of fractional differential equations in Banach spaces. We consider initial value problems for sequential ψ -Caputo fractional Langevin equations, boundary value problems involving impulses with power law kernels, a variable order Caputo thermistor model, and variable order Riemann-Liouville boundary value problems with multi-point data. The analysis is carried out in the weak topology framework using the Pettis integral, the De Blasi measure of weak noncompactness, and fixed-point theorems of Mönch, Schauder, Banach contraction, and Krasnoselskii. Existence is proved via Mönch's theorem, uniqueness via the Banach contraction principle, and generalized Ulam-Hyers-Rassias stability is established. For the variable order thermistor problem, existence and uniqueness are obtained by splitting the order into piecewise constant subintervals and applying Schauder's and Banach's theorems. For the Riemann-Liouville problem, the method of upper and lower solutions combined with Schauder's theorem yields positive solutions in a fractional Sobolev space. Numerical examples illustrate the theoretical findings.

Keywords: ψ -Caputo fractional derivative, Riemann-Liouville fractional derivative, variable order, impulsive differential equations, thermistor problem, weak topology, Pettis integral, De Blasi measure of weak non-compactness, fixed point theorems (Mönch, Schauder, Banach, Krasnoselskii), existence, uniqueness, Ulam-Hyers-Rassias stability, upper and lower solutions, fractional Sobolev space.

Résumé

Dans cette thèse, nous étudions des résultats d'existence, d'unicité et de stabilité pour plusieurs classes d'équations différentielles fractionnaires dans les espaces de Banach. Nous considérons des problèmes de valeur initiale pour des équations de Langevin fractionnaires séquentielles avec dérivée ψ -Caputo, des problèmes aux limites impulsions à noyau puissance, un modèle de thermistor fractionnaire d'ordre variable, et des problèmes aux limites de Riemann-Liouville d'ordre variable avec données multipoints. L'analyse est menée dans le cadre de la topologie faible en utilisant l'intégrale de Pettis, la mesure de non-compacité faible de De Blasi, et les théorèmes de point fixe de Mönch, Schauder, contraction de Banach et Krasnoselskii. L'existence est prouvée par le théorème de Mönch, l'unicité par le principe de contraction de Banach, et la stabilité généralisée d'Ulam-Hyers-Rassias est établie. Pour le problème du thermistor à ordre variable, l'existence et l'unicité sont obtenues en divisant l'ordre en sous-intervalles constants par morceaux et en appliquant les théorèmes de Schauder et de Banach. Pour le problème de Riemann-Liouville, la méthode des sous et sur solutions combinée au théorème de Schauder donne des solutions positives dans un espace de Sobolev fractionnaire. Des exemples numériques illustrent les résultats théoriques.

Mots-clés : dérivée fractionnaire ψ -Caputo, dérivée fractionnaire de Riemann-Liouville, ordre variable, équations différentielles impulsives, problème du thermistor, topologie faible, intégrale de Pettis, mesure de non-compacité faible de De Blasi, théorèmes de point fixe (Mönch, Schauder, Banach, Krasnoselskii), existence, unicité, stabilité d'Ulam-Hyers-Rassias, sous- et sur-solutions, espace de Sobolev fractionnaire.

ملخص

في هذه الأطروحة، ندرس نتائج الوجود والوحدة والاستقرار لعدة فئات من المعادلات التفاضلية الكسرية في فضاءات باناخ. نعتبر مسائل القيمة الابتدائية لمعادلات لانجفان الكسرية المتتالية ذات المشتقة ψ كابوتو، مسائل القيم الحدودية النبضية ذات النواة الأسية، نموذج الثرمستور الكسري ذي الرتبة المتغيرة، ومسائل القيم الحدودية لريمانليوفيل ذات الرتبة المتغيرة مع شروط متعددة النقاط. يتم التحليل في إطار الطوبولوجيا الضعيفة باستخدام تكامل بيتيس، مقياس دي بلاسي للاكتناز الضعيف، ونظريات النقطة الثابتة لـ مونش، شاودر، انكماش باناخ، وكراسنوسيلسكي. تم إثبات الوجود بواسطة نظرية مونش، والوحدة بواسطة مبدأ انكماش باناخ، كما تم إثبات استقرار أولامهايرسراسياس المعمم. بالنسبة لمسألة الثرمستور ذات الرتبة المتغيرة، تم الحصول على الوجود والوحدة بتقسيم الرتبة إلى فترات ثابتة جزئياً وتطبيق نظريتي شاودر و باناخ. بالنسبة لمسألة ريمانليوفيل، تعطي طريقة الحلين العلوي والسفلي مع نظرية شاودر حلاً موجبة في فضاء سوبوليف الكسري. أمثلة عددية توضح النتائج النظرية.

الكلمات المفتاحية:

المعادلات التفاضلية الكسرية؛ مشتق ψ كابوتو؛ المشغلات الكسرية ذات الرتبة المتغيرة؛ مسألة الثرمستور؛ مسائل القيمة الحدية متعددة النقاط؛ الوجود والتميز؛ استقرار أولامهايرز؛ نظرية نقطة ثابت مونش؛ عدم الاكتناز الضعيف؛ مشتق ريمانليوفيل؛ الحلول العليا والسفلى؛ الأنظمة المزدوجة؛ نظريات نقطة الثابت؛ فضاءات باناخ.

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General Introduction

Historical Development of Fractional Calculus

The conceptual foundations of fractional calculus emerged alongside classical calculus. The origins of the discipline trace back to a 1695 correspondence between Leibniz and L'Hôpital, wherein Leibniz first contemplated the interpretation of $\frac{d^n y}{dx^n}$ for non-integer n , particularly for $n = \frac{1}{2}$.

Euler (1738) made significant progress by observing that the derivative formula for power functions:

$$\frac{d^p x^k}{dx^p} = \frac{\Gamma(k+1)}{\Gamma(k-p+1)} x^{k-p} \quad (1)$$

naturally extends to non-integer orders. This insight provided one of the earliest analytic continuations of differential operators.

The field advanced substantially through several key contributions:

- **Lacroix (1819)**: First explicit formula for fractional derivatives of x^m
- **Abel (1823-26)**: Applied fractional operators to solve the tautochrone problem
- **Liouville (1832)**: Developed the first systematic theory using exponential series
- **Riemann (1847)**: Proposed the Riemann-Liouville fractional integral

The 20th century saw rigorous formalization:

- **Weyl's (1917)** periodic fractional operators
- **Grünwald-Letnikov (1867-1938)** difference-based approach
- **Caputo's (1967)** regularized derivative for differential equations
- Emergence of applications in viscoelasticity (1940s) and control theory (1970s)

Fractional derivatives extend beyond integer orders to encompass real, variable, distributed, and even complex values [39, 41, 50, 53]. Their non-local nature distinguishes them from classical differential operators, making them particularly valuable for modeling complex phenomena. This has driven significant research into solving fractional differential equations, establishing fractional calculus as a mature field with dedicated conferences and literature. The discipline's formalization was marked by Ross's pioneering 1974 conference [59] and Oldham & Spanier's foundational monograph [51]. Subsequent works, including the comprehensive reference by Samko et al. [63] and contributions by various authors [18, 46], have expanded both theoretical foundations and practical applications. Recent developments include kernel-based operators like the ψ -Riemann-Liouville [34] and ψ -Caputo derivatives [5], with growing applications across physics, engineering, and numerical analysis [47].

Today, fractional calculus constitutes a vibrant research field with applications spanning anomalous diffusion, signal processing, and biological systems modeling. The historical evolution demonstrates how theoretical mathematics can develop through persistent inquiry into fundamental operator extensions.

Application Domains with Examples

Fractional calculus finds diverse applications across scientific and engineering disciplines. We highlight two representative cases:

Rheology

The characterization of viscoelastic materials through fractional calculus has revolutionized their analysis in damping and insulation applications. This mathematical framework provides superior modeling capabilities for energy-dissipation phenomena compared to classical integer-order approaches [49, 7, 9].

The conventional stress-strain relationship for such materials typically requires multiple terms:

$$\sigma(z) + \sum_{p=1}^P \alpha_p \frac{d^p}{dt^p} \sigma(z) = K_0 \epsilon(z) + \sum_{q=1}^Q K_q \frac{d^q}{dt^q} \epsilon(z), \quad (2)$$

where σ represents normal stress, ϵ indicates strain, and $\{\alpha_p, K_q\}$ are material-dependent coefficients. Fractional calculus yields a remarkably concise alternative:

$$(1 + \mu \mathcal{D}_z^\delta) \sigma(z) = (K_0 + K_1 \mathcal{D}_z^\delta) \epsilon(z), \quad (3)$$

where \mathcal{D}_z^δ denotes the Riemann-Liouville fractional derivative of order δ , and the entire material response is captured by just four parameters (μ, K_0, K_1, δ) [65, 8]. This reduction stems from fractional operators' inherent ability to represent the power-law relaxation spectra observed in polymeric systems [10].

Mathematical expressions involving derivatives and integrals, along with their generalizations to fractional orders, constitute powerful tools in modern mathematical analysis, both theoretical and practical. The foundation of this field, strengthened by concepts from topological studies, functional analytical frameworks, and the theory of fixed points, has undergone substantial progress. Its utility extends to numerous scientific and technical domains, including the dynamics of mechanical systems and theoretical physics. Such equations are indispensable for constructing models in areas like the study of materials that exhibit both viscous and elastic properties, electrochemical processes, systems control, electromagnetic field theory, and fluid flow through permeable substances. The expanding corpus of scholarly work, highlighted by comprehensive texts and investigations into both ordinary and partial fractional equations, confirms their critical role in advancing abstract understanding and solving practical problems. [22, 48]

Coupled fractional-order systems have emerged as a powerful modeling framework for interconnected dynamical processes across multiple disciplines, in which systems of differential equations are linked through shared variables or derivative terms [1]. These formulations prove particularly valuable for capturing complex behaviors in viscoelastic materials, biological networks [77].

Research on multi point boundary value problems (BVPs) has garnered substantial interest, primarily due to their extensive applications in various fields of physics and applied mathematics. The foundational work in this area was established by C.P. Gupta [24], who provided criteria for the resolvability of three-point nonlinear BVPs involving second-order ODEs. This initial research prompted further investigations, leading to the generalization of these results to 'm'-point problems governed by nonlinear growth conditions, as seen in the work of Feng et al. [23] and Ma et al. [44]. The field has since advanced through several key developments, including the use of bifurcation methods to obtain nodal solutions (Sun et al. [66]), analyses focused on positive solutions (Zhang et al. [71]), and significant extensions to settings within Banach spaces (Zhao et al. [75]) and

fractional differential equations (Lv [43]). Finally, a framework for addressing systems of such equations was introduced by Henderson et al. [29]

The theory of non-compactness measures has evolved significantly since De Blasi's introduction of the weak non-compactness measure [20]. Banas and Goebel's subsequent development of the robust measure of non-compactness [11] established a foundational framework that has been extensively applied in nonlinear analysis. Important refinements and applications have been demonstrated by [25], particularly in the study of differential equations. Recent advances employing these techniques have yielded substantial results in functional analysis and operator theory, as evidenced by works in [14, 54]. These developments continue to inform contemporary research in nonlinear functional analysis and its applications to fractional calculus.

Stability of Functional Equations: A Brief Overview

The study of stability in functional equations traces its origins to Ulam's seminal 1940 lecture at the University of Wisconsin, where he first posed the fundamental question of when approximate solutions guarantee the existence of exact solutions [67]. This profound inquiry led to Hyers' 1941 breakthrough result in Banach spaces [30], establishing what is now called Ulam-Hyers stability. The theory was substantially generalized by Rassias in 1978 through his work on variable parameters [58], creating the more versatile Ulam-Hyers-Rassias stability framework. These foundational concepts have since been extended to various classes of equations, including differential, integral, and particularly fractional differential equations.

In our current work, we apply these stability principles to analyze solutions to nonlinear fractional systems 2.1.

Background and Motivation

Fractional calculus has become a powerful tool in modeling complex phenomena across various scientific and engineering disciplines, including viscoelasticity, stochastic processes, signal processing, and control theory. Different fractional operators such as Riemann-Liouville, Hadamard, Hilfer, and conformable derivatives have been studied extensively. A particularly interesting generalization involves fractional operators with kernels dependent on an unknown function, known as fractional integrals and derivatives with respect to

another function.

The historical development of stochastic differential equations can be followed to the beginning of the twentieth century. In 1908, Langevin characterised the fluctuation effects in Brownian motion and introduced the equation that now bears his name. The Langevin model is formulated as follows:

$$m \frac{d^2 u(z)}{dz^2} = -\beta \frac{du(z)}{dz} + F(z) + \eta(z)$$

where m corresponds to the particle mass, β is the viscous friction coefficient, F denotes an external force, and η represents a random force. Typically, one may substitute η with $\frac{dB(z)}{dz}$, thereby converting equation (1) into the following system:

$$\begin{aligned} m dv(z) &= -\beta v(z) dz + F(z) dz + dB(z) \\ \frac{du(z)}{dz} &= v(z) \end{aligned}$$

Here, B indicates classical Brownian motion, which is a stochastic process. The Langevin equation has been extensively employed in various disciplines. Its applications include modeling evacuation processes [36], examining fluid suspensions [28], among others. Additional applications in physical chemistry and electrical engineering can be found in [17]. The classical analysis of the Langevin equation relies on Brownian motion under the assumption that the process is Markovian; this means that the random forces are independent and consequently lack memory.

Although the classical Langevin equation plays an important role in many areas, it is insufficient for characterizing specific behaviors such as anomalous diffusion [21] (including both super-diffusion and sub-diffusion), power-law patterns, and long-range interactions. To overcome these shortcomings, fractional Langevin equations have been proposed in [17, 35, 45].

In the second chapter, we analyze the existence, uniqueness, and generalized Ulam-Hyers-Rassias stability of solutions to a class of Langevin differential equations in an appropriate Banach space, using fixed-point techniques within the weak topology framework.

Thesis overview

This thesis is divided into four chapters.

- **Chapter 1:** Notations and Preliminaries

In the first chapter, we collect some notions and results of functional analysis as well as some technical methods used to establish either existence or stability of some non-linear evolution problems. These results are needed to develop further arguments.

- **Chapter 2:** In this chapter, we focus our study on the existence, uniqueness, and Ulam-Hyers stability of a class of Langevin differential equations under the ψ -Caputo operators. To prove existence results, we apply two classical fixed point theorems: Mönch's fixed point theorem and for the uniqueness result, we use the Banach fixed point theorem. We then continue with our results by studying the Ulam-Hyers stability of solutions. Finally, we extend our analysis to impulsive problems and coupled systems.

Based on techniques from [2, 5], we are interested in the study of certain criteria of existence, uniqueness, and Ulam-Hyers stability for the following problem:

$$\begin{cases} \mathcal{D}^{2\gamma, \psi} v(z) = f(z, v(z), \mathcal{D}^{\gamma, \psi} v(z)), & z \in J := [0, T], \\ v(0) = v_0, \quad \mathcal{D}^{\gamma, \psi} v(0) = v_1, \end{cases} \quad (4)$$

where $T > 0$ is a fixed time horizon, $v_0, v_1 \in E \setminus \{\theta_E\}$ are initial conditions (θ_E being the zero vector in E), $\mathcal{D}^{\gamma, \psi}$ denotes the Caputo-type fractional derivative of order $\gamma \in (0, 1)$ with respect to the function ψ , and the sequential derivative $\mathcal{D}^{2\gamma, \psi}$ is defined through composition as $\mathcal{D}^{2\gamma, \psi} := \mathcal{D}^{\gamma, \psi} \circ \mathcal{D}^{\gamma, \psi}$. The nonlinearity $f : \mathcal{J} \times E \times E \rightarrow E$ satisfies appropriate regularity conditions, and $(E, \|\cdot\|)$ is a general Banach space.

- **Chapter 3:** This chapter extends thermistor problem analysis to variable-order Caputo fractional calculus. Generalizing Mazzouz and Henderson's constant-order model [37], we employ fixed-point methods to establish existence and uniqueness conditions for solutions involving the variable-order derivative, given by:

$$\begin{cases} {}^c\mathcal{D}^{\gamma(z)}v(z) = \frac{\lambda f(z, v(z))}{\left(\int_{v_1}^{v_2} f(b, v(b))db\right)^2}, \\ 0 < \gamma(z) < 1, \quad z \in J = [\varepsilon_1, \varepsilon_2], \quad \varepsilon_1 \geq 0, \quad v(\varepsilon_1) = v_{\varepsilon_1} \in \mathbb{R}. \end{cases} \quad (5)$$

The key distinction and primary challenge of our study lie in the operator ${}^c\mathcal{D}^{\gamma(z)}$, which is the variable-order Caputo fractional derivative of order $\gamma(z)$. This generalization allows the fractional order itself to change with the independent variable (often time), modeling more intricate physical phenomena but introducing significant analytical complexity. The core objective of this chapter is to rigorously address this complexity and establish the well-posedness of the variable-order problem (3.1).

- **Chapter 4:** In this chapter, we extend the analysis of fractional boundary value problems to the variable order case. Our investigation is motivated by the work presented in [26], where the authors studied the existence of solutions for a boundary value problem involving the constant order Riemann-Liouville fractional derivative subject to integral boundary conditions:

$$\begin{cases} \mathcal{D}_{0+}^{\gamma}v(z) + f(z, v(z), \mathcal{D}_{0+}^{\theta}v(z)) = 0, \quad 0 < z < 1, \\ \lim_{z \rightarrow 0+} z^{i-\gamma}v(z) = 0, \quad i = 2, \dots, n, \quad v(1) = \sum_{c=0}^m \lambda_c I_{0+}^{\beta}v(\eta_c). \end{cases} \quad (6)$$

Here, $\mathcal{D}_{0+}^{\gamma}$ and $\mathcal{D}_{0+}^{\theta}$ represent the constant-order Riemann-Liouville fractional derivatives, I_{0+}^{β} denotes the Riemann-Liouville fractional integral of order $\beta > 0$, and $n - 1 \leq \gamma < n$.

Building upon this foundation and previous discussions, we introduce new qualitative results for the following variable order Riemann-Liouville fractional boundary value problem with integral conditions:

$$\begin{cases} \mathcal{D}_{0+}^{\gamma(z)}v(z) + f\left(z, v(z), \mathcal{D}_{0+}^{\theta(z)}v(z)\right) = 0, \quad 0 < z < 1, \\ \lim_{z \rightarrow 0+} z^{i-\gamma(z)}v(z) = 0, \quad i = 2, \dots, n, \quad v(1) = \sum_{c=0}^m \lambda_c I_{0+}^{\beta(z)}v(\eta_c). \end{cases} \quad (7)$$

We consider the variable-order Riemann-Liouville fractional derivatives $\mathcal{D}_{0+}^{\gamma(z)}$ and $\mathcal{D}_{0+}^{\theta(z)}$, along with the fractional integral $I_{0+}^{\beta(z)}$ of order $\beta(z) > 0$. The orders are defined such that $\gamma(z)$ satisfies $n - 1 \leq \gamma(z) < n$ for $z \in (0, 1]$.

The transition from constant orders in problem (4.1) to variable orders in problem (7) represents a significant generalization that enables more precise modeling of systems with memory properties that evolve over time or space.

The final chapter employs the method of upper and lower solutions to prove existence results for various boundary value problems. This approach applies to integer-order, nonlinear, and multi-point problems (see [40, 42, 76, 69, 33]) and is extended here to variable-order fractional calculus. We establish the existence of positive solutions in Sobolev spaces for problems with variable-order Riemann-Liouville fractional derivatives, highlighting the framework's enhanced modeling capabilities for heterogeneous processes and its theoretical and practical significance.

Chapter 1

Preliminaries

This foundational chapter establishes the mathematical framework essential for subsequent developments in this thesis. We systematically present:

Assume that $(E, \|\cdot\|)$ is a Banach space and E^* its topological dual. Denote by $C(J, E)$ the Banach space of all continuous functions from $J = [0, T]$ into E with the norm

$$\|v\|_\infty = \sup_{z \in \mathfrak{J}} \|v(z)\|,$$

and by

- $L^1(J, E)$ the Banach space of Bochner integrable functions $v : J \rightarrow E$ equipped with the norm

$$\|v\|_{L^1} = \int_0^T \|v(s)\| ds.$$

- $L^1(J, \mathbb{R})$ denotes the space of measurable functions $v : J \rightarrow \mathbb{R}$ that are Lebesgue integrable, equipped with the norm

$$\|v\|_{L^1} = \int_0^T |v(\tau)| d\tau. \tag{1.1}$$

- Consider the Banach space $\Upsilon := C(J, \mathbb{R}) \times C(J, \mathbb{R})$ with the norm

$$\|(u, v)\|_\Upsilon = \|u\|_\infty + \|v\|_\infty. \tag{1.2}$$

- Let $J = [0, T] \subset (0, \infty)$ and consider a finite partition $0 = z_0 < z_1 < \dots < z_k < z_{k+1} = T$, which defines the subintervals $J_0 = [z_0, z_1], J_1 = (z_1, z_2], \dots, J_k = (z_k, z_{k+1}]$.

Definition 1.1 (The Banach Space $\mathcal{PC}(J, \mathbb{R}^n)$). *The space of piecewise continuous functions is defined by*

$$\mathcal{PC}(J, \mathbb{R}^n) = \left\{ \Omega : J \rightarrow \mathbb{R}^n \left| \begin{array}{l} (i) \Omega|_{J_q} \in C(I_q, \mathbb{R}^n) \text{ for } q = 0, \dots, k, \\ (ii) \forall q \in \{1, \dots, k\}, \Omega(z_q^+) \in \mathbb{R}^n \text{ exists,} \\ (iii) \forall q \in \{1, \dots, k\}, \Omega(z_q^-) = \Omega(z_q) \end{array} \right. \right\}.$$

This space is a Banach space when equipped with the norm $\|\Omega\|_{\mathcal{PC}} = \sup_{z \in J} |\Omega(z)|$.

1.1 Elements of Fractional Calculus Theory

1.2 Integration and Differentiation in Riemann-Liouville Sense

Definition 1.2. [34] *Let $\gamma > 0$ be a real number and $v : [\varepsilon_1, \varepsilon_2] \rightarrow \mathbb{R}$ be integrable. The operator $I_{\varepsilon_1^+}^\gamma$, representing the LRLFI of order γ , is defined as*

$$I_{\varepsilon_1^+}^\gamma v(z) = \frac{1}{\Gamma(\gamma)} \int_{\varepsilon_1}^z (z-t)^{\gamma-1} v(t) dt,$$

provided the integral exists finitely. Here $\Gamma(\gamma)$ denotes the gamma function.

Definition 1.3 (Riemann-Liouville Fractional Derivative). [34] *Let $\gamma > 0$, $k-1 < \gamma \leq k$ where $k \in \mathbb{N}$, and let $v : [\varepsilon_1, \varepsilon_2] \rightarrow \mathbb{R}$ be an integrable function. The left-sided Riemann-Liouville fractional derivative of order γ is defined by*

$$\mathcal{D}_{\varepsilon_1^+}^\gamma v(z) = \frac{1}{\Gamma(k-\gamma)} \left(\frac{d}{dz} \right)^k \int_{\varepsilon_1}^z (z-s)^{k-\gamma-1} v(s) ds, \quad (1.3)$$

provided the integral exists finitely.

Definition 1.4. [34] *Let $\gamma > 0$ and let $n = \lfloor \gamma \rfloor + 1$. For a function $v \in C^n([\varepsilon_1, \varepsilon_2], \mathbb{R})$ (i.e., v is n -times continuously differentiable), the Caputo fractional derivative of order γ starting at ε is defined by*

$${}^c\mathcal{D}_{\varepsilon_1^+}^\gamma v(z) = \frac{1}{\Gamma(n-\gamma)} \int_{\varepsilon_1}^z (z-s)^{n-\gamma-1} v^{(n)}(s) ds, \quad z > \varepsilon_1,$$

where $v^{(n)}$ denotes the n -th derivative of v . For $\gamma = n \in \mathbb{N}$, we set ${}^c\mathcal{D}_{\varepsilon_1^+}^n v(z) = v^{(n)}(z)$.

Lemma 1.1. [34] *Let $\gamma > 0$ and let $n = \lfloor \gamma \rfloor + 1$. Suppose $v \in C^n([\varepsilon_1, \varepsilon_2], \mathbb{R})$ satisfies*

$${}^c\mathcal{D}_{\varepsilon_1^+}^\gamma v(z) = 0 \quad \text{for all } z \in (\varepsilon_1, \varepsilon_2].$$

Then there exist constants $\rho_0, \rho_1, \dots, \rho_{n-1} \in \mathbb{R}$ such that

$$v(z) = \rho_0 + \rho_1(z - \varepsilon_1) + \rho_2(z - \varepsilon_1)^2 + \dots + \rho_{n-1}(z - \varepsilon_1)^{n-1}.$$

Moreover, for any $\gamma_1, \gamma_2 > 0$ and v as above, the following properties hold:

$$I_{\varepsilon_1^+}^{\gamma} \left({}^c\mathcal{D}_{\varepsilon_1^+}^{\gamma} v(z) \right) = v(z) - \sum_{k=0}^{n-1} \frac{v^{(k)}(\varepsilon_1)}{k!} (z - \varepsilon_1)^k,$$

$${}^c\mathcal{D}_{\varepsilon_1^+}^{\gamma} \left(I_{\varepsilon_1^+}^{\gamma} v(z) \right) = v(z),$$

$$I_{\varepsilon_1^+}^{\gamma_1} I_{\varepsilon_1^+}^{\gamma_2} v(z) = I_{\varepsilon_1^+}^{\gamma_2} I_{\varepsilon_1^+}^{\gamma_1} v(z) = I_{\varepsilon_1^+}^{\gamma_1 + \gamma_2} v(z).$$

Definition 1.5. [34] A function $v : [\varepsilon_1, \varepsilon_2] \rightarrow \mathbb{R}$ is said to belong to the fractional space $AC^n[\varepsilon_1, \varepsilon_2]$ if it admits a representation of the form

$$v(z) = \frac{1}{\Gamma(\gamma)} \int_{\varepsilon_1}^z (z - s)^{\gamma-1} \phi(s) ds + \sum_{j=0}^{k-1} c_j (z - \varepsilon)^j \quad (1.4)$$

for some $\phi \in L^1[\varepsilon_1, \varepsilon_2]$ and constants $c_j \in \mathbb{R}$.

Definition 1.6. [34] Let $\gamma > 0$, ψ be an increasing and positive monotone function, having a continuous derivative on J such that $\psi'(z) \neq 0$ for all $z \in J$. The left-sided ψ -RLFI of a function $v \in L^1$ with respect to ψ of order γ is defined by

$$I^{\gamma, \psi} v(z) = \frac{1}{\Gamma(\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{\gamma-1} v(s) ds.$$

Lemma 1.2. [34] Let $\gamma, \beta > 0$ and $v \in L^1$, then the semigroup property $I^{\gamma, \psi} I^{\beta, \psi} v(z) = I^{\gamma+\beta, \psi} v(z)$ is valid.

Definition 1.7. [5] Let $m \in \mathbb{N}$, v, ψ are m times continuously differentiable on J with ψ increasing and $\psi'(z) \neq 0$ for all $z \in J$. The left ψ -CFD of v of order γ is given by

$$\mathcal{D}^{\gamma, \psi} v(z) = \frac{1}{\Gamma(m - \gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{m-\gamma-1} \left(\frac{1}{\psi'(s)} \frac{d}{ds} \right)^m v^{[m]}(s) ds,$$

where $m = \lceil \gamma \rceil$ for $\gamma \notin \mathbb{N}$, $m = \gamma$ for $\gamma \in \mathbb{N}$. In particular, if $\gamma \in (0, 1)$, we have

$$\mathcal{D}^{\gamma, \psi} v(z) = \frac{1}{\Gamma(1 - \gamma)} \int_0^z (\psi(z) - \psi(s))^{-\gamma} v'(s) ds.$$

Theorem 1.1. [5] Suppose that v is m times continuously differentiable on J . Then, for $\gamma \in (0, 1)$, we have

$$I^{\psi, \gamma} \mathcal{D}^{\gamma, \psi} v(z) = v(z) - v(0). \quad (1.5)$$

Lemma 1.3. [52] Let $\gamma > 0$ and ψ be as in Definition 1.7. For any function $v \in L^1(J, \mathbb{R})$, the ψ -Riemann-Liouville fractional integral $I^{\gamma, \psi} v$ is well-defined and belongs to $AC(J, \mathbb{R})$. Moreover, the ψ -Caputo fractional derivative of this integral satisfies

$$\mathcal{D}^{\gamma, \psi} (I^{\gamma, \psi} v)(z) = v(z) \quad \text{for almost every } z \in J. \quad (1.6)$$

1.3 Integration and Differentiation in the Riemann-Liouville Variable-Order Sense

Definition 1.8 ([61, 68]). Let Γ denote the gamma function, $\varepsilon_1 < \varepsilon_2$ be real numbers and let $J = [\varepsilon_1, \varepsilon_2]$. For a function $\gamma : J \rightarrow (0, \infty)$, the left-sided Riemann-Liouville fractional integral of variable order $\gamma(\cdot)$ of a function v is defined for $z \in (\varepsilon_1, \varepsilon_2]$ as

$$I_{\varepsilon_1^+}^{\gamma(z)} v(z) = \frac{1}{\Gamma(\gamma(z))} \int_{\varepsilon_1}^z (z - \tau)^{\gamma(z)-1} v(\tau) d\tau.$$

Definition 1.9 ([61, 68]). For $-\infty < \varepsilon_1 < \varepsilon_2 < +\infty$, let $\gamma : J \rightarrow (n - 1, n)$ with $n \in \mathbb{N}$. The left-sided Riemann-Liouville fractional derivative of variable order $\gamma(\cdot)$ for a function v is given by

$$\mathcal{D}_{\varepsilon_1^+}^{\gamma(z)} v(z) = \left(\frac{d}{dz} \right)^n \int_{\varepsilon_1}^z \frac{(z - \tau)^{n-\gamma(\tau)-1}}{\Gamma(n - \gamma(\tau))} v(\tau) d\tau = \left(\frac{d}{dz} \right)^n I_{\varepsilon_1^+}^{n-\gamma(z)} v(z), \quad z > \varepsilon_1. \quad (1.7)$$

Definition 1.10. [78] The variable order Caputo derivative of order $\gamma(z)$ of a function v is defined as

$${}^C \mathcal{D}^{\gamma(z)} v(z) = \frac{1}{\Gamma(n - \gamma(z))} \int_0^z (z - s)^{n-\gamma(z)-1} v^{(n)}(s) ds, \quad z > 0, \quad n - 1 < \gamma(z) < n, \quad (1.8)$$

where $n = \lceil \gamma(z) \rceil \in \mathbb{N}$.

The following properties are derived from existing literature:

Lemma 1.4 ([34]). Let $\gamma > 0$, $\varepsilon_1 \geq 0$, $v \in L^1(J)$, and $\mathcal{D}_{\varepsilon_1^+}^\gamma v \in L^1(J)$. If $\mathcal{D}_{\varepsilon_1^+}^\gamma v = 0$, then

$$v(z) = \sum_{k=1}^n c_k (z - \varepsilon_1)^{\gamma-k},$$

where $n = \lceil \gamma \rceil$ and $c_k \in \mathbb{R}$ for $k = 1, \dots, n$.

Lemma 1.5 ([34]). For $\gamma > 0$, $\varepsilon_1 \geq 0$, $v \in L^1(J)$, and $\mathcal{D}_{\varepsilon_1^+}^\gamma v \in L^1(J)$,

$$I_{\varepsilon_1^+}^\gamma \mathcal{D}_{\varepsilon_1^+}^\gamma v(z) = v(z) + \sum_{k=1}^n d_k (z - \varepsilon_1)^{\gamma-k},$$

where $n = \lceil \gamma \rceil$ and $d_k \in \mathbb{R}$ for $k = 1, \dots, n$.

Lemma 1.6 ([34]). Let $\gamma > 0$, $\varepsilon_1 \geq 0$, $v \in L^1(J)$, and $\mathcal{D}_{\varepsilon_1^+}^\gamma v \in L^1(J)$. Then

$$\mathcal{D}_{\varepsilon_1^+}^\gamma I_{\varepsilon_1^+}^\gamma v(z) = v(z).$$

Lemma 1.7 ([34]). For $\gamma_1, \gamma_2 > 0$, $\varepsilon_1 \geq 0$, and $v \in L^1(J)$,

$$I_{\varepsilon_1^+}^{\gamma_1} I_{\varepsilon_1^+}^{\gamma_2} v(z) = I_{\varepsilon_1^+}^{\gamma_2} I_{\varepsilon_1^+}^{\gamma_1} v(z) = I_{\varepsilon_1^+}^{\gamma_1 + \gamma_2} v(z).$$

Remark 1.1 ([72, 74]). In general, for variable-order fractional operators with distinct functions $\gamma_1(z)$ and $\gamma_2(z)$, the semigroup property fails:

$$I_{\varepsilon_1^+}^{\gamma_1(z)} I_{\varepsilon_1^+}^{\gamma_2(z)} v(z) \neq I_{\varepsilon_1^+}^{\gamma_1(z) + \gamma_2(z)} v(z).$$

Example 1.1. Construct a simple example to demonstrate that the two sides of the equation are indeed different.

1. Calculation of the right-hand side (RHS): Given $I_{0^+}^{M(z)+R(z)} 1 = I_{0^+}^{z+0.5} 1$.

$$RHS = \frac{1}{\Gamma(z+0.5)} \int_0^z (z-t)^{(z+0.5)-1} dt = \frac{1}{\Gamma(z+0.5)} \int_0^z (z-t)^{z-0.5} dt.$$

Substitute $u = z - t$:

$$\int_0^z (z-t)^{z-0.5} dt = \int_0^z u^{z-0.5} du = \frac{z^{z+0.5}}{z+0.5}.$$

Thus

$$RHS = \frac{1}{\Gamma(z+0.5)} \cdot \frac{z^{z+0.5}}{z+0.5}.$$

2. Calculation of the left-hand side (LHS): Given $I_{0^+}^{M(z)} I_{0^+}^{R(z)} 1 = I_{0^+}^{0.5} I_{0^+}^z 1$.

First compute the inner integral $g(z) = I_{0^+}^z 1$:

$$g(z) = \frac{1}{\Gamma(z)} \int_0^z (z-t)^{z-1} dt = \frac{1}{\Gamma(z)} \cdot \frac{z^z}{z} = \frac{z^{z-1}}{\Gamma(z)}.$$

Now apply the outer integral $I_{0^+}^{0.5}$ to $g(z)$:

$$LHS = I_{0^+}^{0.5} g(z) = \frac{1}{\Gamma(0.5)} \int_0^z (z-t)^{-0.5} \cdot \frac{t^{t-1}}{\Gamma(t)} dt = \frac{1}{\sqrt{\pi}} \int_0^z (z-t)^{-0.5} \frac{t^{t-1}}{\Gamma(t)} dt.$$

It is immediately clear that $LHS \neq RHS$.

Definition 1.11. A generalized interval $J \subseteq \mathbb{R}$ is defined as any of the following: a standard interval (open, closed, or half-open), a singleton set $\{a\}$ where $a \in \mathbb{R}$, or the empty set \emptyset . This definition follows the conventions established in prior works [73].

A partition \mathcal{P} of J is a finite collection of disjoint generalized intervals such that every element $x \in I$ belongs to exactly one subset $E \in \mathcal{P}$.

A function $g : J \rightarrow \mathbb{R}$ is called piecewise constant with respect to \mathcal{P} if g remains constant on each subset E within the partition.

Lemma 1.8 ([15]). *Let F be a bounded set in $L^p(0, 1)$, where $1 \leq p < \infty$. Assume that:*

$$(i) \lim_{|h| \rightarrow 0} \|\mathcal{T}_h f - f\|_p = 0 \text{ uniformly for } f \in F,$$

$$(ii) \lim_{\varepsilon \rightarrow 0} \int_{1-\varepsilon}^1 |f(\varepsilon)|^p d\varepsilon = 0 \text{ uniformly for } f \in F,$$

where $\mathcal{T}_h f(\varepsilon) = f(\varepsilon + h)$. Then F is relatively compact in $L^p(0, 1)$.

In what follows let $J = [\varepsilon_1, \varepsilon_2]$ be a nonempty interval of \mathbb{R} . Introduce the Sobolev space

$$W^{1,1}(J) = \{v \in L^1(J) \mid v' \in L^1(J)\},$$

equipped with the norm

$$\|v\|_{W^{1,1}} = \|v\|_{L^1} + \|v'\|_{L^1}.$$

Here v' denotes the distributional derivative of v . The space of absolutely continuous functions $AC(J)$ coincides with $W^{1,1}(J)$.

Definition 1.12 ([13]). *The Riemann-Liouville fractional Sobolev space is given by*

$$W_{\text{RL}, \varepsilon_1^+}^{s,1} = \left\{ v \in L^1(J) \mid I_{\varepsilon_1^+}^{1-s} v \in W^{1,1}(J) \right\}, \quad 0 < s < 1.$$

$W_{\text{RL}, \varepsilon_1^+}^{s,1}$ is a Banach space endowed with the norm

$$\|v\|_{W_{\text{RL}, \varepsilon_1^+}^{s,1}} = \|v\|_{L^1} + \left\| I_{\varepsilon_1^+}^{1-s} v \right\|_{W^{1,1}}.$$

Bergounioux et al. in [13] give more details.

1.4 Measure of weak noncompactness

Let \mathcal{M}_E denote the family of all nonempty and bounded subsets of E , and \mathcal{N}_E its subfamily consisting of all relatively weakly compact sets. The symbol B_δ stands for the ball $B(0_E, \delta)$, and we write $\overline{\Omega}$, $\text{conv}\Omega$ to denote the closure and convex hull of a set Ω , respectively.

Definition 1.13. *The De Blasi measure of weak non-compactness is the application $\varpi : \mathcal{M}_E \rightarrow \mathbb{R}^+$ defined by*

$$\varpi(\Omega) = \inf\{\delta > 0 : \text{there exists } W \in \mathcal{N}_E \text{ such that } \Omega \subset W + B_\delta\}.$$

Lemma 1.9 ([12]). *Let D and Q be two elements of \mathcal{M}_E . Then:*

1. $\varpi(D) = 0 \iff D$ is relatively weakly compact.
2. $D \subseteq Q \Rightarrow \varpi(D) \leq \varpi(Q)$.
3. $\varpi(\text{conv}D) = \varpi(D)$.
4. $\varpi(D \cup Q) = \max\{\varpi(D), \varpi(Q)\}$.
5. $\varpi(\lambda D) = |\lambda|\varpi(D), \quad \forall \lambda \in \mathbb{R}$.
6. $\varpi(D + Q) \leq \varpi(D) + \varpi(Q)$.
7. If $(D_n)_{n \geq 0}$ is a decreasing sequence of nonempty bounded and weakly closed subsets of \mathcal{M}_E with $\lim_{n \rightarrow \infty} \varpi(D_n) = 0$, then $D_\infty = \bigcap_{n=0}^{\infty} D_n$ is nonempty and $\varpi(D_\infty) = 0$.

Remark 1.2. If E is a reflexive Banach space, then $\varpi(D) = 0$ for every $D \in \mathcal{M}_E$.

Recall that a sequence $(z_m)_{m \in \mathbb{N}}$ in E is said to converge weakly to $z \in E$ if for every $\phi \in E^*$, we have $\phi(z_m)$ converge to $\phi(z)$. In this case, we write $z_m \rightharpoonup z$ weakly.

Definition 1.14. An operator $L : E \rightarrow E$ is said to be weakly sequentially continuous on E if for every sequence $(z_m)_{m \in \mathbb{N}}$ with $z_m \rightharpoonup z$, we have $Lz_m \rightharpoonup Lz$.

Definition 1.15 ([4, 57]). A function $v : J \rightarrow E$ is Pettis integrable over J if and only if there exists an element v_E in E such that

$$\int_J \phi(v(z)) dz = \phi(v_E), \quad (1.9)$$

for every ϕ in E^* , wherein the integral on the right is the Lebesgue integral. By definition,

$$\int_J v(z) dz = v_E. \quad (1.10)$$

Remark 1.3. It is important to note that from (1.9) and (1.10), we have

$$\int_J \phi(v(z)) dz = \phi \left(\int_J v(z) dz \right), \quad (1.11)$$

for every ϕ in E^* .

Lemma 1.10. [16] If $v(\cdot)$ is Pettis integrable and $h(\cdot)$ is a measurable and essentially bounded real-valued function, then the product $v(\cdot)h(\cdot)$ is Pettis integrable.

Theorem 1.2 ([25, 54]). *Let $Q \subset C(J, E)$ be a bounded and equicontinuous set. Then*

$$\varpi(Q) = \sup_{z \in J} \varpi(Q(z)) = \varpi(Q(J)), \quad (1.12)$$

$$\varpi \left(\int_0^z Q(s) ds \right) \leq \int_0^z \varpi(Q(s)) ds, \quad (1.13)$$

where

$$Q(z) = \{v(z) : v \in Q\}, \quad Q(J) = \bigcup_{z \in J} \{v(z) : v \in Q\}.$$

Remark 1.4. *Note that $\varpi(Q)$ and $\varpi(Q(z))$ are the De Blasi measures of weak noncompactness of the sets Q and $Q(z)$ in the spaces $C(J, E)$ and E , respectively.*

Lemma 1.11. [56] *Let H , K , and G be continuous functions defined on J . If K is non-negative and if G satisfies the integral inequality*

$$G(z) \leq H(z) + \int_0^z K(s)G(s) ds, \quad \forall z \in J,$$

then

$$G(z) \leq H(z) + \int_0^z H(s)K(s) \exp \left(\int_s^z K(\tau) d\tau \right) ds, \quad z \in J.$$

Definition 1.16 (Heaviside function H). *The Heaviside function is given as follows:*

$$H(z) = \begin{cases} 1 & \text{if } z \geq 0 \\ 0 & \text{o.w.} \end{cases} \quad (1.14)$$

Proposition 1.1. [6] *Let Ω be a compact subset of \mathbb{R}^n , and let ω be a bounded subset of $L^1(\Omega, \mathcal{H})$, where \mathcal{H} is a finite-dimensional Banach space. Then, $\kappa(\cdot)$ possesses the following form:*

$$\varpi(\Omega) = \limsup_{\epsilon \rightarrow 0} \left\{ \sup_{\phi \in \omega} \left[\int_D \|\phi\| dt : \text{meas}(D) \leq \epsilon \right] \right\}. \quad (1.15)$$

for any nonempty subset D of Ω , $\text{meas}(\cdot)$ represents the Lebesgue measure.

Let $B : \mathcal{S}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$. Following [31, 32], assume that:

I_1 : If $(\varphi_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}(B)$ is a weakly convergent sequence in \mathcal{H} . Then

$(B\varphi_n)_{n \in \mathbb{N}}$ has a strongly convergent subsequence in \mathcal{H} .

I_2 : If $(\varphi_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}(B)$ is a weakly convergent sequence in \mathcal{H} . Then

$(B\varphi_n)_{n \in \mathbb{N}}$ has a weakly convergent subsequence in \mathcal{H} .

Remark 1.5. 1. A strongly continuous operator φ satisfies the condition (I_1).

2. An operator B satisfies the condition (I_2) if and only if it maps relatively weak compact sets into relatively weak compact sets.

Definition 1.17. [19] Let E and \mathcal{H} be two Banach spaces. A function $v : J \times E \rightarrow \mathcal{H}$, is called a "Carathéodory function" if it satisfies the following two conditions:

1. "Measurability in z ", that is:

$$z \mapsto v(z, x) \text{ is measurable for all } x \in \mathcal{H};$$

2. "Continuity in x ", that is:

$$x \mapsto v(z, x) \text{ is continuous for all } z \in J.$$

1.5 Some Fixed Point Theorems

In this section, we give some fixed point theorems that will be used in the following chapters.

Theorem 1.3 (Banach Fixed Point Theorem (Stefan Banach)). [3] Let (E, d) be a complete metric space and $T : E \rightarrow E$ be a contraction mapping. Then T has a unique fixed point in E [3].

Theorem 1.4 (Arzelà-Ascoli [64]). A subset $\Omega \subset C(J, \mathbb{R}^n)$ is relatively compact if and only if the following two conditions hold:

1. Ω is **bounded**: There exists a constant $M > 0$ such that for every $v \in \Omega$,

$$\|v\|_\infty = \sup_{z \in J} \|u(z)\| \leq M.$$

2. Ω is **equicontinuous**: For every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for any $z_1, z_2 \in J$ and for every $u \in \Omega$,

$$|z_1 - z_2| < \delta \implies \|v(z_1) - v(z_2)\| < \epsilon.$$

Theorem 1.5. (Schauder's fixed point theorem [62]) Let D be a convex subset of a Banach space E and $\mathfrak{N} : D \rightarrow D$ be a completely continuous map. Then \mathfrak{N} has at least one fixed point in D .

Theorem 1.6. (*Mönch fixed point theorem*)[55] Let E be a Banach space, Q a closed, convex subset of E , and $z_0 \in Q$. Assume $L : Q \rightarrow Q$ is a weakly sequentially continuous map with the following property:

$$\Omega \subset Q \text{ and } \Omega = \text{conv}(L\Omega \cup \{z_0\}) \implies \Omega \text{ is weakly compact,} \quad (1.16)$$

then L has a fixed point in Q .

Theorem 1.7. (*Krasnoselskii's fixed point theorem*[27]) Let D be a bounded, convex, and closed nonempty subset of a Banach space \mathcal{H} . Assume that $\mathcal{K} : D \rightarrow \mathcal{H}$ and $\mathcal{J} : D \rightarrow \mathcal{H}$ are two operators:

- \mathcal{K} is continuous and satisfies (I1);
- \mathcal{J} is a contraction and satisfies (I2);
- $\mathcal{K}D + \mathcal{J}D \subseteq D$;
- there exists $\lambda \in [0, 1[$ such that $\varpi(\mathcal{K}S + \mathcal{J}S) \leq \lambda\varpi(S)$ for all $S \subseteq D$.

Then, there is $v \in D$ so that $\mathcal{K}v + \mathcal{J}v = v$.

Chapter 2

On Initial Value Problems for Sequential Fractional Differential Equations in Weak Topologies

Fractional differential equations (FDEs) have gained significant attention due to their broad applications in science and engineering, particularly in modeling complex phenomena like Brownian motion, financial markets, and biological systems. This work examines an initial value problem (IVP) for a sequential fractional Langevin equation in Banach spaces, given by:

$$\begin{cases} \mathcal{D}^{2\gamma, \psi} v(z) = f(z, v(z), \mathcal{D}^{\gamma, \psi} v(z)), & z \in [0, T], \\ v(0) = v_0, \quad \mathcal{D}^{\gamma, \psi} v(0) = v_1, \end{cases} \quad (2.1)$$

where $\mathcal{D}^{\gamma, \psi}$ is the Caputo fractional derivative with respect to a function ψ , and f is a nonlinear operator. The study focuses on *weak solutions* in infinite-dimensional Banach spaces, where classical (strong) solutions may not exist. By leveraging weak topology techniques, we establish existence, uniqueness, and stability results, extending prior work on fractional and abstract differential equations. This framework has implications for both theoretical analysis and practical applications, including particle dynamics, fluid mechanics, and complex systems modeling. This chapter addresses the fundamental challenge of solving differential equations in infinite-dimensional spaces, where traditional "strong" solutions often fail to exist. To break this impasse, we pivot to the framework of weak topology, which allows us to prove the existence of more generalized weak solutions.

Our approach is both innovative and robust: we leverage the Pettis integral and fuse

the analytical power of the De Blasi measure of weak non-compactness with the strategic application of Mönch's fixed point theorem. This powerful combination allows us to establish groundbreaking results regarding the existence and stability of solutions to complex fractional systems, providing a crucial mathematical toolkit for problems beyond the reach of classical method

This research stands out for its novelty, as it integrates and generalizes various types of fractional derivatives for multiple values of the function ψ as follows:

1. $\psi(z) = z$, then the system (2.1) reduces to the Caputo-type system.
2. $\psi(z) = \ln z$, then the system (2.1) reduces to the Caputo-Hadamard-type system.
3. $\psi(z) = z^\sigma$, then the system (2.1) reduces to the Caputo-Erdélyi-Kober-type system.
4. $\psi(z) = \frac{z^\sigma}{\sigma}$, $\sigma > 0$, then the system (2.1) reduces to the Caputo-Katugampola-type system.

We consider the Banach space $C^\gamma(J, E)$ described as:

$$C^\gamma(J, E) = \{v \in C(J, E) \mid \mathcal{D}^{\gamma, \psi} v \in C(J, E)\} \quad (2.2)$$

equipped with the norm

$$\|v\|_\gamma = \|v\|_\infty + \|\mathcal{D}^{\gamma, \psi} v\|_\infty.$$

Remark 2.1. A sequence $(v_m)_{m \in \mathbb{N}} \subset C^\gamma(J, E)$ converges weakly to $v \in C^\gamma(J, E)$ if for every $\phi \in E^*$ and for each $z \in J$ we have $\phi(v_m(z)) \rightarrow \phi(v(z))$ and $\phi(\mathcal{D}^{\gamma, \psi} v_m(z)) \rightarrow \phi(\mathcal{D}^{\gamma, \psi} v(z))$. In this case, a set $\Omega \subset C^\gamma(J, E)$ is relatively weakly compact if Ω and $\mathcal{D}^{\gamma, \psi} \Omega$ are relatively weakly compact, where

$$\mathcal{D}^{\gamma, \psi} \Omega = \{\mathcal{D}^{\gamma, \psi} v \mid v \in \Omega\}.$$

Remark 2.2. According to Theorem 1.2, a set $\Omega \subset C^\gamma(J, E)$ is relatively weakly compact if Ω , $\mathcal{D}^{\gamma, \psi} \Omega$ are bounded, equicontinuous and for each $z \in J$, $\Omega(z)$ and $\mathcal{D}^{\gamma, \psi} \Omega(z)$ are relatively weakly compact in E .

The IVP (2.1) will be examined under the following assumptions:

(H1) For every $z \in J$, the function $f(z, \cdot, \cdot)$ is weak sequentially continuous.

(H2) For any $u, v \in C^\gamma(J, E)$, the function $f(\cdot, u, v)$ is Pettis integrable over J .

(H3) There exists a continuous function $p : J \rightarrow \mathbb{R}^+$ such that for each $z \in J$:

$$\varpi(f(z, \mathcal{D}, \mathcal{Q})) \leq p(z) \max \{ \varpi(\mathcal{D}), \varpi(\mathcal{Q}) \}, \quad \text{for all } \mathcal{D}, \mathcal{Q} \in \mathcal{M}_E,$$

$$\|f(z, u, v)\| \leq p(z)(\|u\| + \|v\|), \quad \text{for all } u, v \in E. \quad (2.3)$$

Theorem 2.1. *Assume that the hypotheses (H1)-(H3) hold. If*

$$\lambda = p^* \left(\frac{|\psi(T) - \psi(0)|^{2\gamma}}{\Gamma(2\gamma + 1)} + \frac{|\psi(T) - \psi(0)|^\gamma}{\Gamma(\gamma + 1)} \right) < 1, \quad p^* = \sup_{z \in J} p(z),$$

then the IVP (2.1) has a solution in $C^\gamma(J, E)$.

Proof. According to (1.5) and (1.6), the IVP (2.1) is equivalent to the following integral equation:

$$\begin{aligned} v(z) &= v_0 + \frac{v_1}{\Gamma(\gamma + 1)} (\psi(z) - \psi(0))^\gamma \\ &\quad + \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{2\gamma-1} f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s)) ds, \end{aligned} \quad (2.4)$$

where the integral in (2.4) is the Pettis integral.

Consider the operator $\mathcal{L} : C^\gamma(J, E) \rightarrow C^\gamma(J, E)$ defined by

$$\begin{aligned} \mathcal{L}v(z) &= v_0 + \frac{v_1}{\Gamma(\gamma + 1)} (\psi(z) - \psi(0))^\gamma \\ &\quad + \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{2\gamma-1} f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s)) ds. \end{aligned}$$

By assumption (H2) and Lemma 1.10, the operator \mathcal{L} is well defined. To prove that $\mathcal{L}v \in C^\gamma(J, E)$, fix $z_0 \in J$ and $v \in C^\gamma(J, E)$. For $z \in J$ with $|z - z_0| < \theta$, $\theta > 0$, we have:

$$\begin{aligned} \|\mathcal{L}v(z) - \mathcal{L}v(z_0)\| &\leq \frac{\|v_1\|}{\Gamma(\gamma + 1)} |(\psi(z) - \psi(0))^\gamma - (\psi(z_0) - \psi(0))^\gamma| \\ &\quad + \frac{1}{\Gamma(2\gamma)} \left\| \int_0^z \psi'(s) (\psi(z) - \psi(s))^{2\gamma-1} f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s)) ds \right. \\ &\quad \left. - \int_0^{z_0} \psi'(s) (\psi(z_0) - \psi(s))^{2\gamma-1} f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s)) ds \right\| \\ &\quad + \frac{1}{\Gamma(2\gamma)} \left\| \int_0^{z_0} \psi'(s) (\psi(z) - \psi(s))^{2\gamma-1} f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s)) ds \right. \\ &\quad \left. - \int_0^{z_0} \psi'(s) (\psi(z_0) - \psi(s))^{2\gamma-1} f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s)) ds \right\| \end{aligned}$$

thus

$$\begin{aligned}
& \|\mathcal{L}v(z) - \mathcal{L}v(z_0)\| \\
& \leq \frac{\|v_1\|}{\Gamma(\gamma+1)} |(\psi(z) - \psi(0))^\gamma - (\psi(z_0) - \psi(0))^\gamma| \\
& \quad + \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) |(\psi(z) - \psi(s))^{2\gamma-1} - (\psi(z_0) - \psi(s))^{2\gamma-1}| \|f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s))\| ds \\
& \quad + \frac{1}{\Gamma(2\gamma)} \int_{z_0}^z \psi'(s) (\psi(z_0) - \psi(s))^{2\gamma-1} \|f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s))\| ds
\end{aligned}$$

hence

$$\begin{aligned}
& \|\mathcal{L}v(z) - \mathcal{L}v(z_0)\| \\
& \leq \frac{\|v_1\|}{\Gamma(\gamma+1)} |(\psi(z) - \psi(0))^\gamma - (\psi(z_0) - \psi(0))^\gamma| \\
& \quad + \frac{p^* \|v\|_\gamma}{\Gamma(2\gamma+1)} [|(\psi(z) - \psi(z_0))^{2\gamma}| - |(\psi(z) - \psi(0))^{2\gamma} - (\psi(z_0) - \psi(0))^{2\gamma}|] \\
& \quad + \frac{p^* \|v\|_\gamma}{\Gamma(2\gamma+1)} |(\psi(z) - \psi(z_0))^{2\gamma}|
\end{aligned}$$

this yields

$$\begin{aligned}
\|\mathcal{L}v(z) - \mathcal{L}v(z_0)\| & \leq \frac{1}{\Gamma(\gamma)} \left\| \int_0^z \psi'(s) (\psi(z) - \psi(s))^{\gamma-1} f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s)) ds \right. \\
& \quad \left. - \int_0^z \psi'(s) (\psi(z_0) - \psi(s))^{\gamma-1} f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s)) ds \right\| \\
& \quad + \frac{1}{\Gamma(\gamma)} \left\| \int_0^z \psi'(s) (\psi(z_0) - \psi(s))^{\gamma-1} f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s)) ds \right. \\
& \quad \left. - \int_0^{z_0} \psi'(s) (\psi(z_0) - \psi(s))^{\gamma-1} f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s)) ds \right\|
\end{aligned}$$

this gives

$$\begin{aligned}
& \|\mathcal{L}v(z) - \mathcal{L}v(z_0)\| \\
& \leq \frac{1}{\Gamma(\gamma)} \int_0^z \psi'(s) |(\psi(z) - \psi(s))^{\gamma-1} - (\psi(z_0) - \psi(s))^{\gamma-1}| \|f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s))\| ds \\
& \quad + \frac{1}{\Gamma(\gamma)} \int_{z_0}^z \psi'(s) (\psi(z_0) - \psi(s))^{\gamma-1} \|f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s))\| ds
\end{aligned}$$

then

$$\begin{aligned} \|\mathcal{L}v(z) - \mathcal{L}v(z_0)\| &\leq \frac{p^*\|v\|_\gamma}{\Gamma(\gamma+1)} [|(\psi(z) - \psi(z_0))^\gamma| - |(\psi(z) - \psi(0))^\gamma| - |(\psi(z_0) - \psi(0))^\gamma|] \\ &\quad + \frac{p^*\|v\|_\gamma}{\Gamma(\gamma+1)} |(\psi(z) - \psi(z_0))^\gamma|. \end{aligned}$$

Since the function ψ is continuous on J , we infer that

$$\|\mathcal{L}v(z) - \mathcal{L}v(z_0)\| \rightarrow 0, \quad \|\mathcal{D}^{\gamma,\psi}\mathcal{L}v(z) - \mathcal{D}^{\gamma,\psi}\mathcal{L}v(z_0)\| \rightarrow 0 \text{ as } \theta \rightarrow 0.$$

Now, we establish an existence result for the operator equation $v(z) = \mathcal{L}v(z)$, $z \in J$ in the space $C^\gamma(J, E)$ by using Theorem 1.6.

For $v \in C^\gamma(J, E)$ and $z \in J$, we have

$$\begin{aligned} \|\mathcal{L}v(z)\| &\leq \|v_0\| + \frac{\|v_1\|}{\Gamma(\gamma+1)} |\psi(z) - \psi(0)|^\gamma \\ &\quad + \frac{1}{\Gamma(2\gamma)} \int_0^z |\psi'(s)| |\psi(z) - \psi(s)|^{2\gamma-1} \|f(s, v(s), \mathcal{D}^{\gamma,\psi}v(s))\| ds \\ &\leq \|v_0\| + \frac{\|v_1\|}{\Gamma(\gamma+1)} |\psi(z) - \psi(0)|^\gamma + \frac{p^*\|v\|_\gamma}{\Gamma(2\gamma+1)} |\psi(z) - \psi(0)|^{2\gamma}, \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{D}^{\gamma,\psi}\mathcal{L}v(z)\| &\leq \|v_1\| + \frac{1}{\Gamma(\gamma)} \int_0^z |\psi'(s)| |\psi(z) - \psi(s)|^{\gamma-1} \|f(s, v(s), \mathcal{D}^{\gamma,\psi}v(s))\| ds \\ &\leq \|v_1\| + \frac{p^*\|v\|_\gamma}{\Gamma(\gamma+1)} |\psi(z) - \psi(0)|^\gamma. \end{aligned}$$

Then

$$\begin{aligned} \|\mathcal{L}v(z)\| + \|\mathcal{D}^{\gamma,\psi}\mathcal{L}v(z)\| &\leq \|v_0\| + \|v_1\| \left(1 + \frac{1}{\Gamma(\gamma+1)} |\psi(z) - \psi(0)|^\gamma \right) \\ &\quad + p^* \left(\frac{|\psi(z) - \psi(0)|^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{|\psi(z) - \psi(0)|^\gamma}{\Gamma(\gamma+1)} \right) \|v\|_\gamma, \end{aligned}$$

hence

$$\|\mathcal{L}v(z)\|_\gamma \leq \|v_0\| + \|v_1\| \left(1 + \frac{1}{\Gamma(\gamma+1)} |\psi(z) - \psi(0)|^\gamma \right) + \lambda \|v\|_\gamma. \quad (2.5)$$

Obviously, the operator \mathcal{L} transforms B_α into itself, where

$$\alpha = \left(\|v_0\| + \|v_1\| \left(1 + \frac{1}{\Gamma(\gamma+1)} |\psi(T) - \psi(0)|^\gamma \right) \right) (1 - \lambda)^{-1}.$$

In view of previous estimates, we have that for all $v \in B_\alpha$ and for each $z, y \in J$ with

$|z - y| < \theta, \theta > 0$:

$$\begin{aligned}
& \|\mathcal{L}v(z) - \mathcal{L}v(y)\|_\gamma \\
& \leq \frac{\|v_1\|}{\Gamma(\gamma + 1)} \left| |\psi(z) - \psi(0)|^\gamma - |\psi(y) - \psi(0)|^\gamma \right| \\
& \quad + \frac{p^* \|v\|_\gamma}{\Gamma(2\gamma + 1)} \left| |\psi(z) - \psi(y)|^{2\gamma} - \left| |\psi(z) - \psi(0)|^{2\gamma} - |\psi(y) - \psi(0)|^{2\gamma} \right| \right| \\
& \quad + \frac{p^* \|v\|_\gamma}{\Gamma(2\gamma + 1)} |\psi(z) - \psi(y)|^{2\gamma} \\
& \quad + \frac{p^* \|v\|_\gamma}{\Gamma(\gamma + 1)} \left| |\psi(z) - \psi(y)|^\gamma - \left| |\psi(z) - \psi(0)|^\gamma - |\psi(y) - \psi(0)|^\gamma \right| \right| \\
& \quad + \frac{p^* \|v\|_\gamma}{\Gamma(\gamma + 1)} |\psi(z) - \psi(y)|^\gamma
\end{aligned}$$

hence

$$\begin{aligned}
& \|\mathcal{L}v(z) - \mathcal{L}v(y)\|_\gamma \\
& \leq \frac{\|v_1\|}{\Gamma(\gamma + 1)} \left| |\psi(z) - \psi(0)|^\gamma - |\psi(y) - \psi(0)|^\gamma \right| \\
& \quad + \frac{p^* \alpha}{\Gamma(2\gamma + 1)} \left| |\psi(z) - \psi(y)|^{2\gamma} - \left| |\psi(z) - \psi(0)|^{2\gamma} - |\psi(y) - \psi(0)|^{2\gamma} \right| \right| \\
& \quad + \frac{p^* \alpha}{\Gamma(2\gamma + 1)} |\psi(z) - \psi(y)|^{2\gamma} \\
& \quad + \frac{p^* \alpha}{\Gamma(\gamma + 1)} \left| |\psi(z) - \psi(y)|^\gamma - \left| |\psi(z) - \psi(0)|^\gamma - |\psi(y) - \psi(0)|^\gamma \right| \right| \\
& \quad + \frac{p^* \alpha}{\Gamma(\gamma + 1)} |\psi(z) - \psi(y)|^\gamma.
\end{aligned}$$

Clearly the right side tends to 0 as $\theta \rightarrow 0$ because the function ψ is continuous on J . Moreover, this convergence is independent of the function v , which implies that the set $\mathcal{L}B_\alpha$ is equicontinuous.

In what follows, we show that the operator \mathcal{L} is weakly sequentially continuous. Let $(v_m)_{m \in \mathbb{N}} \subset B_\alpha$ be a weakly convergent sequence and $\phi \in E^*$. Then for $z \in J$, we have:

$$\begin{aligned}
\phi(\mathcal{L}v_m(z)) &= \phi \left(v_0 + \frac{v_1}{\Gamma(\gamma + 1)} (\psi(z) - \psi(0))^\gamma \right) \\
& \quad + \phi \left(\frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{2\gamma-1} f(s, v_m(s), \mathcal{D}^{\gamma, \psi} v_m(s)) ds \right).
\end{aligned}$$

By (H2) and the properties of the Pettis integral, we obtain:

$$\begin{aligned}
\phi(\mathcal{L}v_m(z)) &= \phi \left(v_0 + \frac{v_1}{\Gamma(\gamma + 1)} (\psi(z) - \psi(0))^\gamma \right) \\
& \quad + \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{2\gamma-1} \phi \left(f(s, v_m(s), \mathcal{D}^{\gamma, \psi} v_m(s)) \right) ds.
\end{aligned}$$

Using assumption (H1) and the Lebesgue dominated convergence theorem yields:

$$\begin{aligned} \lim_{m \rightarrow \infty} \phi(\mathcal{L}v_m(z)) &= \phi \left(v_0 + \frac{v_1}{\Gamma(\gamma + 1)} (\psi(z) - \psi(0))^\gamma \right) \\ &\quad + \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{2\gamma-1} \lim_{m \rightarrow \infty} \phi(f(s, v_m(s), \mathcal{D}^{\gamma, \psi} v_m(s))) ds \end{aligned}$$

thus

$$\begin{aligned} \lim_{m \rightarrow \infty} \phi(\mathcal{L}v_m(z)) &= \phi \left(v_0 + \frac{v_1}{\Gamma(\gamma + 1)} (\psi(z) - \psi(0))^\gamma \right) \\ &\quad + \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{2\gamma-1} \phi(f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s))) ds \\ &= \phi(\mathcal{L}v(z)). \end{aligned}$$

Similarly, we have:

$$\begin{aligned} \lim_{m \rightarrow \infty} \phi(\mathcal{D}^{\gamma, \psi} \mathcal{L}v_m(z)) &= \phi(v_1) + \frac{1}{\Gamma(\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{\gamma-1} \\ &\quad \times \lim_{m \rightarrow \infty} \phi(f(s, v_m(s), \mathcal{D}^{\gamma, \psi} v_m(s))) ds \\ &= \phi \left(v_1 + \frac{1}{\Gamma(\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{\gamma-1} f(s, v_m(s), \mathcal{D}^{\gamma, \psi} v_m(s)) ds \right) \\ &= \phi(\mathcal{D}^{\gamma, \psi} \mathcal{L}v(z)). \end{aligned}$$

This establishes the weak sequential continuity of the operator \mathcal{L} .

It remains to show that the operator \mathcal{L} satisfies property (1.16). Let $\Omega \subset B_\alpha$ such that for $z \in J$:

$$\begin{aligned} \Omega(z) &= \text{conv}(\mathcal{L}\Omega)(z) \cup \{0_E\}, \\ \mathcal{D}^{\gamma, \psi} \Omega(z) &= \text{conv}(\mathcal{D}^{\gamma, \psi} \mathcal{L}\Omega)(z) \cup \{0_E\}. \end{aligned}$$

Then:

$$\begin{aligned} &\varpi(\Omega(z)) + \varpi(\mathcal{D}^{\gamma, \psi} \Omega(z)) \\ &= \varpi(\text{conv}(\mathcal{L}\Omega)(z) \cup \{0_E\}) + \varpi(\text{conv}(\mathcal{D}^{\gamma, \psi} \mathcal{L}\Omega)(z) \cup \{0_E\}) \\ &\leq \varpi \left(v_0 + \frac{v_1}{\Gamma(\gamma + 1)} (\psi(z) - \psi(0))^\gamma + v_1 \right) \\ &\quad + \varpi \left(\frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{2\gamma-1} f(s, \Omega(s), \mathcal{D}^{\gamma, \psi} \Omega(s)) ds \right) \\ &\quad + \varpi \left(\frac{1}{\Gamma(\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{\gamma-1} f(s, \Omega(s), \mathcal{D}^{\gamma, \psi} \Omega(s)) ds \right). \end{aligned}$$

Since $\mathcal{L}B_\alpha$ and $\mathcal{D}^{\gamma,\psi}\mathcal{L}B_\alpha$ are equicontinuous, the sets Ω and $\mathcal{D}^{\gamma,\psi}\Omega$ are equicontinuous. Therefore, the functions $z \mapsto \varpi(\Omega(z))$ and $z \mapsto \varpi(\mathcal{D}^{\gamma,\psi}\Omega(z))$ are continuous on J . Using (1.13) and properties of the function ϖ (Lemma 1.9), we get:

$$\begin{aligned} & \varpi(\Omega(z)) + \varpi(\mathcal{D}^{\gamma,\psi}\Omega(z)) \\ & \leq \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s)(\psi(z) - \psi(s))^{2\gamma-1} \varpi(f(s, \Omega(s), \mathcal{D}^{\gamma,\psi}\Omega(s))) ds \\ & \quad + \frac{1}{\Gamma(\gamma)} \int_0^z \psi'(s)(\psi(z) - \psi(s))^{\gamma-1} \varpi(f(s, \Omega(s), \mathcal{D}^{\gamma,\psi}\Omega(s))) ds \\ & \leq \left(\frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s)(\psi(z) - \psi(s))^{2\gamma-1} ds \right. \\ & \quad \left. + \frac{1}{\Gamma(\gamma)} \int_0^z \psi'(s)(\psi(z) - \psi(s))^{\gamma-1} ds \right) \\ & \quad \times \sup_{s \in J} p(s) (\varpi(\Omega(s)) + \varpi(\mathcal{D}^{\gamma,\psi}\Omega(s))). \end{aligned}$$

In view of (1.12), we have:

$$(\varpi(\Omega) + \varpi(\mathcal{D}^{\gamma,\psi}\Omega)) \leq \lambda (\varpi(\Omega) + \varpi(\mathcal{D}^{\gamma,\psi}\Omega)). \quad (2.6)$$

Since $\lambda < 1$, we deduce that $\varpi(\Omega) + \varpi(\mathcal{D}^{\gamma,\psi}\Omega) = 0$. This implies that the set Ω is relatively weakly compact in the space $C^\gamma(J, E)$. Consequently, by Theorem 4.1, the operator \mathcal{L} has a fixed point in B_α which is a solution of equation (2.4). \square

2.1 Uniqueness and Stability Results

In this section, we investigate the uniqueness and the generalized Ulam-Hyers-Rassias stability of the solution

Theorem 2.2. *Assume that the function $f : J \times E \times E \rightarrow E$ satisfies the hypotheses (H1)-(H2) with $f(z, 0_E, 0_E) = 0_E$ and there exists a continuous function $p : J \rightarrow \mathbb{R}^+$ such that for each $z \in J$:*

$$\varpi(f(z, \mathcal{D}, \mathcal{Q})) \leq p(z) \max\{\varpi(\mathcal{D}), \varpi(\mathcal{Q})\}, \text{ for all } \mathcal{D}, \mathcal{Q} \in \mathcal{M}_E, \quad (2.7)$$

$$\|f(z, u_1, v_1) - f(z, u_2, v_2)\| \leq p(z)(\|u_1 - u_2\| + \|v_1 - v_2\|) \text{ for all } u_1, u_2, v_1, v_2 \in E.$$

If

$$\lambda = p^* \left(\frac{|\psi(T) - \psi(0)|^{2\gamma}}{\Gamma(2\gamma + 1)} + \frac{|\psi(T) - \psi(0)|^\gamma}{\Gamma(\gamma + 1)} \right) < 1, \quad p^* = \sup_{z \in J} p(z),$$

then the IVP (2.1) has a unique solution in $C^\gamma(J, E)$.

Proof. From the inequality (2.7), for each $z \in J$, we have

$$\|f(z, u, v)\| \leq p(z)(\|u\| + \|v\|) + \|f(z, 0_E, 0_E)\| \text{ for all } u, v \in E,$$

so the function f satisfies inequality (2.3). Theorem 4 guarantees the existence of a solution to the IVP (2.1) in the space $C^\gamma(J, E)$.

Let v, w be two solutions of IVP (2.1). Then, for $z \in J$, we get:

$$\begin{aligned} \|v(z) - w(z)\| &\leq \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) |\psi(z) - \psi(s)|^{2\gamma-1} \\ &\quad \times \|f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s)) - f(s, w(s), \mathcal{D}^{\gamma, \psi} w(s))\| ds \\ &\leq \frac{p^* |\psi(z) - \psi(0)|^{2\gamma}}{\Gamma(2\gamma + 1)} \left(\sup_{z \in J} \|v(z) - w(z)\| + \sup_{z \in J} \|\mathcal{D}^{\gamma, \psi} v(z) - \mathcal{D}^{\gamma, \psi} w(z)\| \right) \\ &= \frac{p^* |\psi(z) - \psi(0)|^{2\gamma}}{\Gamma(2\gamma + 1)} \|v - w\|_\gamma, \end{aligned}$$

and

$$\begin{aligned} &\|\mathcal{D}^{\gamma, \psi} v(z) - \mathcal{D}^{\gamma, \psi} w(z)\| \\ &\leq \frac{1}{\Gamma(\gamma)} \int_0^z \psi'(s) |\psi(z) - \psi(s)|^{\gamma-1} \\ &\quad \times \|f(s, v(s), \mathcal{D}^{\gamma, \psi} v(s)) - f(s, w(s), \mathcal{D}^{\gamma, \psi} w(s))\| ds \\ &\leq \frac{p^* |\psi(z) - \psi(0)|^\gamma}{\Gamma(\gamma + 1)} \left(\sup_{z \in J} \|v(z) - w(z)\| + \sup_{z \in J} \|\mathcal{D}^{\gamma, \psi} v(z) - \mathcal{D}^{\gamma, \psi} w(z)\| \right) \\ &= \frac{p^* |\psi(z) - \psi(0)|^\gamma}{\Gamma(\gamma + 1)} \|v - w\|_\gamma. \end{aligned}$$

Hence $\|v - w\|_\gamma \leq \lambda \|v - w\|_\gamma$. Since $\lambda < 1$, we deduce that $\|v - w\|_\gamma = 0$. \square

Let $g : J \times E \rightarrow E$ be a continuous operator and $\zeta : J \rightarrow \mathbb{R}^+$ be continuous. Consider the following differential equation:

$$w'(z) = g(z, w(z)), \quad z \in J. \quad (2.8)$$

Definition 2.1. [60] *The equation (2.8) is generalized Ulam-Hyers-Rassias stable with respect to ζ if there exists $L_{g, \zeta} > 0$ such that for each solution w of the following differential inequality*

$$\|w'(z) - g(z, w(z))\| \leq \zeta(z), \quad z \in J, \quad (2.9)$$

there exists a solution v of (2.8) with

$$\|w(z) - v(z)\| \leq L_{g, \zeta} \zeta(z), \quad z \in J.$$

Based on the above definition, we give the definition of the generalized Ulam-Hyers-Rassias stability of solutions for the IVP (2.1).

Definition 2.2. *The IVP (2.1) is generalized Ulam-Hyers-Rassias stable with respect to ζ if there exists $L_{f,\zeta} > 0$ such that for each solution w of the following differential inequality*

$$\|\mathcal{D}^{2\gamma,\psi}w(z) - f(z, w(z), \mathcal{D}^{\gamma,\psi}w(z))\| \leq \zeta(z), \quad z \in J, \quad (2.10)$$

with initial conditions

$$w(0) = v_0, \quad \mathcal{D}^{\gamma,\psi}w(0) = v_1, \quad (2.11)$$

there exists a solution v of the IVP (2.1) with

$$\|w(z) - v(z)\| + \|\mathcal{D}^{\gamma,\psi}w(z) - \mathcal{D}^{\gamma,\psi}v(z)\| \leq L_{f,\zeta}\zeta(z), \quad z \in J.$$

Remark 2.3. *Let the function $w \in C(J, E)$ be a solution of the differential inequality (2.10)-(2.11). Then, there exists a function $\vartheta \in C(J, E)$ which depends on w such that:*

(a) For $z \in J$, $\|\vartheta(z)\| \leq \zeta(z)$.

(b)

$$\begin{cases} \mathcal{D}^{2\gamma,\psi}w(z) = f(z, w(z), \mathcal{D}^{\gamma,\psi}w(z)) + \vartheta(z), & z \in J, \\ \text{with initial conditions} \\ w(0) = v_0, \quad \mathcal{D}^{\gamma,\psi}w(0) = v_1. \end{cases} \quad (2.12)$$

Lemma 2.1. *The function w is a solution of the problem (2.12) if and only if w satisfies the following integral equation:*

$$\begin{aligned} w(z) &= v_0 + \frac{v_1}{\Gamma(\gamma+1)}(\psi(z) - \psi(0))^\gamma \\ &\quad + \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s)(\psi(z) - \psi(s))^{2\gamma-1} f(s, w(s), \mathcal{D}^{\gamma,\psi}w(s)) ds \\ &\quad + \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s)(\psi(z) - \psi(s))^{2\gamma-1} \vartheta(s) ds. \end{aligned} \quad (2.13)$$

Theorem 2.3. *Assume the conditions of Theorem 2.2 are satisfied. If the differential inequality (2.10)-(2.11) has a solution such that for $z \in J$, the inequalities $I^{2\gamma,\psi}\zeta(z) \leq \varrho\zeta(z)$, $I^{\gamma,\psi}\zeta(z) \leq \varrho'\zeta(z)$ hold, then the IVP (2.1) is generalized Ulam-Hyers-Rassias stable with respect to ζ .*

Proof. Let the function v be the unique solution of the IVP (2.1) and let w be a solution of the integral equation (2.13). Then, for $z \in J$, we have:

$$\begin{aligned}
\|w(z) - v(z)\| &\leq \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) |\psi(z) - \psi(s)|^{2\gamma-1} \\
&\quad \times \|f(s, w(s), \mathcal{D}^{\gamma,\psi} w(s)) - f(s, v(s), \mathcal{D}^{\gamma,\psi} v(s))\| ds \\
&\quad + \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) |\psi(z) - \psi(s)|^{2\gamma-1} \|\vartheta(s)\| ds \\
&\leq \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) |\psi(z) - \psi(s)|^{2\gamma-1} p(s) \\
&\quad \times (\|w(s) - v(s)\| + \|\mathcal{D}^{\gamma,\psi} w(s) - \mathcal{D}^{\gamma,\psi} v(s)\|) ds \\
&\quad + \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) |\psi(z) - \psi(s)|^{2\gamma-1} \zeta(s) ds,
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{D}^{\gamma,\psi} w(s) - \mathcal{D}^{\gamma,\psi} v(s)\| &\leq \frac{1}{\Gamma(\gamma)} \int_0^z \psi'(s) |\psi(z) - \psi(s)|^{\gamma-1} \\
&\quad \times \|f(s, w(s), \mathcal{D}^{\gamma,\psi} w(s)) - f(s, v(s), \mathcal{D}^{\gamma,\psi} v(s))\| ds \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_0^z \psi'(s) |\psi(z) - \psi(s)|^{\gamma-1} \|\vartheta(s)\| ds \\
&\leq \frac{1}{\Gamma(\gamma)} \int_0^z \psi'(s) |\psi(z) - \psi(s)|^{\gamma-1} p(s) \\
&\quad \times (\|w(s) - v(s)\| + \|\mathcal{D}^{\gamma,\psi} w(s) - \mathcal{D}^{\gamma,\psi} v(s)\|) ds \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_0^z \psi'(s) |\psi(z) - \psi(s)|^{\gamma-1} \zeta(s) ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|w(z) - v(z)\| + \|\mathcal{D}^{\gamma,\psi} w(s) - \mathcal{D}^{\gamma,\psi} v(s)\| \\
&\leq \int_0^z \psi'(s) p(s) \left(\frac{|\psi(z) - \psi(s)|^{2\gamma-1}}{\Gamma(2\gamma)} + \frac{|\psi(z) - \psi(s)|^{\gamma-1}}{\Gamma(\gamma)} \right) \\
&\quad \times (\|w(s) - v(s)\| + \|\mathcal{D}^{\gamma,\psi} w(s) - \mathcal{D}^{\gamma,\psi} v(s)\|) ds + \varrho \zeta(z) + \varrho' \zeta(z).
\end{aligned}$$

Applying Grönwall's inequality (Lemma 1.11), we get

$$\begin{aligned}
&\|w(z) - v(z)\| + \|\mathcal{D}^{\gamma,\psi} w(s) - \mathcal{D}^{\gamma,\psi} v(s)\| \\
&\leq \varrho \zeta(z) + \varrho' \zeta(z) \\
&\quad + (\varrho + \varrho') \int_0^z \psi'(s) p(s) \left(\frac{|\psi(z) - \psi(s)|^{2\gamma-1}}{\Gamma(2\gamma)} + \frac{|\psi(z) - \psi(s)|^{\gamma-1}}{\Gamma(\gamma)} \right) \zeta(s) \\
&\quad \times \exp \left(\int_s^z \psi'(\tau) p(\tau) \left(\frac{|\psi(z) - \psi(\tau)|^{2\gamma-1}}{\Gamma(2\gamma)} + \frac{|\psi(z) - \psi(\tau)|^{\gamma-1}}{\Gamma(\gamma)} \right) d\tau \right) ds
\end{aligned}$$

thus

$$\begin{aligned}
& \|w(z) - v(z)\| + \|\mathcal{D}^{\gamma,\psi}w(s) - \mathcal{D}^{\gamma,\psi}v(s)\| \\
& \leq \varrho\zeta(z) + \varrho'\zeta(z) \\
& \quad + p^*(\varrho + \varrho') \int_0^z \psi'(s) \left(\frac{|\psi(z) - \psi(s)|^{2\gamma-1}}{\Gamma(2\gamma)} + \frac{|\psi(z) - \psi(s)|^{\gamma-1}}{\Gamma(\gamma)} \right) \zeta(s) \\
& \quad \times \exp \left(p^* \left(\frac{|\psi(z) - \psi(s)|^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{|\psi(z) - \psi(s)|^\gamma}{\Gamma(\gamma+1)} \right) \right) ds \\
& \leq \varrho\zeta(z) + \varrho'\zeta(z) + p^*M(\varrho + \varrho')(\varrho\zeta(z) + \varrho'\zeta(z)),
\end{aligned}$$

where

$$M = \sup_{z,s \in J} \exp \left(p^* \left(\frac{|\psi(z) - \psi(s)|^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{|\psi(z) - \psi(s)|^\gamma}{\Gamma(\gamma+1)} \right) \right).$$

Then

$$\|w(z) - v(z)\| + \|\mathcal{D}^{\gamma,\psi}w(s) - \mathcal{D}^{\gamma,\psi}v(s)\| \leq \zeta(z) ((\varrho + \varrho') + p^*M(\varrho + \varrho')^2).$$

Consequently, the IVP (2.1) is generalized Ulam-Hyers-Rassias stable on J with respect to ζ with

$$L_{f,\zeta} = (\varrho + \varrho') + p^*M(\varrho + \varrho')^2.$$

2.2 Example

Consider the following IVP:

$$\begin{cases} \mathcal{D}^{\frac{3}{2},\psi}v(z) = \frac{e^{-2z-3}v(z)}{\|v(z)\| + 2z + 1} + \frac{z\mathcal{D}^{\frac{3}{4},\psi}v(z)}{\|\mathcal{D}^{\gamma,\psi}v(z)\| + z^4 + 1} + z, & z \in J = [0, \pi] \\ v(0) = v_0, \quad \mathcal{D}^{\frac{3}{4},\psi}v(0) = v_1, & v_0, v_1 \in \mathbb{R}^p \setminus \{\mathbf{0}_{\mathbb{R}^p}\}, \end{cases} \quad (2.14)$$

where $\mathcal{D}^{\alpha,\psi}$ denotes the ψ -Caputo fractional derivative of order α with $\psi(z) = \ln(z+1)$.

The IVP (2.14) is a special case of IVP (2.1), where

$$f \left(z, v(z), \mathcal{D}^{\frac{3}{4},\psi}v(z) \right) = \frac{e^{-2z-3}}{\|v(z)\| + 2z + 1} v(z) + \frac{z}{\|\mathcal{D}^{\frac{3}{4},\psi}v(z)\| + z^4 + 1} \mathcal{D}^{\frac{3}{4},\psi}v(z) + z. \quad (2.15)$$

Let us show that the assumptions of Theorem 2.1 hold. Observe that for all $z \in J$, $u, v \in \mathbb{R}^p$, $f(z, u, v) \in \mathbb{R}^p$ and for $u, v, u', v' \in \mathbb{R}^p$, we have:

$$\|f(z, u, v) - f(z, u', v')\| \leq p(z)(\|u - v\| + \|u' - v'\|),$$

where $p(z) = e^{-2z-3}$. Since \mathbb{R}^p is a reflexive Banach space and $f : J \times \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ is continuous, then f satisfies (H1)-(H2) with $\varpi(f(z, \mathcal{D}, \mathcal{Q})) = 0$ for every $\mathcal{D}, \mathcal{Q} \in \mathcal{M}_{\mathbb{R}^p}$. Also,

$$\lambda = e^{-3} \left(\frac{(\ln(\pi + 1))^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2} + 1\right)} + \frac{(\ln(\pi + 1))^{\frac{3}{4}}}{\Gamma\left(\frac{3}{4} + 1\right)} \right) < 1.$$

Hence the IVP (2.1) has a unique solution in the space $C^{\frac{3}{4}}(J, \mathbb{R}^p)$.

Now, let $\zeta(z) = \frac{1}{(z+1)^2}$ such that the differential inequality

$$\begin{aligned} & \left\| \mathcal{D}^{\frac{3}{2}, \psi} w(x) - \frac{e^{-2z-3}}{\|w(z)\| + 2z + 1} w(z) - \frac{z}{\|\mathcal{D}^{\gamma, \psi} w(z)\| + z^4 + 1} \mathcal{D}^{\frac{3}{4}, \psi} w(z) - z \right\| \\ & \leq \frac{1}{(z+1)^2}, \end{aligned} \tag{2.16}$$

with initial conditions

$$w(0) = v_0, \quad \mathcal{D}^{\frac{3}{4}, \psi} w(0) = v_1, \tag{2.17}$$

has a solution w in $C^{\frac{3}{4}}(J, \mathbb{R}^p)$.

For $z \in J$, we have:

$$\begin{aligned} I^{\psi, 2\gamma} \zeta(z) &= \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{2\gamma-1} \zeta(s) ds \\ &= \frac{1}{\Gamma(2\gamma)} \int_0^z \frac{1}{s+1} (\ln(z+1) - \ln(s+1))^{2\gamma-1} \left(\frac{1}{(s+1)^2} \right) ds \\ &\leq \frac{1}{\Gamma(2\gamma)} \int_0^z \frac{1}{(s+1)} (\ln(z+1) - \ln(s+1))^{2\gamma-1} ds \\ &= \frac{1}{\Gamma(2\gamma+1)} \frac{(\ln(z+1))^{2\gamma}}{(z+1)^2} (z+1)^2 \\ &\leq \frac{1}{\Gamma(2\gamma+1)} \sup_{z \in J} \left\{ \frac{(\ln(z+1))^{2\gamma}}{(z+1)^2} \right\} \zeta(z) \\ &\leq \frac{1}{\Gamma(2\gamma+1)} \frac{(\ln(\pi+1))^{2\gamma}}{(\pi+1)^2} \zeta(z) = \varrho \zeta(z). \end{aligned}$$

In the same way:

$$\begin{aligned} I^{\psi, \gamma} \zeta(z) &= \frac{1}{\Gamma(\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{\gamma-1} \zeta(s) ds \\ &= \frac{1}{\Gamma(\gamma)} \int_0^z \frac{1}{s+1} (\ln(z+1) - \ln(s+1))^{\gamma-1} \left(\frac{1}{(s+1)^2} \right) ds \end{aligned}$$

thus

$$\begin{aligned}
 I^{\psi, \gamma} \zeta(z) &\leq \frac{1}{\Gamma(\gamma)} \int_0^z \frac{1}{(s+1)} (\ln(z+1) - \ln(s+1))^{\gamma-1} ds \\
 &= \frac{1}{\Gamma(\gamma+1)} \frac{(\ln(z+1))^\gamma}{(z+1)^2} (z+1)^2 \\
 &\leq \frac{1}{\Gamma(\gamma+1)} \sup_{z \in J} \left\{ \frac{(\ln(z+1))^\gamma}{(z+1)^2} \right\} \zeta(z) \\
 &\leq \frac{1}{\Gamma(\gamma+1)} \frac{(\ln(\pi+1))^\gamma}{(\pi+1)^2} \zeta(z) = \varrho' \zeta(z).
 \end{aligned}$$

Consequently, together with Theorem 6, the IVP (2.14) is generalized Ulam-Hyers-Rassias stable on J with respect to ζ . \square

2.3 Weak Solutions for a Coupled System of Fractional Differential Equations

Consider the following coupled system of two ψ -Caputo fractional differential equations:

$$\begin{cases}
 \mathcal{D}^{2\gamma, \psi} v(z) = f(z, v(z), w(z), \mathcal{D}^{\gamma, \psi} v(z), \mathcal{D}^{\gamma, \psi} w(z)), & z \in [0, T], \\
 \mathcal{D}^{2\gamma, \psi} w(z) = g(z, v(z), w(z), \mathcal{D}^{\gamma, \psi} v(z), \mathcal{D}^{\gamma, \psi} w(z)), & z \in [0, T], \\
 v(0) = v_0, \quad \mathcal{D}^{\gamma, \psi} v(0) = v_1, \\
 w(0) = w_0, \quad \mathcal{D}^{\gamma, \psi} w(0) = w_1,
 \end{cases} \quad (2.18)$$

The solution to the coupled system (2.18) is a pair of functions $(v, w) \in C^\gamma(J, E) \times C^\gamma(J, E)$ that satisfy the following system of nonlinear Volterra integral equations on the product space $\mathcal{X} = C^\gamma(J, E) \times C^\gamma(J, E)$. The operator $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) : \mathcal{X} \rightarrow \mathcal{X}$ is defined by:

$$\mathcal{L}(v, w)(z) = \left(\mathcal{L}_1(v, w)(z), \mathcal{L}_2(v, w)(z) \right),$$

where the components are given by:

$$\begin{aligned}
 \mathcal{L}_1(v, w)(z) &= v_0 + \frac{v_1}{\Gamma(\gamma+1)} (\psi(z) - \psi(0))^\gamma \\
 &\quad + \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{2\gamma-1} f(s, v(s), w(s), \mathcal{D}^{\gamma, \psi} v(s), \mathcal{D}^{\gamma, \psi} w(s)) ds,
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{L}_2(v, w)(z) &= w_0 + \frac{w_1}{\Gamma(\gamma+1)} (\psi(z) - \psi(0))^\gamma \\
 &\quad + \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{2\gamma-1} g(s, v(s), w(s), \mathcal{D}^{\gamma, \psi} v(s), \mathcal{D}^{\gamma, \psi} w(s)) ds.
 \end{aligned}$$

where

$$\begin{aligned} v(z) &= v_0 + \frac{v_1}{\Gamma(\gamma + 1)}(\psi(z) - \psi(0))^\gamma \\ &\quad + \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s)(\psi(z) - \psi(s))^{2\gamma-1} f(s, v(s), w(s), \mathcal{D}^{\gamma,\psi}v(s), \mathcal{D}^{\gamma,\psi}w(s)) ds, \end{aligned} \quad (2.19)$$

$$\begin{aligned} w(z) &= w_0 + \frac{w_1}{\Gamma(\gamma + 1)}(\psi(z) - \psi(0))^\gamma \\ &\quad + \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s)(\psi(z) - \psi(s))^{2\gamma-1} g(s, v(s), w(s), \mathcal{D}^{\gamma,\psi}v(s), \mathcal{D}^{\gamma,\psi}w(s)) ds. \end{aligned} \quad (2.20)$$

For the coupled system, the solution pair (v, w) is sought in the product Banach space:

$$\mathcal{X} = C^\gamma(J, E) \times C^\gamma(J, E),$$

which consists of all ordered pairs (v, w) such that $v, w \in C^\gamma(J, E)$, where

$$C^\gamma(J, E) = \{u \in C(J, E) \mid \mathcal{D}^{\gamma,\psi}u \in C(J, E)\}.$$

We equip the space \mathcal{X} with the maximum norm $\|\cdot\|_{\mathcal{X}}$ defined by

$$\|(v, w)\|_{\mathcal{X}} = \max\{\|v\|_\gamma, \|w\|_\gamma\}, \quad (2.21)$$

where the norm $\|\cdot\|_\gamma$ on $C^\gamma(J, E)$ is given by

$$\|u\|_\gamma = \|u\|_\infty + \|\mathcal{D}^{\gamma,\psi}u\|_\infty.$$

This choice of norm makes $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ a Banach space. The initial value problem for the coupled system will be examined under the following assumptions:

(H1) For every $z \in J$, the functions $f(z, \cdot, \cdot, \cdot, \cdot)$ and $g(z, \cdot, \cdot, \cdot, \cdot)$ are weak sequentially continuous.

(H2) For any $v, w \in C^\gamma(J, E)$, the functions

$$f(\cdot, v(\cdot), w(\cdot), \mathcal{D}^{\gamma,\psi}v(\cdot), \mathcal{D}^{\gamma,\psi}w(\cdot)), \quad g(\cdot, v(\cdot), w(\cdot), \mathcal{D}^{\gamma,\psi}v(\cdot), \mathcal{D}^{\gamma,\psi}w(\cdot)),$$

are Pettis integrable over J .

(H3) There exist continuous functions $p_f, p_g : J \rightarrow \mathbb{R}^+$ such that for each $z \in J$ and all bounded subsets $\mathcal{D}_1, \mathcal{D}_2, \mathcal{Q}_1, \mathcal{Q}_2 \subset E$:

$$\varpi(f(z, \mathcal{D}_1, \mathcal{D}_2, \mathcal{Q}_1, \mathcal{Q}_2)) \leq p_f(z) \max\{\varpi(\mathcal{D}_1), \varpi(\mathcal{D}_2), \varpi(\mathcal{Q}_1), \varpi(\mathcal{Q}_2)\},$$

$$\|f(z, u_1, u_2, v_1, v_2)\| \leq p_f(z) (\|u_1\| + \|u_2\| + \|v_1\| + \|v_2\|), \quad \forall u_1, u_2, v_1, v_2 \in E.$$

Similarly, for the function g , there exists a continuous function $p_g : J \rightarrow \mathbb{R}^+$ such that for each $z \in J$:

$$\begin{aligned} \varpi(g(z, \mathcal{D}_1, \mathcal{D}_2, \mathcal{Q}_1, \mathcal{Q}_2)) &\leq p_g(z) \max \{ \varpi(\mathcal{D}_1), \varpi(\mathcal{D}_2), \varpi(\mathcal{Q}_1), \varpi(\mathcal{Q}_2) \}, \\ \|g(z, u_1, u_2, v_1, v_2)\| &\leq p_g(z) (\|u_1\| + \|u_2\| + \|v_1\| + \|v_2\|), \quad \forall u_1, u_2, v_1, v_2 \in E. \end{aligned}$$

(H4) The constants $p_f^* = \sup_{z \in J} p_f(z)$ and $p_g^* = \sup_{z \in J} p_g(z)$ are finite, and the following inequality holds:

$$\lambda = \max \{ p_f^*, p_g^* \} \left(\frac{|\psi(T) - \psi(0)|^{2\gamma}}{\Gamma(2\gamma + 1)} + \frac{|\psi(T) - \psi(0)|^\gamma}{\Gamma(\gamma + 1)} \right) < 1.$$

Theorem 2.4. *Assume that the hypotheses (H1)-(H4) hold for the coupled system. Then, the coupled IVP has a solution (v, w) in the space $\mathcal{X} = C^\gamma(J, E) \times C^\gamma(J, E)$.*

Proof. By assumption (H2) and the properties of the Pettis integral, the operator $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$ is well-defined on \mathcal{X} . To prove that $\mathcal{L}_1(v, w) \in C^\gamma(J, E)$ (and similarly for \mathcal{L}_2), we fix $z_0 \in J$ and $(v, w) \in \mathcal{X}$. The continuity proof for \mathcal{L}_1 follows the same structure as the single-equation case, but the norm of f is now estimated using (H3):

$$\begin{aligned} &\|f(s, v(s), w(s), \mathcal{D}^{\gamma, \psi} v(s), \mathcal{D}^{\gamma, \psi} w(s))\| \\ &\leq p_f(s) (\|v(s)\| + \|w(s)\| + \|\mathcal{D}^{\gamma, \psi} v(s)\| + \|\mathcal{D}^{\gamma, \psi} w(s)\|) \leq 4p_f^* \|(v, w)\|_{\mathcal{X}}. \end{aligned}$$

The same applies to g for \mathcal{L}_2 , with p_g^* . The rest of the continuity argument, relying on the continuity of ψ , remains unchanged.

Now, we establish an existence result for the operator equation $(v, w) = \mathcal{L}(v, w)$ in the space \mathcal{X} by using theorem 1.6

For $(v, w) \in \mathcal{X}$ and $z \in J$, we have the following estimates for the first component:

$$\begin{aligned} \|\mathcal{L}_1(v, w)(z)\| &\leq \|v_0\| + \frac{\|v_1\|}{\Gamma(\gamma + 1)} |\psi(z) - \psi(0)|^\gamma + \frac{4p_f^* \|(v, w)\|_{\mathcal{X}}}{\Gamma(2\gamma + 1)} |\psi(z) - \psi(0)|^{2\gamma}, \\ \|\mathcal{D}^{\gamma, \psi} \mathcal{L}_1(v, w)(z)\| &\leq \|v_1\| + \frac{4p_f^* \|(v, w)\|_{\mathcal{X}}}{\Gamma(\gamma + 1)} |\psi(z) - \psi(0)|^\gamma. \end{aligned}$$

Therefore, combining these:

$$\begin{aligned} \|\mathcal{L}_1(v, w)\|_\gamma &\leq \|v_0\| + \|v_1\| \left(1 + \frac{|\psi(z) - \psi(0)|^\gamma}{\Gamma(\gamma + 1)} \right) \\ &\quad + 4p_f^* \left(\frac{|\psi(z) - \psi(0)|^{2\gamma}}{\Gamma(2\gamma + 1)} + \frac{|\psi(z) - \psi(0)|^\gamma}{\Gamma(\gamma + 1)} \right) \|(v, w)\|_{\mathcal{X}}. \end{aligned}$$

A similar estimate holds for the second component:

$$\begin{aligned} \|\mathcal{L}_2(v, w)\|_\gamma &\leq \|w_0\| + \|w_1\| \left(1 + \frac{|\psi(z) - \psi(0)|^\gamma}{\Gamma(\gamma + 1)}\right) \\ &\quad + 4p_g^* \left(\frac{|\psi(z) - \psi(0)|^{2\gamma}}{\Gamma(2\gamma + 1)} + \frac{|\psi(z) - \psi(0)|^\gamma}{\Gamma(\gamma + 1)}\right) \|(v, w)\|_{\mathcal{X}}. \end{aligned}$$

Using the maximum norm

$$\|\mathcal{L}(v, w)\|_{\mathcal{X}} = \max\{\|\mathcal{L}_1(v, w)\|_\gamma, \|\mathcal{L}_2(v, w)\|_\gamma\},$$

we get:

$$\begin{aligned} \|\mathcal{L}(v, w)\|_{\mathcal{X}} &\leq \max\{\|v_0\| + \|v_1\|, \|w_0\| + \|w_1\|\} \left(1 + \frac{|\psi(T) - \psi(0)|^\gamma}{\Gamma(\gamma + 1)}\right) \\ &\quad + 4 \max\{p_f^*, p_g^*\} \left(\frac{|\psi(T) - \psi(0)|^{2\gamma}}{\Gamma(2\gamma + 1)} + \frac{|\psi(T) - \psi(0)|^\gamma}{\Gamma(\gamma + 1)}\right) \|(v, w)\|_{\mathcal{X}}. \end{aligned}$$

Let

$$\begin{aligned} M &= \max\{\|v_0\| + \|v_1\|, \|w_0\| + \|w_1\|\} \left(1 + \frac{|\psi(T) - \psi(0)|^\gamma}{\Gamma(\gamma + 1)}\right), \\ \lambda &= 4 \max\{p_f^*, p_g^*\} \left(\frac{|\psi(T) - \psi(0)|^{2\gamma}}{\Gamma(2\gamma + 1)} + \frac{|\psi(T) - \psi(0)|^\gamma}{\Gamma(\gamma + 1)}\right). \end{aligned}$$

By (H4), $\lambda < 1$. Thus, we have:

$$\|\mathcal{L}(v, w)\|_{\mathcal{X}} \leq M + \lambda \|(v, w)\|_{\mathcal{X}}.$$

The operator \mathcal{L} transforms the ball

$$B_{\alpha'} = \{(v, w) \in \mathcal{X} : \|(v, w)\|_{\mathcal{X}} \leq \alpha'\},$$

into itself, where

$$\alpha' = \frac{M}{(1 - \lambda)}.$$

The proof of equicontinuity for the set $\mathcal{L}(B_{\alpha'})$ follows the same pattern as the single-equation case. The right-hand sides of the inequalities will tend to zero independently of the choice of $(v, w) \in B_{\alpha'}$ due to the continuity of ψ and the uniform bound $\|(v, w)\|_{\mathcal{X}} \leq \alpha'$.

To show that \mathcal{L} is weakly sequentially continuous, let $(v_m, w_m) \xrightarrow{w} (v, w)$ in \mathcal{X} . For any $\phi \in E^*$ and $z \in J$, we consider $\phi(\mathcal{L}_1(v_m, w_m)(z))$ defined by

$$\begin{aligned} &\phi(\mathcal{L}_1(v_m, w_m)(z)) \\ &= \phi \left(v_0 + \frac{v_1}{\Gamma(\gamma + 1)} (\psi(z) - \psi(0))^\gamma \right) \\ &\quad + \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{2\gamma-1} \phi(f(s, v_m(s), w_m(s), \mathcal{D}^{\gamma, \psi} v_m(s), \mathcal{D}^{\gamma, \psi} w_m(s))) ds. \end{aligned}$$

By (H1), f is weakly sequentially continuous, so the integrand converges point wise:

$$\phi(f(s, v_m(s), w_m(s), \mathcal{D}^{\gamma, \psi} v_m(s), \mathcal{D}^{\gamma, \psi} w_m(s))) \rightarrow \phi(f(s, v(s), w(s), \mathcal{D}^{\gamma, \psi} v(s), \mathcal{D}^{\gamma, \psi} w(s))).$$

Furthermore, the integrand is dominated by an integrable function (from (H3)). Thus, by the Lebesgue Dominated Convergence Theorem for the Pettis integral, we conclude:

$$\lim_{m \rightarrow \infty} \phi(\mathcal{L}_1(v_m, w_m)(z)) = \phi(\mathcal{L}_1(v, w)(z)).$$

A similar argument applies to \mathcal{L}_2 and to the derivatives $\mathcal{D}^{\gamma, \psi} \mathcal{L}_1$ and $\mathcal{D}^{\gamma, \psi} \mathcal{L}_2$, establishing the weak sequential continuity of \mathcal{L} . This establishes the weak sequential continuity of the operator \mathcal{L} .

It remains to show that the operator \mathcal{L} satisfies property (1.16). Let $\Omega \subset B_{\alpha'}$ such that for $z \in J$:

$$\begin{aligned} \Omega(z) &= \text{conv}(\mathcal{L}\Omega)(z) \cup \{0_E\}, \\ \mathcal{D}^{\gamma, \psi} \Omega(z) &= \text{conv}(\mathcal{D}^{\gamma, \psi} \mathcal{L}\Omega)(z) \cup \{0_E\}. \end{aligned}$$

Then, for \mathcal{L}_1

$$\begin{aligned} &\varpi(\Omega_1(z)) + \varpi(\mathcal{D}^{\gamma, \psi} \Omega_1(z)) \\ &= \varpi(\text{conv}(\mathcal{L}_1 \Omega)(z) \cup \{0_E\}) + \varpi(\text{conv}(\mathcal{D}^{\gamma, \psi} \mathcal{L}_1 \Omega_1)(z) \cup \{0_E\}) \\ &\leq \varpi \left(v_0 + \frac{v_1}{\Gamma(\gamma + 1)} (\psi(z) - \psi(0))^\gamma + v_1 \right) \\ &\quad + \varpi \left(\frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{2\gamma-1} f(s, \Omega_1(s), \mathcal{D}^{\gamma, \psi} \Omega_1(s)) ds \right) \\ &\quad + \varpi \left(\frac{1}{\Gamma(\gamma_1)} \int_0^z \psi'(s) (\psi(z) - \psi(s))^{\gamma-1} f(s, \Omega_1(s), \mathcal{D}^{\gamma, \psi} \Omega_1(s)) ds \right). \end{aligned}$$

Since $\mathcal{L}_1 B_{\alpha'}$ and $\mathcal{D}^{\gamma, \psi} \mathcal{L}_1 B_{\alpha'}$ are equicontinuous, the sets Ω_1 and $\mathcal{D}^{\gamma, \psi} \Omega_1$ are equicontinuous. Therefore, the functions $z \mapsto \varpi(\Omega_1(z))$ and $z \mapsto \varpi(\mathcal{D}^{\gamma, \psi} \Omega_1(z))$ are continuous on J . Using (1.13) and properties of the function ϖ (Lemma 1.9), we get:

$$\begin{aligned}
& \varpi(\Omega_1(z)) + \varpi(\mathcal{D}^{\gamma,\psi}\Omega_1(z)) \\
& \leq \frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s)(\psi(z) - \psi(s))^{2\gamma-1} \varpi(f(s, \Omega_1(s), \mathcal{D}^{\gamma,\psi}\Omega_1(s))) ds \\
& \quad + \frac{1}{\Gamma(\gamma)} \int_0^z \psi'(s)(\psi(z) - \psi(s))^{\gamma-1} \varpi(f(s, \Omega_1(s), \mathcal{D}^{\gamma,\psi}\Omega_1(s))) ds \\
& \leq \left(\frac{1}{\Gamma(2\gamma)} \int_0^z \psi'(s)(\psi(z) - \psi(s))^{2\gamma-1} ds \right. \\
& \quad \left. + \frac{1}{\Gamma(\gamma)} \int_0^z \psi'(s)(\psi(z) - \psi(s))^{\gamma-1} ds \right) \\
& \quad \times \sup_{s \in J} p(s) (\varpi(\Omega_1(s)) + \varpi(\mathcal{D}^{\gamma,\psi}\Omega_1(s))).
\end{aligned}$$

A similar argument applies to \mathcal{L}_2 and to the derivatives $\mathcal{D}^{\gamma,\psi}\mathcal{L}_1$ and $\mathcal{D}^{\gamma,\psi}\mathcal{L}_2$. In view of (1.12), we have:

$$\begin{aligned}
& \max\{\varpi(\Omega_1) + \varpi(\mathcal{D}^{\gamma,\psi}\Omega_1), \varpi(\Omega_2) + \varpi(\mathcal{D}^{\gamma,\psi}\Omega_2)\} \\
& \leq \lambda \max\{\varpi(\Omega_1) + \varpi(\mathcal{D}^{\gamma,\psi}\Omega_1), \varpi(\Omega_2) + \varpi(\mathcal{D}^{\gamma,\psi}\Omega_2)\}.
\end{aligned} \tag{2.22}$$

Since $\lambda < 1$, we deduce that $\varpi(\Omega) + \varpi(\mathcal{D}^{\gamma,\psi}\Omega) = 0$. This implies that the set Ω is relatively weakly compact in the space $C^\gamma(J, E)$. Consequently, by Theorem 4.1, the operator \mathcal{L} has a fixed point in $B_{\alpha'}$ which is a solution of equation (2.4). \square

2.4 Example

Consider the following coupled system of IVPs:

$$\left\{ \begin{array}{l}
\mathcal{D}^{\frac{3}{2},\psi}v_1(z) = \frac{e^{-2z-3}}{\|v_1(z)\| + \|v_2(z)\| + 2z + 1}v_1(z) + \frac{z}{\|\mathcal{D}^{\frac{3}{4},\psi}v_1(z)\| + z^4 + 1}\mathcal{D}^{\frac{3}{4},\psi}v_2(z) + z, \\
\mathcal{D}^{\frac{3}{2},\psi}v_2(z) = \frac{\sin z}{\|v_1(z)\| + \|v_2(z)\| + z^2 + 2}v_2(z) + \frac{e^{-z}}{\|\mathcal{D}^{\frac{3}{4},\psi}v_2(z)\| + 1}\mathcal{D}^{\frac{3}{4},\psi}v_1(z) + \cos z, \\
z \in J = [0, \pi] \\
v_1(0) = v_0^{(1)}, \quad \mathcal{D}^{\frac{3}{4},\psi}v_1(0) = v_1^{(1)}, \\
v_2(0) = v_0^{(2)}, \quad \mathcal{D}^{\frac{3}{4},\psi}v_2(0) = v_1^{(2)}, \\
v_0^{(i)}, v_1^{(i)} \in \mathbb{R}^p \setminus \{\mathbf{0}_{\mathbb{R}^p}\}, \quad i = 1, 2,
\end{array} \right. \tag{2.23}$$

where $\mathcal{D}^{\alpha,\psi}$ denotes the ψ -Caputo fractional derivative of order α with $\psi(z) = \ln(z + 1)$. The coupled system (2.23) is a special case of the general coupled system, where the nonlinearities are given by:

$$\begin{aligned} & f_1 \left(z, v_1(z), v_2(z), \mathcal{D}_{\frac{3}{4}, \psi} v_1(z), \mathcal{D}_{\frac{3}{4}, \psi} v_2(z) \right) \\ &= \frac{e^{-2z-3}}{\|v_1(z)\| + \|v_2(z)\| + 2z + 1} v_1(z) + \frac{z}{\|\mathcal{D}_{\frac{3}{4}, \psi} v_1(z)\| + z^4 + 1} \mathcal{D}_{\frac{3}{4}, \psi} v_2(z) + z, \end{aligned} \quad (2.24)$$

$$\begin{aligned} & f_2 \left(z, v_1(z), v_2(z), \mathcal{D}_{\frac{3}{4}, \psi} v_1(z), \mathcal{D}_{\frac{3}{4}, \psi} v_2(z) \right) \\ &= \frac{\sin z}{\|v_1(z)\| + \|v_2(z)\| + z^2 + 2} v_2(z) + \frac{e^{-z}}{\|\mathcal{D}_{\frac{3}{4}, \psi} v_2(z)\| + 1} \mathcal{D}_{\frac{3}{4}, \psi} v_1(z) + \cos z. \end{aligned} \quad (2.25)$$

Let us show that the assumptions of Theorem 1.6 hold for the coupled system. Observe that for all $z \in J$, $u_1, u_2, v_1, v_2 \in \mathbb{R}^p$, $f_1(z, u_1, u_2, v_1, v_2), f_2(z, u_1, u_2, v_1, v_2) \in \mathbb{R}^p$ and for $u_1, u_2, v_1, v_2, u'_1, u'_2, v'_1, v'_2 \in \mathbb{R}^p$, we have:

$$\begin{aligned} & \|f_1(z, u_1, u_2, v_1, v_2) - f_1(z, u'_1, u'_2, v'_1, v'_2)\| \\ & \leq p_1(z)(\|u_1 - u'_1\| + \|u_2 - u'_2\| + \|v_1 - v'_1\| + \|v_2 - v'_2\|), \\ & \|f_2(z, u_1, u_2, v_1, v_2) - f_2(z, u'_1, u'_2, v'_1, v'_2)\| \\ & \leq p_2(z)(\|u_1 - u'_1\| + \|u_2 - u'_2\| + \|v_1 - v'_1\| + \|v_2 - v'_2\|), \end{aligned}$$

where $p_1(z) = e^{-2z-3}$ and $p_2(z) = \max\{|\sin z|, e^{-z}\}$. Since \mathbb{R}^p is a reflexive Banach space and $f_1, f_2 : J \times \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ are continuous, then f_1 and f_2 satisfy (H1)-(H2) with $\varpi(f_i(z, \mathcal{D}_1, \mathcal{D}_2, \mathcal{Q}_1, \mathcal{Q}_2)) = 0$ for every $\mathcal{D}_1, \mathcal{D}_2, \mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{M}_{\mathbb{R}^p}$, $i = 1, 2$. Also,

$$\lambda = \max\{e^{-3}, 1\} \left(\frac{(\ln(\pi + 1))^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{(\ln(\pi + 1))^{\frac{3}{4}}}{\Gamma(\frac{7}{4})} \right) < 1.$$

Hence the coupled system (2.23) has a unique solution in the space $C^{\frac{3}{4}}(J, \mathbb{R}^p) \times C^{\frac{3}{4}}(J, \mathbb{R}^p)$.

2.5 Weak solutions for boundary value problems involving power law kernel functions in Caputo impulsive fractional D.E

Impulsive fractional differential equations are essential for modeling real world phenomena that experience sudden, abrupt changes such as vaccination campaigns in epidemiology, drug dosing in medicine, mechanical shocks in engineering, or market interventions in economics. The ψ -Caputo impulsive framework studied in this thesis provides a rigorous mathematical tool to capture such effects, extending the applicability of fractional calculus to complex systems where classical models fall short.

In this chapter, we present some hypotheses to prove the existence of solutions to (P). To this end, the following assumptions are considered.

Lemma 2.2. *A function $v \in PC(J, \mathbb{R}^n)$ is said to be a solution of (P) if v satisfies the equation*

$$(P) = \begin{cases} \psi, C_{0,z} \mathcal{D}^\alpha v(z) + f(z, v(z)) = 0, & 1 < \alpha < 2, \\ \Delta(v(z_k)) = v(z_k^+) - v(z_k^-) = I_k(v(z_k)), & (1.1) \\ v(c) = 0, \quad v(d) - \sum_{i=1}^{n-2} k_i v(\xi_i) = d', & n > 2, \quad (1.2) \end{cases} \quad (2.26)$$

where $k_i \in (0, \infty)$, $d' > 0$, $\xi_i \in J$, and $f : J \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Carathéodory function.

The domain of z is the interval $J = [c, d]$, excluding the impulsive points z_1, z_2, \dots, z_n . Also, ${}^C_{\psi} \mathcal{D}_{0,z}^\alpha$ denotes the ψ -Caputo fractional derivative of order α . The functions $I_k \in C(J)$ for $k = 1, \dots, n$ are continuous. Furthermore,

$$d = z_{k+1} > z_k > \dots > z_1 > z_0 = c,$$

and

$$v(z_k^+) = \lim_{z \rightarrow z_k^+} v(z), \quad v(z_k^-) = \lim_{z \rightarrow z_k^-} v(z), \quad k \in \{1, \dots, n\}.$$

Solution on the first interval $[c, z_1]$.

On $[c, z_1]$, there are no impulses. The ψ -Caputo fractional derivative of order $\alpha \in (1, 2)$ is defined as:

$${}^{\psi, C}_c \mathcal{D}^\alpha v(z) = {}^{\psi} I_c^{2-\alpha} \left(\frac{1}{\psi'(z)} \frac{d}{dz} \right)^2 v(z).$$

Given the equation:

$${}^{\psi, C}_c \mathcal{D}^\alpha v(z) = -f(z, v(z)).$$

Apply the ψ -fractional integral of order α to both sides:

$${}^{\psi} I_c^\alpha ({}^{\psi, C}_c \mathcal{D}^\alpha v(z)) = -{}^{\psi} I_c^\alpha (f(z, v(z))).$$

Using the identity for the ψ -Caputo derivative:

$${}^{\psi} I_c^\alpha ({}^{\psi, C}_c \mathcal{D}^\alpha v(z)) = v(z) - v(c) - c_0(\psi(z) - \psi(c)).$$

Since $v(c) = 0$, we have:

$$v(z) - c_0(\psi(z) - \psi(c)) = -{}^{\psi} I_c^\alpha (f(z, v(z))).$$

Thus,

$$v(z) = c_0(\psi(z) - \psi(c)) - \frac{1}{\Gamma(\alpha)} \int_c^z \psi'(s)(\psi(z) - \psi(s))^{\alpha-1} f(z, v(z)) ds.$$

Then:

$$v(z) = c_0(\psi(z) - \psi(c)) - \frac{1}{\Gamma(\alpha)} \int_c^z \psi'(s)(\psi(z) - \psi(s))^{\alpha-1} f(z, v(z)) ds. \quad (2.27)$$

Solution on $(z_1, z_2]$ with impulse at z_1 .

For $z \in (z_1, z_2]$, the solution must account for the impulse at z_1 . The general solution for $z > z_1$ is still given by:

$$v(z) = c_0(\psi(z) - \psi(c)) - \frac{1}{\Gamma(\alpha)} \int_c^z \psi'(s)(\psi(z) - \psi(s))^{\alpha-1} f(z, v(z)) ds.$$

However, at z_1 , there is a jump:

$$v(z_1^+) = v(z_1^-) + I_1(v(z_1)).$$

From (2.27) at $z = z_1^-$:

$$v(z_1^-) = c_0(\psi(z_1) - \psi(c)) - \frac{1}{\Gamma(\alpha)} \int_c^{z_1} \psi'(s)(\psi(z_1) - \psi(s))^{\alpha-1} f(z, v(z)) ds.$$

Thus,

$$v(z_1^+) = c_0(\psi(z_1) - \psi(c)) - \frac{1}{\Gamma(\alpha)} \int_c^{z_1} \psi'(s)(\psi(z_1) - \psi(s))^{\alpha-1} f(z, v(z)) ds + I_1(v(z_1)).$$

For $z > z_1$, the solution is:

$$v(z) = v(z_1^+) + c_0(\psi(z) - \psi(z_1)) - \frac{1}{\Gamma(\alpha)} \int_{z_1}^z \psi'(s)(\psi(z) - \psi(s))^{\alpha-1} f(z, v(z)) ds.$$

Substituting $v(z_1^+)$:

$$v(z) = \left[c_0(\psi(z_1) - \psi(c)) - \frac{1}{\Gamma(\alpha)} \int_c^{z_1} \psi'(s)(\psi(z_1) - \psi(s))^{\alpha-1} f(z, v(z)) ds + I_1(v(z_1)) \right] + c_0(\psi(z) - \psi(z_1)) - \frac{1}{\Gamma(\alpha)} \int_{z_1}^z \psi'(s)(\psi(z) - \psi(s))^{\alpha-1} f(z, v(z)) ds.$$

Note that:

$$c_0(\psi(z_1) - \psi(c)) + c_0(\psi(z) - \psi(z_1)) = c_0(\psi(z) - \psi(c)).$$

Therefore,

$$v(z) = c_0(\psi(z) - \psi(c)) + I_1(v(z_1)) - \frac{1}{\Gamma(\alpha)} \int_c^z \psi'(s)(\psi(z) - \psi(s))^{\alpha-1} f(z, v(z)) ds.$$

Solution on $(z_j, z_{j+1}]$ for $j \geq 2$.

By induction, for $z \in (z_j, z_{j+1}]$, the solution is:

$$v(z) = c_0(\psi(z) - \psi(c)) + \sum_{m=1}^j I_m(v(z_m)) - \frac{1}{\Gamma(\alpha)} \int_c^z \psi'(s)(\psi(z) - \psi(s))^{\alpha-1} f(z, v(z)) ds.$$

In particular, for $z \in (z_k, d]$:

$$v(z) = c_0(\psi(z) - \psi(c)) + \sum_{j=1}^k I_j(v(z_j)) - \frac{1}{\Gamma(\alpha)} \int_c^z \psi'(s)(\psi(z) - \psi(s))^{\alpha-1} f(z, v(z)) ds. \quad (2.28)$$

To account for the Heaviside effect, we write:

$$v(z) = c_0(\psi(z) - \psi(c)) + \sum_{j=1}^k I_j(v(z_j))H(z - z_j) - \frac{1}{\Gamma(\alpha)} \int_c^z \psi'(s)(\psi(z) - \psi(s))^{\alpha-1} f(z, v(z)) ds,$$

where H is the Heaviside function.

Apply the boundary condition to determine c_0 .

We have:

$$v(d) - \sum_{i=1}^{n-2} k_i v(\xi_i) = d'.$$

From (2.28), evaluate at $z = d$:

$$v(d) = c_0(\psi(d) - \psi(c)) + \sum_{j=1}^k I_j(v(z_j)) - \frac{1}{\Gamma(\alpha)} \int_c^d \psi'(s)(\psi(d) - \psi(s))^{\alpha-1} f(z, v(z)) ds. \quad (2.29)$$

For each ξ_i , let $m_i = \max\{j : z_j < \xi_i\}$. Then:

$$v(\xi_i) = c_0(\psi(\xi_i) - \psi(c)) + \sum_{j=1}^{m_i} I_j(v(z_j)) - \frac{1}{\Gamma(\alpha)} \int_c^{\xi_i} \psi'(s)(\psi(\xi_i) - \psi(s))^{\alpha-1} f(z, v(z)) ds. \quad (2.30)$$

Substitute (2.29) and (2.30) into the boundary condition. Thus:

$$c_0 = \frac{d' - \left(\sum_{j=1}^k I_j(v(z_j)) - \sum_{i=1}^{n-2} k_i \sum_{j=1}^{m_i} I_j(v(z_j)) \right) + \left(I_d - \sum_{i=1}^{n-2} k_i I_{\xi_i} \right)}{(\psi(d) - \psi(c)) - \sum_{i=1}^{n-2} k_i (\psi(\xi_i) - \psi(c))}. \quad (2.31)$$

The unique solution of (P) is:

$$v(z) = c_0(\psi(z) - \psi(c)) + \sum_{j=1}^k I_j(v(z_j))H(z - z_j) - \frac{1}{\Gamma(\alpha)} \int_c^z \psi'(s)(\psi(z) - \psi(s))^{\alpha-1} f(z, v(z)) ds,$$

with c_0 given by 2.31.

We can write the equation in the form of two operators defined from $PC(J, \mathbb{R}^n)$ by

$$v = \mathcal{K}v + \mathcal{J}v.$$

2.5. WEAK SOLUTIONS FOR BOUNDARY VALUE PROBLEMS INVOLVING
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This is expressed as follows:

$$\mathcal{K}v(z) = -\frac{1}{\Gamma(\alpha)} \int_c^z \psi'(s)(\psi(z) - \psi(s))^{\alpha-1} f(z, v(z)) ds,$$

and

$$\mathcal{J}v(z) = c_0(\psi(z) - \psi(c)) + \sum_{j=1}^k I_j(v(z_j))H(z - z_j),$$

with c_0 given by (2.31).

Our main task is to apply Krasnoselskii's fixed point theorem. To this end, we consider the following hypotheses:

- (c_1) The function $f : J \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Carathéodory function and there exist constants $L, c > 0$ such that

$$\|f(z, v(z))\| \leq L\|v(z)\| + c.$$

- (c_2) There exist constants $\gamma_j > 0$ for $j = 1, \dots, n$ such that

$$\|I_j(u) - I_j(v)\| \leq \gamma_j\|u - v\|, \quad \text{for } I_j \in C(\mathbb{R}^n, \mathbb{R}^n),$$

and

$$\|f(z, u(z)) - f(z, v(z))\| \leq L\|u - v\|, \quad \forall u, v \in \mathbb{R}^n.$$

- (c_3) There exists a constant $M > 0$ such that

$$\|f(z, v(z))\| \leq M,$$

and

$$\Delta = (\psi(d) - \psi(c)) - \sum_{i=1}^{n-2} k_i(\psi(\xi_i) - \psi(c)) > 0.$$

Lemma 2.3. [38] *Let \mathcal{H} be a finite-dimensional Banach space. Suppose that (c_2) holds. Then $\mathcal{J}v$ satisfies condition I_2 .*

Theorem 2.5. *Suppose that (c_1)–(c_3) are verified. Then the BVP has at least one solution in $PC(J, \mathbb{R}^n)$, provided*

$$\left(B\|\psi\|_\infty + \sum_{j=1}^k \gamma_j \right) < 1.$$

Proof. We show that the operators \mathcal{K} and \mathcal{J} satisfy all the conditions of Krasnoselskii's fixed point theorem. We need four steps. Let

$$\theta \geq \frac{A\|\psi\|_\infty + \frac{M(\psi(d) - \psi(c))^\alpha}{\Gamma(\alpha + 1)}}{1 - \left(B\|\psi\|_\infty + \sum_{j=1}^k \gamma_j \right)}. \quad (2.32)$$

Clearly, θ is positive as mentioned in 2.32. Let us consider the convex, bounded, and closed subset of $PC(J, \mathbb{R}^n)$ defined by

$$D_\theta := \{v \in PC(J, \mathbb{R}^n) : \|v\|_{PC} \leq \theta\}.$$

Step 1: \mathcal{K} is continuous.

Let $(v_n)_{n \in \mathbb{N}}$ be a sequence in D_θ converging to v . There exists an integer l such that

$$\|v_n\|_{PC} \leq \theta \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad \|v\|_{PC} \leq l,$$

so $v_n \in D_\theta$ and $v \in D_\theta$. According to Lebesgue's Dominated Convergence Theorem, we get:

$$\begin{aligned} \|\mathcal{K}(v_n) - \mathcal{K}(v)\| &= \left\| \frac{1}{\Gamma(\alpha)} \int_c^z \psi'(s)(\psi(z) - \psi(s))^{\alpha-1} (f(s, v_n(s)) - f(s, v(s))) ds \right\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_c^z \psi'(s)(\psi(z) - \psi(s))^{\alpha-1} \|f(s, v_n(s)) - f(s, v(s))\| ds \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence, \mathcal{K} is continuous.

Let $(v_n)_{n \in \mathbb{N}}$ be a weakly convergent sequence in D_θ . According to Lemma 2.3, we can deduce that the subsequence $(f(v_{n_k}))_{k \in \mathbb{N}}$ of $(f(v_n))$ weakly converges. Moreover, given the boundedness of the integral operator, the sequence $\frac{1}{\Gamma(\alpha)} \int_c^z \psi'(s)(\psi(z) - \psi(s))^{\alpha-1} f(s, v_{n_k}(s)) ds$ converges pointwise for almost all $z \in J$. By using the Vitali convergence theorem ([6], p.150), we deduce that the sequence $(\mathcal{K}(v_{n_k}))_{k \in \mathbb{N}}$ converges strongly in D_θ . Thus, we establish that \mathcal{K} satisfies property (I_1) .

Step 2: Measure of non-compactness estimate.

We assert that the second condition of Krasnoselskii's theorem is indeed fulfilled. Let G be a bounded subset of D_θ . For every $v \in G$ and for any $\epsilon > 0$ and nonempty subset k of J , the following holds:

$$\begin{aligned} \int_G \|\mathcal{K}v(z)\| dz &\leq \int_G \frac{1}{\Gamma(\alpha)} \int_c^z \psi'(s)(\psi(z) - \psi(s))^{\alpha-1} \|f(s, v(s))\| ds dz \\ &\leq \frac{(\psi(d) - \psi(c))^\alpha}{\Gamma(\alpha + 1)} \int_G M dz \\ &\leq \frac{M(\psi(d) - \psi(c))^\alpha}{\Gamma(\alpha + 1)} |G|. \end{aligned}$$

Given that a set containing only one element is weakly compact, we may invoke 2.33 to conclude that

$$\limsup_{\epsilon \rightarrow 0} \left\{ \int_k \|\Phi(z)\| dz : \text{mes}(k) \leq \epsilon \right\} = 0. \quad (2.33)$$

Then,

$$\varpi(\mathcal{K}(G)) \leq \frac{ML(\psi(d) - \psi(c))^\alpha}{\Gamma(\alpha + 1)} \varpi(G).$$

For the operator \mathcal{J} , we have:

$$\mathcal{J}v(z) = c_0(\psi(z) - \psi(c)) + \sum_{j=1}^k I_j(v(z_j))H(z - z_j),$$

with

$$|c_0| \leq \frac{1}{\Delta} |d'| + \sum_{j=1}^k \gamma_j \|v\| + \sum_{i=1}^{n-2} |k_i| \sum_{j:z_j < \xi_i} \gamma_j \|v\| \quad (2.34)$$

$$+ \frac{M}{\Gamma(\alpha + 1)} \left[(\psi(d) - \psi(c))^\alpha + \sum_{i=1}^{n-2} |k_i| (\psi(\xi_i) - \psi(c))^\alpha \right]. \quad (2.35)$$

Let

$$A = \frac{|d'| + \frac{M}{\Gamma(\alpha + 1)} [(\psi(d) - \psi(c))^\alpha + \sum_{i=1}^{n-2} |k_i| (\psi(\xi_i) - \psi(c))^\alpha]}{\Delta},$$

$$B = \frac{\sum_{j=1}^k \gamma_j + \sum_{i=1}^{n-2} |k_i| \sum_{j:z_j < \xi_i} \gamma_j}{\Delta}.$$

So $|c_0| \leq A + B\|v\|$.

For any measurable $G \subseteq D_\theta$:

$$\begin{aligned} \int_G \|\mathcal{J}v(z)\| dz &\leq \int_G |c_0| |\psi(z) - \psi(c)| dz + \int_G \sum_{j=1}^k |I_j(v(z_j))| H(z - z_j) dz \\ &\leq (A + B\|v\|) \int_G |\psi(z) - \psi(c)| dz + \int_G \sum_{j=1}^k \gamma_j \|v\| dz. \end{aligned}$$

For the measure of noncompactness, we have:

$$\varpi(\mathcal{J}(G)) \leq \left(B\|\psi\|_\infty + \sum_{j=1}^k \gamma_j \right) \varpi(G).$$

Consequently,

$$\varpi(\mathcal{K}(G) + \mathcal{J}(G)) \leq \left(B\|\psi\|_\infty + \sum_{j=1}^k \gamma_j + \frac{ML(\psi(d) - \psi(c))^\alpha}{\Gamma(\alpha + 1)} \right) \varpi(G).$$

Step 3: Contraction property of \mathcal{J} .

For the operator \mathcal{J} :

$$|c_0(u) - c_0(v)| \leq \frac{1}{\Delta} \left[\left(\sum_{j=1}^k \gamma_j + \sum_{i=1}^{n-2} |k_i| \sum_{j:z_j < \xi_i} \gamma_j \right) + \frac{L}{\Gamma(\alpha + 1)} \left((\psi(d) - \psi(c))^\alpha + \sum_{i=1}^{n-2} |k_i| (\psi(\xi_i) - \psi(c))^\alpha \right) \right] \|u - v\|.$$

Let

$$\Lambda = \frac{1}{\Delta} \left[\left(\sum_{j=1}^k \gamma_j + \sum_{i=1}^{n-2} |k_i| \sum_{j:z_j < \xi_i} \gamma_j \right) + \frac{L}{\Gamma(\alpha + 1)} \left((\psi(d) - \psi(c))^\alpha + \sum_{i=1}^{n-2} |k_i| (\psi(\xi_i) - \psi(c))^\alpha \right) \right].$$

Then

$$\|\mathcal{J}u - \mathcal{J}v\| \leq |c_0(u) - c_0(v)| \cdot |\psi(z) - \psi(c)| + \sum_{j=1}^k |I_j(u(z_j)) - I_j(v(z_j))| \leq \left(\Lambda \|\psi\|_\infty + \sum_{j=1}^k \gamma_j \right) \|u - v\|.$$

Therefore, \mathcal{J} is a contraction with constant

$$\mathfrak{F} = \Lambda \|\psi\|_\infty + \sum_{j=1}^k \gamma_j.$$

Step 4: Invariance of D_θ .

Now,

$$\begin{aligned} \|(\mathcal{K} + \mathcal{J})v\| &\leq |c_0| \cdot |\psi(z) - \psi(c)| + \sum_{j=1}^k |I_j(v(z_j))| + \frac{1}{\Gamma(\alpha)} \int_c^z \psi'(s) (\psi(z) - \psi(s))^{\alpha-1} |f(s, v(s))| ds \\ &\leq (A + B\|v\|) \|\psi\|_\infty + \left(\sum_{j=1}^k \gamma_j \right) \|v\| + \frac{M}{\Gamma(\alpha)} \int_c^z \psi'(s) (\psi(z) - \psi(s))^{\alpha-1} ds. \end{aligned}$$

Since

$$\int_c^z \psi'(s) (\psi(z) - \psi(s))^{\alpha-1} ds = \frac{(\psi(z) - \psi(c))^\alpha}{\alpha},$$

we have

$$\|(\mathcal{K} + \mathcal{J})v\| \leq (A + B\|v\|) \|\psi\|_\infty + \left(\sum_{j=1}^k \gamma_j \right) \|v\| + \frac{M(\psi(z) - \psi(c))^\alpha}{\Gamma(\alpha + 1)}.$$

For $v \in D_\theta$:

$$\|(\mathcal{K} + \mathcal{J})v\| \leq A \|\psi\|_\infty + \left(B \|\psi\|_\infty + \sum_{j=1}^k \gamma_j \right) \theta + \frac{M(\psi(d) - \psi(c))^\alpha}{\Gamma(\alpha + 1)}.$$

We require

$$A\|\psi\|_\infty + \left(B\|\psi\|_\infty + \sum_{j=1}^k \gamma_j \right) \theta + \frac{M(\psi(d) - \psi(c))^\alpha}{\Gamma(\alpha + 1)} \leq \theta.$$

This is exactly the condition defining θ in (2.32). Hence $\mathcal{K} + \mathcal{J}$ maps D_θ into itself. By Krasnoselskii's fixed point theorem, the BVP (P) has at least one solution $v \in PC(J, \mathbb{R}^n)$. \square

2.6 Example

Consider problem (P) with the following data:

$$\alpha = \frac{3}{2}, \quad c = 0.5, \quad d = 1. \quad (2.36)$$

Let:

$$\psi(z) = z, \quad z_1 = 0.55, \quad \xi_i = 0.6, \quad k_i = \frac{1}{3^i} \text{ for } i = 1, \dots, 10. \quad (2.37)$$

Define:

$$f(z, v(z)) = \frac{1}{e^z + 9} \left(1 + \frac{|v(z)|}{1 + |v(z)|} \right), \quad I_1(v(z_1)) = \frac{1}{100} v(z_1).$$

Then:

- $|f(z, v(z))| \leq \frac{1}{e^{0.5} + 9} \cdot 2 \approx 0.186$, so take $M = 1$ (bounded).
- The function is Lipschitz with $L = 1$ (since the derivative is bounded).
- $|I_1(u) - I_1(v)| \leq \frac{1}{100} |u - v|$, so $\gamma_1 = 0.01$.

Now, compute the contraction constant \mathfrak{F} for \mathcal{J} :

$$\mathfrak{F} = \Lambda \|\psi\|_\infty + \sum_{j=1}^k \gamma_j.$$

We have:

$$\Delta = (\psi(d) - \psi(c)) - \sum_{i=1}^{10} k_i (\psi(\xi_i) - \psi(c)) = 0.5 - \sum_{i=1}^{10} \frac{1}{3^i} \cdot 0.1 = 0.5 - 0.1 \cdot 0.5 = 0.45.$$

$$\Lambda = \frac{1}{0.45} \left[\left(\gamma_1 + \sum_{i=1}^{10} |k_i| \gamma_1 \right) + \frac{L}{\Gamma(2.5)} \left((0.5)^{3/2} + \sum_{i=1}^{10} |k_i| (0.1)^{3/2} \right) \right].$$

With $\Gamma(2.5) \approx 1.329$, $0.5^{3/2} \approx 0.3536$, $0.1^{3/2} \approx 0.03162$, and $\sum_{i=1}^{10} \frac{1}{3^i} \approx 0.5$:

$$\Lambda = \frac{1}{0.45} \left[(0.01 + 0.005) + \frac{1}{1.329} (0.3536 + 0.5 \cdot 0.03162) \right] = \frac{1}{0.45} (0.015 + 0.278) \approx 0.651.$$

Then:

$$\mathfrak{F} = 0.651 \cdot 1 + 0.01 = 0.661 < 1.$$

Thus, \mathcal{J} is a contraction. Then we require: $\left(B\|\psi\|_\infty + \sum_{j=1}^k \gamma_j \right) < 1$. Compute:

$$\sum_{i=1}^{10} |k_i| \sum_{j:z_j < \xi_i} \gamma_j = 0.01 \cdot \sum_{i=1}^{10} \frac{1}{3^i}.$$

The geometric series:

$$\sum_{i=1}^{10} \frac{1}{3^i} = \frac{1/3(1 - (1/3)^{10})}{1 - 1/3} = \frac{1}{2} \left(1 - \frac{1}{59049} \right) \approx 0.5.$$

So:

$$\sum_{i=1}^{10} |k_i| \sum_{j:z_j < \xi_i} \gamma_j \approx 0.01 \cdot 0.5 = 0.005.$$

Then:

$$B = \frac{\sum_{j=1}^k \gamma_j + \sum_{i=1}^{n-2} |k_i| \sum_{j:z_j < \xi_i} \gamma_j}{\Delta} = \frac{0.01 + 0.005}{0.45} = \frac{0.015}{0.45} = \frac{1}{30} \approx 0.03333.$$

Now:

$$B\|\psi\|_\infty + \sum_{j=1}^k \gamma_j = 0.03333 \cdot 1 + 0.01 = 0.04333 < 1.$$

By Krasnoselskii's fixed point theorem, the BVP (P) has at least one solution $v \in PC(J, \mathbb{R}^n)$.

Chapter 3

Existence of Solutions to the Variable Order Caputo Fractional Thermistor Problem

A thermistor is a temperature-dependent resistor. For years, its complex behavior has been analytically shown to have unique solutions. Recently, computational models like finite element analysis have been used to study its electro-thermal coupling, capturing the feedback between Joule heating, electrical conduction, and heat dissipation.

PTC thermistors increase resistance with temperature, serving as self-regulating current limiters or resettable fuses. NTC thermistors decrease resistance with heat, making them precise sensors in automotive systems and home appliances for reliable temperature control.

In this chapter, we extend the classical thermistor model to the framework of variable-order fractional calculus. The fractional order $\gamma(z)$ is allowed to vary with the independent variable z , which enables a more accurate description of memory and hereditary effects in materials with evolving properties. The governing equation is given by the following initial value problem involving the variable-order Caputo fractional derivative:

$$\begin{cases} {}^c\mathcal{D}^{\gamma(z)}v(z) = \frac{\lambda f(z, v(z))}{\left(\int_{v_1}^{v_2} f(b, v(b))db\right)^2}, \\ 0 < \gamma(z) < 1, \quad z \in J = [v_1, v_2], \quad v_1 \geq 0, \\ v(v_1) = v_{v_1} \in \mathbb{R}. \end{cases} \quad (3.1)$$

3.1 Main Results

This section contains the main results of our research. Our first goal is to obtain an integral representation for a solution to our problem. We first need to point out that $[\gamma(z)] = 0$ so that in what follows, $n = [\gamma(z)] + 1 = 1$.

Take

$$\mathcal{P} = \{[v_1, z_1], (z_1, z_2], (z_2, z_3], \dots, (z_{k-1}, v_2]\}$$

to be a partition of the interval $[v_1, v_2]$, and consider the function $\gamma(z) : [v_1, v_2] \rightarrow (0, 1)$ that is piecewise constant with respect to \mathcal{P} ; this function is defined as

$$\gamma(z) = \sum_{i=1}^k \gamma_i \chi_i(z), \quad \text{for } z \in [v_1, v_2],$$

where $0 < \gamma_i < 1$ for each $i = 1, 2, \dots, k$, and the constants γ_i are defined over the intervals specified by \mathcal{P} . Here, χ_i denotes the indicator (characteristic) function for the interval $[z_{i-1}, z_i]$, where $z_0 = v_1$, $z_k = v_2$, and is defined by

$$\chi_i(z) = \begin{cases} 1, & \text{if } z \in [z_{i-1}, z_i], \\ 0, & \text{otherwise.} \end{cases}$$

Thus, in view of Definition 4,

$$\begin{aligned} {}^c\mathcal{D}^{\gamma(z)}v(z) &= \frac{1}{\Gamma\left(1 - \sum_{i=1}^k \gamma_i \chi_i(z)\right)} \int_{v_1}^z (z-s)^{1-\sum_{i=1}^k \gamma_i \chi_i(z)-1} \frac{d}{ds}(v(s)) ds \\ &= \frac{1}{\Gamma\left(1 - \sum_{i=1}^k \gamma_i \chi_i(z)\right)} \int_{v_1}^z (z-s)^{-\sum_{i=1}^k \gamma_i \chi_i(z)} v'(s) ds \\ &= \frac{\lambda f(z, v(z))}{\left(\int_{v_1}^{v_2} f(b, v(b)) db\right)^2}. \end{aligned} \tag{3.2}$$

for $0 \leq v_1 \leq z \leq v_2 < +\infty$.

We denote by $Z_i = C([z_{i-1}, z_i], \mathbb{R})$, the class of functions that form a Banach space with the norm

$$\|v\|_{Z_i} = \sup_{z \in [z_{i-1}, z_i]} \{|v(z)| : i \in \{1, 2, \dots, k\}\}.$$

For each $i \in \{2, \dots, k\}$, let the functions $\hat{v}_i \in Z_i$ be such that $\hat{v}_i(z) = 0$ for all $z \in [v_1, z_{i-1}]$. Therefore, in the interval $[v_1, z_1]$ we have

$${}^c\mathcal{D}^{\gamma_1} \hat{v}(z) = \frac{1}{\Gamma(1 - \gamma_1)} \int_{v_1}^z (z-s)^{\gamma_1} \hat{v}'(s) ds = \frac{\lambda f(z, v(z))}{\left(\int_{v_1}^z f(b, \hat{v}(b)) db\right)^2}. \tag{3.3}$$

In the interval $(z_1, z_2]$,

$${}^c\mathcal{D}^{\gamma_2}\hat{v}(z) = \frac{1}{\Gamma(1-\gamma_2)} \int_{z_1}^z (z-s)^{\gamma_2-1} \hat{v}'(s) ds = \frac{\lambda f(z, v(z))}{\left(\int_{z_1}^z f(b, \hat{v}(b)) db\right)^2}. \quad (3.4)$$

In general, in $(z_{i-1}, z_i]$, we have

$${}^c\mathcal{D}^{\gamma_i}\hat{v}(z) = \frac{1}{\Gamma(1-\gamma_i)} \int_{z_{i-1}}^z (z-s)^{\gamma_i-1} \hat{v}'(s) ds = \frac{\lambda f(z, v(z))}{\left(\int_{z_{i-1}}^z f(b, \hat{v}(b)) db\right)^2}. \quad (3.5)$$

Consequently, for each $i \in \{1, 2, \dots, k\}$, we examine the auxiliary constant-order initial value problem

$$\begin{cases} {}^c\mathcal{D}^{\gamma_i}\hat{v}(z) = \frac{\lambda f(z, v(z))}{\left(\int_{z_{i-1}}^z f(b, \hat{v}(b)) db\right)^2}, & z_{i-1} < z < z_i, \\ \hat{v}(v_1) = \hat{v}_{v_1}. \end{cases} \quad (3.6)$$

Definition 3.1 (6). We define the solution of problem (3.1) to be v if there exist functions v_i such that:

- $v_1 \in Z_1$ satisfies (3.3) and $v_1(v_1) = v_1(z_0) = \hat{v}_{v_1}$;
- $v_2 \in Z_2$ satisfies (3.4) and $v_2(z_0) = v_2(z_1) = \hat{v}_{v_1}$;
- ...;
- $v_i \in Z_i$ satisfies (3.5) and $v_i(z_{i-1}) = v_i(z_i) = \hat{v}_{v_1}$ for $i = \{3, \dots, k\}$.

Remark 3.1. We say that problem (2) possesses a unique solution in $C([v_1, v_2], \mathbb{R})$ if the functions \hat{v}_i are unique for each $i \in \{1, 2, \dots, k\}$.

Lemma 3.1 (2). For $i \in \{1, 2, \dots, k\}$, the function v_i is a solution of (12) if and only if it satisfies the integral equation

$$v_i(z) = v_{v_1} + \frac{1}{\Gamma(\gamma_i)} \int_{z_{i-1}}^z (z-s)^{\gamma_i-1} h(s) ds. \quad (3.7)$$

for $z \in (z_{i-1}, z_i]$ for each $i \in \{1, 2, \dots, k\}$.

Proof. Given that v satisfies (12), we rewrite (12) as the equivalent integral equation as follows: For $z_{i-1} < z < z_i$, by Lemma 1,

$$I_{z_{i-1}^+}^{\gamma_i} \left({}^c\mathcal{D}_{z_{i-1}^+}^{\gamma_i} v(z) \right) = I_{z_{i-1}^+}^{\gamma_i} h(z) + \rho_0,$$

where

$$h(z) = \frac{\lambda f(z, v(z))}{\left(\int_{v_1}^{v_2} f(b, v(b)) db\right)^2}.$$

Using the boundary conditions $v(v_1) = v_{v_1}$, we obtain

$$v(z) = v_{v_1} + \frac{1}{\Gamma(\gamma_i)} \int_{z_{i-1}}^z (z-s)^{\gamma_i-1} h(s) ds. \quad (3.8)$$

For the sake of brevity, we set

$$\mathcal{N}_z^{\gamma_i}(s) = \frac{(z-s)^{\gamma_i-1}}{\Gamma(\gamma_i)} \quad \text{and} \quad f_v(z) = f(z, v(z)).$$

It is easy to verify that

$$\int_{z_{i-1}}^z \mathcal{N}_z^{\gamma_i}(s) ds \leq \int_{z_{i-1}}^{z_i} \mathcal{N}_{z_i}^{\gamma_i}(s) ds = \frac{(z_i - z_{i-1})^{\gamma_i}}{\Gamma(\gamma_i + 1)} = \mathcal{N}^*.$$

Prior to presenting our primary findings, we list the following hypotheses that are needed in our analysis.

(C1) f is continuous;

(C2) $|f(z, v_2(z)) - f(z, v_1(z))| \leq \kappa |v_2 - v_1|$, $\kappa > 0$, for all $z \in J$ and $v_1, v_2 \in \mathbb{R}$;

(C3) $0 < c_f \leq f_v(z)$, for all $z \in J$ and $v \in \mathbb{R}$;

(C4) For $i = 1, 2, \dots, k$

$$\frac{|v_{v_1}|(c_f(z_i - z_{i-1}))^2 + \lambda \mathcal{N}^* f_0^*}{(c_f(z_i - z_{i-1}))^2 - \lambda \mathcal{N}^* \kappa} < \delta_i.$$

□

Theorem 3.1 (2). *Assume that conditions (C1)–(C4) hold. Then, the boundary value problem (2) possesses at least one solution in $C([v_1, v_2], \mathbb{R})$.*

Proof. Consider the mapping $G_i : Z_i \rightarrow Z_i$ given by

$$(G_i v)(z) = v_{v_1} + \lambda \int_{z_{i-1}}^z \frac{f_v(s) \mathcal{N}_z^{\gamma_i}(s)}{\left(\int_{z_{i-1}}^{z_i} f_v(b) db\right)^2} ds. \quad (3.9)$$

Let the ball $B_{\delta_i} = \{v \in Z_i : \|v\|_{Z_i} \leq \delta_i\}$ be the nonempty, closed, bounded, convex subset of Z_i , where

$$f_0^* = \sup_{z \in [z_{i-1}, z_i]} |f(z, 0)|.$$

Let the ball $B_{\delta_i} = \{v \in Z_i : \|v\|_{Z_i} \leq \delta_i\}$ be the nonempty, closed, bounded, convex subset of Z_i , where

$$\delta_i \geq \frac{|v_{v_1}| + \frac{\lambda \mathcal{N}^* f_0^*}{(c_f(z_i - z_{i-1}))^2}}{1 - \frac{\lambda \mathcal{N}^* \kappa}{(c_f(z_i - z_{i-1}))^2}}.$$

The proof will be given through several claims.

Claim 1: The function G_i is continuous for each $i \in \{1, 2, \dots, k\}$.

Suppose that $\{v_m\}$ is a sequence such that $v_m \rightarrow v$ in B_{δ_i} . For each $i \in \{1, 2, \dots, k\}$ and for every $z \in [z_{i-1}, z_i]$, we have

$$\begin{aligned} & |(G_i v_m)(z) - (G_i v)(z)| \\ & \leq \lambda \int_{z_{i-1}}^z \mathcal{N}_z^{\gamma_i}(s) \left| \frac{f_{v_m}(s)}{\left(\int_{z_{i-1}}^{z_i} f_{v_m}(b) db\right)^2} - \frac{f_v(s)}{\left(\int_{z_{i-1}}^{z_i} f_v(b) db\right)^2} \right| ds \\ & \leq \lambda \int_{z_{i-1}}^z \mathcal{N}_z^{\gamma_i}(s) \left[\left| \frac{f_{v_m}(s) - f_v(s)}{\left(\int_{z_{i-1}}^{z_i} f_{v_m}(b) db\right)^2} \right| + \left| \frac{f_v(s)}{\left(\int_{z_{i-1}}^{z_i} f_{v_m}(b) db\right)^2} - \frac{f_v(s)}{\left(\int_{z_{i-1}}^{z_i} f_v(b) db\right)^2} \right| \right] ds \\ & \leq \lambda \int_{z_{i-1}}^z \mathcal{N}_z^{\gamma_i}(s) \left[\frac{\kappa |v_m(s) - v(s)|}{\left(\int_{z_{i-1}}^{z_i} f_{v_m}(b) db\right)^2} + |f_v(s)| \left| \frac{\left(\int_{z_{i-1}}^{z_i} f_v(b) db\right)^2 - \left(\int_{z_{i-1}}^{z_i} f_{v_m}(b) db\right)^2}{\left(\int_{z_{i-1}}^{z_i} f_{v_m}(b) db\right)^2 \left(\int_{z_{i-1}}^{z_i} f_v(b) db\right)^2} \right| \right] ds \\ & \leq \frac{\lambda}{(c_f(z_i - z_{i-1}))^2} \int_{z_{i-1}}^z \mathcal{N}_z^{\gamma_i}(s) [\kappa |v_m(s) - v(s)| + \\ & \quad \frac{(\kappa \|v\|_{Z_i} + f_0^*)}{(c_f(z_i - z_{i-1}))^4} \left| \left(\int_{z_{i-1}}^{z_i} f_{v_m}(b) db\right)^2 - \left(\int_{z_{i-1}}^{z_i} f_v(b) db\right)^2 \right|] ds \\ & \leq \lambda \int_{z_{i-1}}^z \mathcal{N}_z^{\gamma_i}(s) \left[\frac{\kappa |v_m(s) - v(s)|}{(c_f(z_i - z_{i-1}))^2} + \right. \\ & \quad \left. \frac{\kappa \|v\|_{Z_i} + f_0^*}{(c_f(z_i - z_{i-1}))^4} \times \left| \left(\int_{z_{i-1}}^{z_i} f_{v_m}(b) db\right)^2 - \left(\int_{z_{i-1}}^{z_i} f_v(b) db\right)^2 \right| \right] ds \\ & \leq \frac{\lambda}{(c_f(z_i - z_{i-1}))^2} \\ & \quad \int_{z_{i-1}}^z \mathcal{N}_z^{\gamma_i}(s) [\kappa |v_m(s) - v(s)| + \\ & \quad \frac{\kappa \|v\|_{Z_i} + f_0^*}{(c_f(z_i - z_{i-1}))^2} \times \left| \left(\int_{z_{i-1}}^{z_i} (f_{v_m}(b) - f_v(b)) db\right) \left(\int_{z_{i-1}}^{z_i} (f_{v_m}(b) + f_v(b)) db\right) \right|] ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\lambda}{(c_f(z_i - z_{i-1}))^2} \int_{z_{i-1}}^z \mathcal{N}_z^{\gamma_i}(s) \left[\kappa |v_m(s) - v(s)| + \right. \\
&\quad \left. \frac{2(\kappa \|v\|_{Z_i} + f_0^*)^2}{(c_f(z_i - z_{i-1}))^2} \int_{z_{i-1}}^{z_i} |f_{v_m}(b) - f_v(b)| db \right] ds \\
&\leq \frac{\lambda \kappa}{(c_f(z_i - z_{i-1}))^2} \int_{z_{i-1}}^z \mathcal{N}_z^{\gamma_i}(s) \left(1 + \frac{2(\kappa \|v\|_{Z_i} + f_0^*)^2}{c_f^2} \right) |v_m(s) - v(s)| ds \\
&\leq \frac{\lambda \kappa (c_f^2 + 2(\kappa \|v\|_{Z_i} + f_0^*)^2)}{(c_f(z_i - z_{i-1}))^4} \int_{z_{i-1}}^z \mathcal{N}_z^{\gamma_i}(s) |v_m(s) - v(s)| ds \\
&\leq \frac{\lambda \kappa (c_f^2 + 2(\kappa \|v\|_{Z_i} + f_0^*)^2) \mathcal{N}^*}{(c_f(z_i - z_{i-1}))^4} \|v_m - v\|_{Z_i}.
\end{aligned}$$

Since $\{v_m\} \rightarrow v$ and f is continuous, the right-hand side of the above inequality tends to zero as $m \rightarrow +\infty$. Therefore,

$$\|(G_i v_m)(z) - (G_i v)(z)\|_{Z_i} \rightarrow 0 \quad \text{as } m \rightarrow +\infty,$$

and so G_i is continuous.

Claim 2: For each $i \in \{1, 2, \dots, k\}$, $G(B_{\delta_i}) \subset B_{\delta_i}$.

We have

$$\begin{aligned}
|(Gv)(z)| &= \left| v_{v_1} + \lambda \int_{z_{i-1}}^z \frac{f_v(s) \mathcal{N}_z^{\gamma_i}(s)}{\left(\int_{z_{i-1}}^{z_i} f_v(b) db \right)^2} ds \right| \\
&\leq |v_{v_1}| + \lambda \int_{z_{i-1}}^z \frac{|f_v(s)| \mathcal{N}_z^{\gamma_i}(s)}{\left(\int_{z_{i-1}}^{z_i} f_v(b) db \right)^2} ds \\
&\leq |v_{v_1}| + \frac{\lambda}{(c_f(z_i - z_{i-1}))^2} \int_{z_{i-1}}^z (|f_v(s) - f_0(s)| + |f_0(s)|) \mathcal{N}_z^{\gamma_i}(s) ds \\
&\leq |v_{v_1}| + \frac{\lambda(\kappa \|v\|_{Z_i} + \max_{s \in J} |f_0(s)|)}{(c_f(z_i - z_{i-1}))^2} \int_{z_{i-1}}^z \mathcal{N}_z^{\gamma_i}(s) ds \\
&\leq |v_{v_1}| + \frac{\lambda \mathcal{N}^* (\kappa \|v\|_{Z_i} + f_0^*)}{(c_f(z_i - z_{i-1}))^2} \leq \delta_i,
\end{aligned}$$

where $f_0^* = \max_{s \in J} |f_0(s)|$, which proves the claim.

Claim 3: G_i is relatively compact for each $i \in \{1, 2, \dots, k\}$.

From Claim 2, $G_i(B_{\delta_i}) \subset B_{\delta_i}$, so $G_i(B_{\delta_i})$ is uniformly bounded. It remains to show the equicontinuity of G_i for each $i \in \{1, 2, \dots, k\}$.

Consider $z_1, z_2 \in (z_{i-1}, z_i]$. Then,

$$\begin{aligned}
& |(G_i v)(z_2) - (G_i v)(z_1)| \\
&= \left| \lambda \int_{z_{i-1}}^{z_2} \frac{f_v(s) \mathcal{N}_{z_2}^{\gamma_i}(s)}{\left(\int_{z_{i-1}}^{z_i} f_v(b) db\right)^2} ds - \lambda \int_{z_{i-1}}^{z_1} \frac{f_v(s) \mathcal{N}_{z_1}^{\gamma_i}(s)}{\left(\int_{z_{i-1}}^{z_i} f_v(b) db\right)^2} ds \right| \\
&\leq \lambda \int_{z_{i-1}}^{z_2} \frac{|f_v(s)| \mathcal{N}_{z_2}^{\gamma_i}(s)}{\left(\int_{z_{i-1}}^{z_i} f_v(b) db\right)^2} ds - \lambda \int_{z_{i-1}}^{z_1} \frac{|f_v(s)| \mathcal{N}_{z_1}^{\gamma_i}(s)}{\left(\int_{z_{i-1}}^{z_i} f_v(b) db\right)^2} ds \\
&\leq \frac{\lambda}{(c_f(z_i - z_{i-1}))^2} \left(\int_{z_{i-1}}^{z_2} |f_v(s)| \mathcal{N}_{z_2}^{\gamma_i}(s) ds - \int_{z_{i-1}}^{z_1} |f_v(s)| \mathcal{N}_{z_1}^{\gamma_i}(s) ds \right) \\
&\leq \frac{\lambda}{(c_f(z_i - z_{i-1}))^2} \left(\int_{z_{i-1}}^{z_1} |f_v(s)| (\mathcal{N}_{z_2}^{\gamma_i}(s) - \mathcal{N}_{z_1}^{\gamma_i}(s)) ds + \int_{z_1}^{z_2} |f_v(s)| \mathcal{N}_{z_2}^{\gamma_i}(s) ds \right)
\end{aligned}$$

hence

$$\begin{aligned}
& |(G_i v)(z_2) - (G_i v)(z_1)| \\
&\leq \frac{\lambda(\kappa \|v\|_{Z_i} + f_0^*)}{(c_f(z_i - z_{i-1}))^2} \left(\int_{z_{i-1}}^{z_1} (\mathcal{N}_{z_2}^{\gamma_i}(s) - \mathcal{N}_{z_1}^{\gamma_i}(s)) ds + \int_{z_1}^{z_2} \mathcal{N}_{z_2}^{\gamma_i}(s) ds \right) \\
&\leq \frac{\lambda(\kappa \|v\|_{Z_i} + f_0^*)}{(c_f(z_i - z_{i-1}))^2} \left[\frac{2(z_2 - z_1)^{\gamma_i}}{\Gamma(\gamma_i + 1)} + \frac{(z_1 - z_{i-1})^{\gamma_i} - (z_2 - z_{i-1})^{\gamma_i}}{\Gamma(\gamma_i + 1)} \right].
\end{aligned}$$

As $z_1 \rightarrow z_2$, the right-hand side of the above inequality approaches zero, so the mapping G_i is equicontinuous. Thus, by the Ascoli-Arzelà theorem, the mapping G_i is relatively compact on B_{δ_i} .

Theorem 1 thus ensures that the auxiliary boundary value problem (12) possesses at least one solution in B_{δ_i} for each $i \in \{1, 2, \dots, k\}$. Consequently, the boundary value problem (2) possesses at least one solution in $C([v_1, v_2], \mathbb{R})$ given by

$$v(z) = \begin{cases} v_1(z) = \hat{v}(z), & z \in [v_1, z_1], \\ v_2(z) = \begin{cases} 0, & z \in [v_1, z_1], \\ \hat{v}_2(z), & z \in (z_1, z_2], \end{cases} \\ \vdots \\ v_k(z) = \begin{cases} 0, & z \in [v_1, z_{k-1}], \\ \hat{v}_k(z), & z \in (z_{k-1}, z_k]. \end{cases} \end{cases}$$

□

Theorem 3.2. *Assume that conditions (C1) and (C2) hold with $0 < c_1 \leq f_v(z) < c_2$ for all $z \in J$ and $v \in \mathbb{R}$. Then, Equation (12) has a unique solution in $[v_1, v_2]$ for each*

$i \in \{1, 2, \dots, k\}$, provided that

$$\frac{\lambda \kappa (c_1^2 + 2c_2^2) \mathcal{N}^*}{(c_1^2 (z_i - z_{i-1}))^2} \leq 1. \quad (16)$$

Proof. As in Claim 2 in the proof of Theorem 2, the mapping $G_i : B_{\delta_i} \rightarrow B_{\delta_i}$ is uniformly bounded. It remains to show that G_i is a contraction.

Consider $i \in \{1, 2, \dots, k\}$ and let $v_i^*, v_i \in B_{\delta_i}$. Then,

$$\begin{aligned} & |(G_i v_i^*)(z) - (G_i v_i)(z)| \\ &= \left| \lambda \int_{z_{i-1}}^z \frac{f_{v_i^*}(s) \mathcal{N}_z^{\gamma_i}(s)}{\left(\int_{z_{i-1}}^{z_i} f_{v_i^*}(b) db\right)^2} ds - \lambda \int_{z_{i-1}}^z \frac{f_{v_i}(s) \mathcal{N}_z^{\gamma_i}(s)}{\left(\int_{z_{i-1}}^{z_i} f_{v_i}(b) db\right)^2} ds \right| \\ &\leq \lambda \int_{z_{i-1}}^{z_i} \mathcal{N}_z^{\gamma_i}(s) \left| \frac{f_{v_i^*}(s)}{\left(\int_{z_{i-1}}^{z_i} f_{v_i^*}(b) db\right)^2} - \frac{f_{v_i}(s)}{\left(\int_{z_{i-1}}^{z_i} f_{v_i}(b) db\right)^2} \right| ds \end{aligned}$$

thus

$$\begin{aligned} & |(G_i v_i^*)(z) - (G_i v_i)(z)| \\ &\leq \lambda \int_{z_{i-1}}^{z_i} \mathcal{N}_z^{\gamma_i}(s) \left[\left| \frac{f_{v_i^*}(s) - f_{v_i}(s)}{\left(\int_{z_{i-1}}^{z_i} f_{v_i^*}(b) db\right)^2} \right| + \left| \frac{f_{v_i}(s)}{\left(\int_{z_{i-1}}^{z_i} f_{v_i^*}(b) db\right)^2} - \frac{f_{v_i}(s)}{\left(\int_{z_{i-1}}^{z_i} f_{v_i}(b) db\right)^2} \right| \right] ds \\ &\leq \lambda \int_{z_{i-1}}^z \mathcal{N}_z^{\gamma_i}(s) \left[\frac{\kappa |v_i^*(s) - v_i(s)|}{\left(\int_{z_{i-1}}^{z_i} f_{v_i^*}(b) db\right)^2} + \frac{\left| \left(\int_{z_{i-1}}^{z_i} f_{v_i}(b) db\right)^2 - \left(\int_{z_{i-1}}^{z_i} f_{v_i^*}(b) db\right)^2 \right|}{\left(\int_{z_{i-1}}^{z_i} f_{v_i^*}(b) db\right)^2 \left(\int_{z_{i-1}}^{z_i} f_{v_i}(b) db\right)^2} |f_{v_i}(s)| \right] ds \\ &\leq \lambda \int_{z_{i-1}}^z \mathcal{N}_z^{\gamma_i}(s) \left[\frac{\kappa |v_i^*(s) - v_i(s)|}{(c_1(z_i - z_{i-1}))^2} + \right. \\ &\quad \left. \frac{c_2}{(c_1(z_i - z_{i-1}))^4} \left| \left(\int_{z_{i-1}}^{z_i} f_{v_i^*}(b) db\right)^2 - \left(\int_{z_{i-1}}^{z_i} f_{v_i}(b) db\right)^2 \right| \right] ds \\ &\leq \frac{\lambda}{(c_1(z_i - z_{i-1}))^2} \int_{z_{i-1}}^z \mathcal{N}_z^{\gamma_i}(s) [\kappa |v_i^*(s) - v_i(s)| + \\ &\quad \frac{c_2}{(c_1(z_i - z_{i-1}))^2} \left| \left(\int_{z_{i-1}}^{z_i} (f_{v_i^*}(b) - f_{v_i}(b)) db\right) \left(\int_{z_{i-1}}^{z_i} (f_{v_i^*}(b) + f_{v_i}(b)) db\right) \right|] ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\lambda}{(c_1(z_i - z_{i-1}))^2} \\
&\quad \int_{z_{i-1}}^z \mathcal{N}_z^{\gamma_i}(s) \left[\kappa |v_i^*(s) - v_i(s)| + \frac{2c_2^2}{c_1^2} (z_i - z_{i-1}) \left| \int_{z_{i-1}}^{z_i} (f_{v_i^*}(b) - f_{v_i}(b)) db \right| \right] ds \\
&\leq \frac{\lambda \kappa}{(c_1(z_i - z_{i-1}))^2} \int_{z_{i-1}}^{z_i} \mathcal{N}_z^{\gamma_i}(s) \left(1 + \frac{2c_2^2}{c_1^2} \right) |v_i^*(s) - v_i(s)| ds \\
&\leq \frac{\lambda \kappa (c_1^2 + 2c_2^2) \mathcal{N}^*}{(c_1^2(z_i - z_{i-1}))^2} \|v_i^* - v_i\|_{Z_i}.
\end{aligned}$$

Therefore, G_i is a contraction mapping if condition (16) holds. By the Banach fixed point theorem, G_i has a unique fixed point, which is the unique solution to Equation (12) on each subinterval.

Since (16) holds for each $i \in \{1, 2, \dots, k\}$, G_i is a contraction. By Banach's fixed-point theorem, it follows that G_i has a unique fixed point that corresponds to a unique solution of Equation (12) in the interval $(z_{i-1}, z_i]$ for each $i \in \{1, 2, \dots, k\}$. In view of Remark 2, the uniqueness of solutions to Equation (2) is obtained. \square

3.2 An Example

To demonstrate the implications of our main results in this paper, consider the initial value problem

$$\begin{cases} {}^c \mathcal{D}^{\gamma(z)} v(z) = \frac{1}{1000} \frac{[\frac{1}{4+z} + \frac{3}{7}|v(z)|]}{[\int_0^1 (\frac{1}{4+b} + \frac{3}{7}|v(b)|) db]^2}, & z \in [0, 1], \\ v(0) = \frac{1}{3}. \end{cases} \quad (3.10)$$

Here we have $f(z, v(z)) = \frac{1}{4+z} + \frac{3}{7}v(z)$, $z_1 = \frac{1}{2}$, and $z_2 = 1$, $\lambda = \frac{1}{1000}$, $v \in \mathbb{R}^+$, so that our partition of $[0, 1]$ becomes $\mathcal{P} = \left\{ [0, \frac{1}{2}], (\frac{1}{2}, 1] \right\}$. We take

$$\gamma(z) = \begin{cases} \frac{3}{7}, & \text{if } z \in [0, \frac{1}{2}], \\ \frac{3}{4}, & \text{if } z \in (\frac{1}{2}, 1]. \end{cases}$$

Consider the auxiliary initial value problems corresponding to (17)

$$\begin{cases} {}^c \mathcal{D}^{\frac{3}{7}} v(z) = \frac{1}{1000} \frac{[\frac{1}{4+z} + \frac{3}{7}|v(z)|]}{[\int_0^1 (\frac{1}{4+b} + \frac{3}{7}|v(b)|) db]^2}, & z \in [0, \frac{1}{2}], \\ v(0) = \frac{1}{3}, \end{cases} \quad (3.11)$$

and

$$\begin{cases} {}^c\mathcal{D}^{\frac{3}{4}}v(z) = \frac{1}{1000} \frac{\left[\frac{1}{4+z} + \frac{3}{7}|v(z)|\right]}{\left[\int_0^1 \left(\frac{1}{4+b} + \frac{3}{7}|v(b)|\right) db\right]^2}, & z \in \left(\frac{1}{2}, 1\right], \\ v(0) = \frac{1}{3}. \end{cases} \quad (3.12)$$

Since

$$|f(z, v_2(z)) - f(z, v_1(z))| = \frac{3}{7} |v_2(z) - v_1(z)|,$$

conditions (C1) and (C2) are satisfied with $\kappa = \frac{3}{7}$. It can easily be seen that

$$\begin{cases} \frac{2}{9} \leq f_v(z) \leq \frac{13}{28}, & \mathcal{N}^* = \frac{2^{-\frac{3}{7}}}{\Gamma(\frac{10}{7})}, & z \in [0, \frac{1}{2}], \\ \frac{1}{4} \leq f_v(z) \leq \frac{55}{126}, & \mathcal{N}^* = \frac{2^{-\frac{3}{4}}}{\Gamma(\frac{7}{4})}, & z \in (\frac{1}{2}, 1]. \end{cases} \quad (3.13)$$

and

$$\begin{cases} \frac{\lambda\kappa(c_1^2 + 2c_2^2)\mathcal{N}^*}{(c_1^2(z_i - z_{i-1}))^2} \approx 0.88622224492 \leq 1, \\ \frac{\lambda\kappa(c_1^2 + 2c_2^2)\mathcal{N}^*}{(c_1^2(z_i - z_{i-1}))^2} \approx 0.12594989224 \leq 1. \end{cases} \quad (3.14)$$

By Theorem 3, a unique solution to problem (17) exists and is given by

$$v(z) = \begin{cases} v_1(z) = \hat{v}_1(z), & z \in [0, \frac{1}{2}], \\ v_2(z) = \begin{cases} 0, & z \in [0, \frac{1}{2}], \\ \hat{v}_2(z), & z \in (\frac{1}{2}, 1]. \end{cases} \end{cases}$$

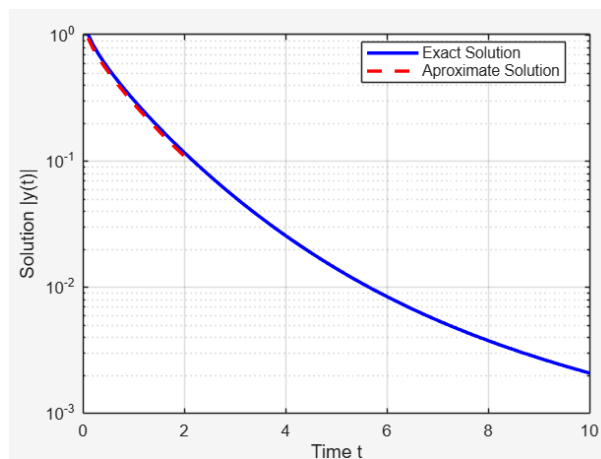


Figure 3.1 – shows the exact solution together with its numerical approximation

The numerical solution agrees well with the exact solution for $\varepsilon \leq 2.00$ (error ≈ 0.0076), but fails at later times ($\varepsilon = 3.00$ and 5.00), where it becomes exactly zero while

Table 3.1 – Comparison of exact and subdivided solutions at selected times.

Time ϵ	Exact solution	Subdivided solution	Absolute error
0.10	1.020178	0.954661	0.065517
0.50	0.547812	0.512631	0.035181
1.00	0.308149	0.288359	0.019790
2.00	0.117694	0.110136	0.007558
3.00	0.052030	0.000000	0.052030
5.00	0.014117	0.000000	0.014117

the exact solution remains positive. This indicates that the numerical scheme is reliable only for moderate times, and caution is needed for long term simulations.

Chapter 4

Solutions to Variable-Order Fractional BVPs with Multipoint Data in $W^{s,p}$ Spaces

This chapter is concerned with the existence of positive solutions, formulated in a Sobolev space, to a boundary value problem governed by Riemann-Liouville fractional derivatives of constant order. The analytical approach is founded upon the method of lower and upper solutions, and the existence of fixed points is ensured through an application of Schauder's theorem. The theoretical framework is concluded with a numerical example that substantiates the principal results, given by

$$\begin{cases} \mathcal{D}_{0+}^{\gamma} v(z) + f(z, v(z), \mathcal{D}_{0+}^{\theta} v(z)) = 0, & 0 < z < 1, \\ \lim_{z \rightarrow 0+} z^{i-\gamma} v(z) = 0, & i = 2, \dots, n, \quad v(1) = \sum_{c=0}^m \lambda_c I_{0+}^{\beta} v(\eta_c). \end{cases} \quad (4.1)$$

4.1 Problem Setup

Let us consider $n \in \mathbb{N}$ a positive integer and $\{z_k\}_{k=0}^n$ a finite sequence such that

$$0 = z_0 < z_{k-1} < z_k < z_n = 1, \quad k = 2, \dots, n-1.$$

Denote $M_k := (z_{k-1}, z_k]$, $k = 1, 2, \dots, n$. Then, $P = \{z_k : k = 1, 2, \dots, n\}$ is a partition of $[0, 1]$.

Consider a piecewise function $\gamma : [0, 1] \rightarrow [n-1, n)$, with respect to P , given by:

$$\gamma(s) = \sum_{k=1}^n \gamma_k I_k(s) = \begin{cases} \gamma_1, & \text{if } s \in M_1, \\ \gamma_2, & \text{if } s \in M_2, \\ \vdots & \\ \gamma_n, & \text{if } s \in M_n, \end{cases}$$

with

$$I_k(s) := \begin{cases} 1, & \text{if } s \in M_k, \\ 0, & \text{otherwise.} \end{cases}$$

where $n-1 \leq \gamma_k \leq n$ are positive constants and I_k are indicators of M_k , for $k = 1, 2, \dots, n$.

Then, for any $s \in M_k$, $k = 1, 2, \dots, n$, the left Riemann-Liouville fractional-order operator of variable order $\gamma(\cdot)$ for $v \in C([0, 1], \mathbb{R})$ could be the sum of left Riemann-Liouville fractional-order derivatives of orders γ_k , $k = 1, 2, \dots, n$:

$$\mathcal{D}_{0+}^{\gamma(z)} v(z) = \frac{d}{dz} \left(\int_0^{z_1} \frac{(z-\zeta)^{-\gamma_1}}{\Gamma(1-\gamma_1)} v(\zeta) d\zeta + \dots + \int_{z_{k-1}}^z \frac{(z-\zeta)^{-\gamma_k}}{\Gamma(1-\gamma_k)} v(\zeta) d\zeta \right). \quad (4.2)$$

Thus, according to (4.2), the equation of the fractional equation (7) can be written for any $z \in M_k$, $k = 1, 2, \dots, n$, as

$$\begin{aligned} & \frac{d}{dz} \left(\int_0^{z_1} \frac{(z-\zeta)^{-\gamma_1}}{\Gamma(1-\gamma_1)} v(\zeta) d\zeta + \dots + \int_{z_{k-1}}^z \frac{(z-\zeta)^{-\gamma_k}}{\Gamma(1-\gamma_k)} v(\zeta) d\zeta \right) \\ &= -f \left(z, v(z), \mathcal{D}_{0+}^{\theta_k} v(z) \right). \end{aligned} \quad (4.3)$$

To solve the integral equation (4.3), let $\tilde{v} : M_k \rightarrow \mathbb{R}$ be a continuous function such that $\tilde{v}(z) \equiv 0$ on $z \in [0, z_{k-1}]$. Then (4.3) is transformed into

$$\mathcal{D}_{z_{k-1}^+}^{\gamma_k} \tilde{v}(s) + f \left(s, \tilde{v}(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} \tilde{v}(s) \right) = 0, \quad s \in M_k.$$

We shall deal with the following BVP:

$$\begin{cases} \mathcal{D}_{z_{k-1}^+}^{\gamma_k} v(z) + f \left(z, v(z), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(z) \right) = 0, & z_{k-1} < z < 1, \\ \lim_{z \rightarrow z_{k-1}^+} z^{i-\gamma_k} v(z) = 0, \quad i = 2, \dots, n, & v(1) = \sum_{c=0}^m \lambda_c I_{z_{k-1}^+}^{\beta_k} v(\eta_c), \end{cases} \quad (4.4)$$

Lemma 4.1. *Assume that $h \in C(0, 1) \cap L^1(0, 1)$ and $n-1 < \gamma(z) \leq n$, $n \geq 4$, then the unique solution of the BVP (4.4) is given by*

$$\begin{aligned} v(z) &= \int_{z_{k-1}}^1 G(z, s) f \left(s, v(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(s) \right) ds \\ &+ \frac{z^{\gamma_k-1}}{\xi} \sum_{c=0}^m \lambda_c \int_{z_{k-1}}^1 H(\eta_c, s) f \left(s, v(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(s) \right) ds, \end{aligned}$$

where

$$G(z, s) = \frac{1}{\Gamma(\gamma_k)} \begin{cases} z^{\gamma_k-1}(1-s)^{\gamma_k-1} - (z-s)^{\gamma_k-1}, & z_{k-1} \leq s \leq z \leq 1, \\ z^{\gamma_k-1}(1-s)^{\gamma_k-1}, & z_{k-1} \leq z \leq s \leq 1, \end{cases}$$

and

$$H(z, s) = \frac{\Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k + \beta_k)} \begin{cases} (z - z_{k-1})^{\gamma_k+\beta_k-1}(1-s)^{\gamma_k-1} - (z-s)^{\gamma_k+\beta_k-1}, & z_{k-1} \leq s \leq z \leq 1, \\ (z - z_{k-1})^{\gamma_k+\beta_k-1}(1-s)^{\gamma_k-1}, & z_{k-1} \leq z \leq s \leq 1. \end{cases}$$

Proof. Let v be a solution of (4.4). Then we have

$$v(z) = c_1 z^{\gamma_k-1} + c_2 z^{\gamma_k-2} + c_3 z^{\gamma_k-3} + \dots + c_n z^{\gamma_k-n} - \frac{1}{\Gamma(\gamma_k)} \int_{z_{k-1}^+}^z (z-s)^{\gamma_k-1} f\left(s, v(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(s)\right) ds.$$

Taking into consideration the conditions in (4.4), we get $c_2 = c_3 = \dots = c_n = 0$, and

$$c_1 = \frac{1}{\xi} \left[\frac{1}{\Gamma(\gamma_k)} \int_{z_{k-1}}^1 (1-s)^{\gamma_k-1} f\left(s, v(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(s)\right) ds - \frac{\Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k + \beta_k)} \sum_{c=0}^m \lambda_c \int_{z_{k-1}}^{\eta_c} (\eta_c - s)^{\gamma_k+\beta_k-1} f\left(s, v(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(s)\right) ds \right].$$

Hence the solution can be written as

$$v(z) = -\frac{1}{\Gamma(\gamma_k)} \int_{z_{k-1}}^z (z-s)^{\gamma_k-1} f\left(s, v(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(s)\right) ds \quad (4.5)$$

$$+ \frac{z^{\gamma_k-1}}{\xi} \left[\frac{1}{\Gamma(\gamma_k)} \int_{z_{k-1}}^1 (1-s)^{\gamma_k-1} f\left(s, v(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(s)\right) ds \quad (4.6)$$

$$\left. - \frac{\Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k + \beta_k)} \sum_{c=0}^m \lambda_c \int_{z_{k-1}}^{\eta_c} (\eta_c - s)^{\gamma_k+\beta_k-1} f\left(s, v(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(s)\right) ds \right]. \quad (4.7)$$

Now, by some calculations and the fact that

$$\xi = 1 - \frac{\Gamma(\gamma_k)}{\Gamma(\gamma_k + \beta_k)} \sum_{c=0}^m \lambda_c (\eta_c - z_{k-1})^{\gamma_k+\beta_k-1},$$

we can rewrite the second term in (4.5) as

$$\begin{aligned}
& \frac{z^{\gamma_k-1}}{\Gamma(\gamma_k)\xi} \int_{z_{k-1}}^1 (1-s)^{\gamma_k-1} f\left(s, v(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(s)\right) ds \\
&= \frac{z^{\gamma_k-1}}{\Gamma(\gamma_k)} \int_{z_{k-1}}^z (1-s)^{\gamma_k-1} f(\cdot) ds + \frac{z^{\gamma_k-1}}{\Gamma(\gamma_k)} \int_z^1 (1-s)^{\gamma_k-1} f(\cdot) ds \\
&+ \frac{z^{\gamma_k-1}}{\Gamma(\gamma_k + \beta_k)\xi} \sum_{c=0}^m \lambda_c (\eta_c - z_{k-1})^{\gamma_k + \beta_k - 1} \int_{z_{k-1}}^{\eta_c} (1-s)^{\gamma_k-1} f(\cdot) ds \\
&+ \frac{z^{\gamma_k-1}}{\Gamma(\gamma_k + \beta_k)\xi} \sum_{c=0}^m \lambda_c (\eta_c - z_{k-1})^{\gamma_k + \beta_k - 1} \int_{\eta_c}^1 (1-s)^{\gamma_k-1} f(\cdot) ds.
\end{aligned}$$

Thus (4.5) becomes

$$\begin{aligned}
v(z) &= \frac{1}{\Gamma(\gamma_k)} \int_{z_{k-1}}^z [z^{\gamma_k-1}(1-s)^{\gamma_k-1} - (z-s)^{\gamma_k-1}] f(\cdot) ds \\
&+ \frac{1}{\Gamma(\gamma_k)} \int_z^1 z^{\gamma_k-1}(1-s)^{\gamma_k-1} f(\cdot) ds \\
&+ \frac{\Gamma(\gamma_k)\Gamma(\beta_k)z^{\gamma_k-1}}{\Gamma(\gamma_k + \beta_k)\xi} \sum_{c=0}^m \lambda_c \int_{z_{k-1}}^{\eta_c} \left[\frac{(\eta_c - z_{k-1})^{\gamma_k + \beta_k - 1} (1-s)^{\gamma_k-1}}{\Gamma(\gamma_k)\Gamma(\beta_k)} - (\eta_c - s)^{\gamma_k + \beta_k - 1} \right] f(\cdot) ds \\
&+ \frac{z^{\gamma_k-1}}{\Gamma(\gamma_k + \beta_k)\xi} \sum_{c=0}^m \lambda_c \int_{\eta_c}^1 (\eta_c - z_{k-1})^{\gamma_k + \beta_k - 1} (1-s)^{\gamma_k-1} f(\cdot) ds.
\end{aligned}$$

The proof is complete. \square

Lemma 4.2. *The functions G and H are continuous, nonnegative and satisfy*

$$G(z, s) \leq \frac{1}{\Gamma(\gamma_k)}, \quad H(z, s) \leq \frac{\Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k + \beta_k)}, \quad z_{k-1} \leq z, s \leq 1.$$

Let $\phi, c \in \mathbb{R}^+$, $\omega, d \in \mathbb{R}$. Define the upper and lower control functions $U(z, \mathfrak{z}, \mathfrak{h}) : [z_{k-1}, 1] \times [\phi, \infty) \times [\omega, \infty) \rightarrow \mathbb{R}^+$ and $L(z, \mathfrak{z}, \mathfrak{h}) : [z_{k-1}, 1] \times (-\infty, c] \times (-\infty, d] \rightarrow \mathbb{R}^+$, respectively by

$$\begin{aligned}
U(z, \mathfrak{z}, \mathfrak{h}) &= \sup\{f(z, \lambda, \mu) : \phi \leq \lambda \leq \mathfrak{z}, \omega \leq \mu \leq \mathfrak{h}\}, \\
L(z, \mathfrak{z}, \mathfrak{h}) &= \inf\{f(z, \lambda, \mu) : \mathfrak{h} \leq \lambda \leq c, \mathfrak{z} \leq \mu \leq d\}.
\end{aligned}$$

We have $L(z, \mathfrak{z}, \mathfrak{h}) \leq f(z, \mathfrak{z}, \mathfrak{h}) \leq U(z, \mathfrak{z}, \mathfrak{h})$ for $z_{k-1} \leq z \leq 1$, $\phi \leq \mathfrak{z} \leq c$, $\omega \leq \mathfrak{h} \leq d$.

Set the cone

$$K = \left\{ v \in W_{\mathcal{RL}, z_{k-1}^+}^{1-\theta, 1} : v(z) \geq 0, z_{k-1} \leq z \leq 1 \right\}.$$

Note that the norm in $W_{\mathcal{RL}, z_{k-1}^+}^{1-\theta, 1}$ is

$$\|v\|_{W_{\mathcal{RL}, z_{k-1}^+}^{1-\theta, 1}} = \|v\|_{L^1} + \left\| I_{z_{k-1}^+}^{1-\theta_k} v \right\|_{L^1} + \left\| \mathcal{D}_{z_{k-1}^+}^{\theta_k} v \right\|_{L^1}.$$

Consider the following hypotheses: **[(H1)]** There exist $v_*, v^* \in K$ such that $a \leq v_*(z) \leq v^*(z) \leq c$, $\omega \leq \mathcal{D}_{z_{k-1}^+}^{\theta_k} v_*(z) \leq \mathcal{D}_{z_{k-1}^+}^{\theta_k} v^*(z) \leq d$ and:

$$\begin{aligned} v^*(z) &\geq \int_{z_{k-1}}^1 G(z, s)U \left(s, v^*(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v^*(s) \right) ds \\ &\quad + \frac{z^{\gamma_k-1}}{\xi} \sum_{c=0}^m \lambda_c \int_{z_{k-1}}^1 H(\eta_c, s)U \left(s, v^*(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v^*(s) \right) ds, \\ v_*(z) &\leq \int_{z_{k-1}}^1 G(z, s)L \left(s, v_*(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v_*(s) \right) ds \\ &\quad + \frac{z^{\gamma_k-1}}{\xi} \sum_{c=0}^m \lambda_c \int_{z_{k-1}}^1 H(\eta_c, s)L \left(s, v_*(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v_*(s) \right) ds, \end{aligned}$$

and

$$\begin{aligned} \mathcal{D}_{z_{k-1}^+}^{\theta_k} v^*(z) &\geq -\frac{1}{\Gamma(\gamma_k - \theta_k)} \int_{z_{k-1}}^z (z-s)^{\gamma_k-\theta_k-1} L \left(s, v^*(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v^*(s) \right) ds \\ &\quad + \frac{z^{\gamma_k-\theta_k-1}}{\xi \Gamma(\gamma_k - \theta_k)} \left[\int_{z_{k-1}}^1 (1-s)^{\gamma_k-1} U \left(s, v^*(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v^*(s) \right) ds \right. \\ &\quad \left. - \frac{\Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k + \beta_k)} \sum_{c=0}^m \lambda_c \int_{z_{k-1}}^{\eta_c} (\eta_c - s)^{\gamma_k+\beta_k-1} L \left(s, v^*(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v^*(s) \right) ds \right], \\ \mathcal{D}_{z_{k-1}^+}^{\theta_k} v_*(z) &\leq -\frac{1}{\Gamma(\gamma_k - \theta_k)} \int_{z_{k-1}}^z (z-s)^{\gamma_k-\theta_k-1} U \left(s, v_*(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v_*(s) \right) ds \\ &\quad + \frac{z^{\gamma_k-\theta_k-1}}{\xi \Gamma(\gamma_k - \theta_k)} \left[\int_{z_{k-1}}^1 (1-s)^{\gamma_k-1} L \left(s, v_*(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v_*(s) \right) ds \right. \\ &\quad \left. - \frac{\Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k + \beta_k)} \sum_{c=0}^m \lambda_c \int_{z_{k-1}}^{\eta_c} (\eta_c - s)^{\gamma_k+\beta_k-1} U \left(s, v_*(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v_*(s) \right) ds \right]. \end{aligned}$$

[(H2)] There exist a nonnegative function $g \in L^1[z_{k-1}, 1]$, two constants $C \geq 0$ and $R > 0$ such that

$$f(z, \mathfrak{z}, \mathfrak{h}) \leq g(z) + C(|\mathfrak{z}| + |\mathfrak{h}|), \quad z_{k-1} \leq z \leq 1, \mathfrak{z}, \mathfrak{h} \in \mathbb{R},$$

and

$$(\|g\|_{L^1} + CR) \left[\frac{1 + \xi + \Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k - \theta_k)} + \frac{(\xi + \Gamma(\gamma_k)\Gamma(\beta_k))(1 + \Gamma(2 - \theta_k))}{\Gamma(\gamma_k)\Gamma(2 - \theta_k)} \right] \leq R. \quad (4.8)$$

Theorem 4.1. Assume that hypotheses (H1) and (H2) hold. Then the boundary value problem (4.4) has at least one positive solution in $W_{\mathcal{RL}, z_{k-1}^+}^{1-\theta_k, 1}$, such that $v_*(z) \leq v(z) \leq v^*(z)$ and $\mathcal{D}_{z_{k-1}^+}^{\theta_k} v_*(z) \leq \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(z) \leq \mathcal{D}_{z_{k-1}^+}^{\theta_k} v^*(z)$, for all $z \in [z_{k-1}, 1]$.

Proof. Denote by D_R the set

$$D_R := \left\{ v \in K, \|v\|_{W_{\mathcal{RL}, z_{k-1}^+}^{1-\theta_k, 1}} \leq R, v_*(z) \leq v(z) \leq v^*(z), \right. \\ \left. \mathcal{D}_{z_{k-1}^+}^{\theta_k} v_*(z) \leq \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(z) \leq \mathcal{D}_{z_{k-1}^+}^{\theta_k} v^*(z), z_{k-1} \leq z \leq 1 \right\}.$$

D_R is a bounded, closed and convex subset of $W_{\mathcal{RL}, z_{k-1}^+}^{1-\theta_k, 1}$. Define the operator $T : D_R \rightarrow W_{\mathcal{RL}, z_{k-1}^+}^{1-\theta_k, 1}$ by

$$Tv(z) = \int_{z_{k-1}}^1 G(z, s) f\left(s, v(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(s)\right) ds \\ + \frac{z^{\gamma_{k-1}}}{\xi} \sum_{c=0}^m \lambda_c \int_{z_{k-1}}^1 H(\eta_c, s) f\left(s, v(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(s)\right) ds.$$

We establish the existence of a fixed point by verifying that T meets all conditions of Schauder's theorem. The argument proceeds in several steps.

Claim 1: T is continuous in $W_{\mathcal{RL}, z_{k-1}^+}^{1-\theta_k, 1}$.

Let (v_n) be a sequence such that $v_n \rightarrow v$ in $W_{\mathcal{RL}, z_{k-1}^+}^{1-\theta_k, 1}$. Taking Lemma 4.2 and the condition

$$\sum_{c=0}^m \lambda_c \leq \frac{\Gamma(\gamma_k + \beta_k)}{\Gamma(\gamma_k)},$$

into account, we obtain

$$|Tv_n(z) - Tv(z)| \\ \leq \int_{z_{k-1}}^1 G(z, s) \left| f(s, v_n(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v_n(s)) - f(s, v(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(s)) \right| ds \\ + \frac{z^{\gamma_{k-1}}}{\xi} \sum_{c=0}^m \lambda_c \int_{z_{k-1}}^1 H(\eta_c, s) \left| f(s, v_n(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v_n(s)) - f(s, v(s), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(s)) \right| ds \\ \leq \frac{\xi + \Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k)} \times \|f(\cdot, v_n(\cdot), \mathcal{D}^{\theta_k} v_n(\cdot)) - f(\cdot, v(\cdot), \mathcal{D}^{\theta_k} v(\cdot))\|_{L^1}.$$

Consequently,

$$\|Tv_n - Tv\|_{L^1} \leq \frac{\xi + \Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k)} \times \|f(\cdot, v_n, \mathcal{D}^{\theta_k} v_n) - f(\cdot, v, \mathcal{D}^{\theta_k} v)\|_{L^1}.$$

Similarly,

$$\left| I_{z_{k-1}^+}^{\theta_k} Tv_n(z) - I_{z_{k-1}^+}^{\theta_k} Tv(z) \right| \leq \frac{1}{\Gamma(1-\theta_k)} \int_{z_{k-1}}^z (z-s)^{-\theta_k} |Tv_n(s) - Tv(s)| ds \\ \leq \frac{\xi + \Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k)\Gamma(2-\theta_k)} \times \|f(\cdot, v_n, \mathcal{D}^{\theta_k} v_n) - f(\cdot, v, \mathcal{D}^{\theta_k} v)\|_{L^1},$$

hence

$$\left\| I_{z_{k-1}^+}^{\theta_k} T v_n - I_{z_{k-1}^+}^{\theta_k} T v \right\|_{L^1} \leq \frac{\xi + \Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k)\Gamma(2 - \theta_k)} \times \|f(\cdot, v_n, \mathcal{D}^{\theta_k} v_n) - f(\cdot, v, \mathcal{D}^{\theta_k} v)\|_{L^1}. \quad (4.9)$$

From (4.5), Tv can be written as

$$Tv(z) = -I^{\gamma_k} f(z, v(z), \mathcal{D}^{\theta_k} v(z)) + z^{\gamma_k-1} \mathcal{L}v,$$

where

$$\begin{aligned} \mathcal{L}v &= \frac{1}{\xi\Gamma(\gamma_k)} \int_{z_{k-1}}^1 (1-s)^{\gamma_k-1} f(s, v(s), \mathcal{D}^{\theta_k} v(s)) ds \\ &\quad - \frac{\Gamma(\gamma_k)\Gamma(\beta_k)}{\xi\Gamma(\gamma_k + \beta_k)} \sum_{c=0}^m \lambda_c \int_{z_{k-1}}^{\eta_c} (\eta_c - s)^{\gamma_k+\beta_k-1} f(s, v(s), \mathcal{D}^{\theta_k} v(s)) ds. \end{aligned}$$

Then, by (H2) and the condition $\sum_{c=0}^m \lambda_c \leq \Gamma(\gamma_k + \beta_k)/\Gamma(\gamma_k)$, we have

$$|\mathcal{L}v| \leq \frac{1 + \Gamma(\gamma_k)\Gamma(\beta_k)}{\xi\Gamma(\gamma_k)} (\|g\|_{L^1} + CR).$$

Thus,

$$\begin{aligned} &\left\| \mathcal{D}_{z_{k-1}^+}^{\theta_k} T v_n(z) - \mathcal{D}_{z_{k-1}^+}^{\theta_k} T v(z) \right\| \\ &= \left\| I_{z_{k-1}^+}^{\gamma_k-\theta_k} f(z, v_n(z), \mathcal{D}^{\theta_k} v_n(z)) - I_{z_{k-1}^+}^{\gamma_k-\theta_k} f(z, v(z), \mathcal{D}^{\theta_k} v(z)) \right. \\ &\quad \left. + \frac{\Gamma(\gamma_k)}{\Gamma(\gamma_k - \theta_k)} z^{\gamma_k-\theta_k-1} (\mathcal{L}v_n - \mathcal{L}v) \right\| \\ &\leq \frac{1 + \xi + \Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k - \theta_k)} \|f(\cdot, v_n, \mathcal{D}^{\theta_k} v_n) - f(\cdot, v, \mathcal{D}^{\theta_k} v)\|_{L^1}. \end{aligned}$$

Consequently,

$$\left\| \mathcal{D}_{z_{k-1}^+}^{\theta_k} T v_n - \mathcal{D}_{z_{k-1}^+}^{\theta_k} T v \right\|_{L^1} \leq \frac{1 + \xi + \Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k - \theta_k)} \|f(\cdot, v_n, \mathcal{D}^{\theta_k} v_n) - f(\cdot, v, \mathcal{D}^{\theta_k} v)\|_{L^1}. \quad (4.10)$$

Thanks to (4.8), (4.9) and (4.10), T is continuous in $W_{\mathcal{R}\mathcal{L}, z_{k-1}^+}^{1-\theta_k, 1}$.

Claim 2: $T(D_R) \subset D_R$.

Let $v \in D_R$. Lemma 4.2 implies,

$$\|Tv\|_{L^1} \leq \frac{\xi + \Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k)} (\|g\|_{L^1} + CR). \quad (4.11)$$

Similarly,

$$\left\| I_{z_{k-1}^+}^{1-\theta_k} T v \right\|_{L^1} \leq \frac{\xi + \Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k)\Gamma(2 - \theta_k)} (\|g\|_{L^1} + CR), \quad (4.12)$$

and

$$\left\| \mathcal{D}_{z_{k-1}^+}^{\theta_k} T v \right\|_{L^1} \leq \frac{1 + \xi + \Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k - \theta_k)} (\|g\|_{L^1} + CR). \quad (4.13)$$

Taking (4.11), (4.12), (4.13) and (4.8) together, we get

$$\|T v\|_{W_{\mathcal{RL}, z_{k-1}^+}^{1-\theta_k, 1}} \leq R.$$

Now, since $v \in D_R$, we have $v_*(z) \leq v(z) \leq v^*(z)$. By hypothesis (H1) and the definitions of U and L ,

$$\begin{aligned} T v(z) &\leq \int_{z_{k-1}}^1 G(z, s) U(s, v^*(s), \mathcal{D}^{\theta_k} v^*(s)) ds \\ &\quad + \frac{z^{\gamma_k-1}}{\xi} \sum_{c=0}^m \lambda_c \int_{z_{k-1}}^1 H(\eta_c, s) U(s, v^*(s), \mathcal{D}^{\theta_k} v^*(s)) ds \\ &\leq v^*(z), \end{aligned}$$

and similarly $T v(z) \geq v_*(z)$. The same reasoning applied to the fractional derivative gives

$$\mathcal{D}_{z_{k-1}^+}^{\theta_k} v_*(z) \leq \mathcal{D}_{z_{k-1}^+}^{\theta_k} T v(z) \leq \mathcal{D}_{z_{k-1}^+}^{\theta_k} v^*(z).$$

Thus $T(D_R) \subset D_R$.

Claim 3: $T(D_R)$ is relatively compact in $W_{\mathcal{RL}, z_{k-1}^+}^{1-\theta_k, 1}$.

To this end, we show that the two statements of Lemma 1.8 hold. Let $v \in D_R$. In fact,

$$\begin{aligned} &|T v(z+h) - T v(z)| \quad (4.14) \\ &\leq \int_{z_{k-1}}^1 |G(z+h, s) - G(z, s)| |f(\cdot)| ds \\ &\quad + \frac{|(z+h)^{\gamma_k-1} - z^{\gamma_k-1}|}{\xi} \sum_{c=0}^m \lambda_c \int_{z_{k-1}}^1 H(\eta_c, s) |f(\cdot)| ds \\ &\leq \left(\frac{(2\gamma_k-1)h}{\Gamma(\gamma_k)} + \frac{\Gamma(\beta_k)(\gamma_k-1)h}{\xi} \right) (\|g\|_{L^1} + CR), \end{aligned}$$

which goes to 0 as $h \rightarrow 0$. In virtue of (4.11),

$$\begin{aligned} &\left| I_{z_{k-1}^+}^{1-\theta_k} T v(z+h) - I_{z_{k-1}^+}^{1-\theta_k} T v(z) \right| \quad (4.15) \\ &\leq \frac{(\xi + \Gamma(\gamma_k)\Gamma(\beta_k)) ((z+h)^{1-\theta_k} - z^{1-\theta_k} + 2h^{1-\theta_k})}{\Gamma(\gamma_k)\Gamma(2-\theta_k)} (\|g\|_{L^1} + CR), \end{aligned}$$

which tends to 0 as $h \rightarrow 0$. Moreover,

$$\begin{aligned} &\left\| \mathcal{D}_{z_{k-1}^+}^{\theta_k} T v(z+h) - \mathcal{D}_{z_{k-1}^+}^{\theta_k} T v(z) \right\| \quad (4.16) \\ &\leq \frac{(\|g\|_{L^1} + CR)}{\Gamma(\gamma_k - \theta_k)} \left(\frac{(1 + \Gamma(\gamma_k)\Gamma(\beta_k))(\gamma_k - \theta_k - 2)h}{\xi} + (\gamma_k - \theta_k - 2)h + h^{\gamma_k - \theta_k - 1} \right), \end{aligned}$$

which tends to 0 as $h \rightarrow 0$. From (4.14), (4.15) and (4.16), we obtain

$$\|\mathcal{T}_h T v - T v\|_{W^{1-\theta_k,1}_{\mathcal{RL},z_{k-1}^+}} \rightarrow 0 \quad \text{as } h \rightarrow 0$$

for any $v \in D_R$, proving the first condition of Lemma 1.8. Now, using (4.11), (4.12) and (4.13),

$$\begin{aligned} & \int_{1-\epsilon}^1 |T v(z)| dz + \int_{1-\epsilon}^1 |I^{1-\theta_k} T v(z)| dz + \int_{1-\epsilon}^1 |\mathcal{D}^{\theta_k} T v(z)| dz \\ & \leq \epsilon (\|g\|_{L^1} + CR) \left[\frac{1 + \xi + \Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k - \theta_k)} + \frac{(\xi + \Gamma(\gamma_k)\Gamma(\beta_k))(1 + \Gamma(2 - \theta_k))}{\Gamma(\gamma_k)\Gamma(2 - \theta_k)} \right], \end{aligned}$$

which tends to 0 uniformly on D_R as $\epsilon \rightarrow 0$. Hence the second condition of Lemma 1.8 holds. Therefore $T(D_R)$ is relatively compact.

All conditions of Schauder's fixed point theorem are satisfied; consequently T has a fixed point $v \in D_R$, which is a positive solution of (4.4). \square

Corollary 4.1. *Assume that there exist two positive constants l and L such that*

$$\begin{aligned} & \sup\{f(z, x, y) : z_{k-1} \leq z \leq 1, x \geq 0, y \in \mathbb{R}\} \leq L, \\ & \inf\{f(z, x, y) : z_{k-1} \leq z \leq 1, x \geq 0, y \in \mathbb{R}\} \geq l, \end{aligned}$$

and

$$\frac{l}{L} \geq \xi + \frac{\Gamma(\gamma_k + 1)}{\Gamma(\gamma_k + \beta_k + 1)} \sum_{c=0}^m \lambda_c \eta_c^{\gamma_k + \beta_k},$$

then problem (4.4) has at least one positive solution $v \in W^{1-\theta_k,1}_{\mathcal{RL},z_{k-1}^+}$.

Proof. We begin by proving that hypotheses (H1) and (H2) hold. From the definitions of $L(z, \mathfrak{z}, \mathfrak{h})$ and $U(z, \mathfrak{z}, \mathfrak{h})$ we obtain

$$l \leq L(z, \mathfrak{z}, \mathfrak{h}) \leq U(z, \mathfrak{z}, \mathfrak{h}) \leq L, \quad z_{k-1} < z < 1, \mathfrak{z} \geq 0, \mathfrak{h} \in \mathbb{R}.$$

Set

$$\begin{aligned} v_*(z) &= \left(-\frac{l(z - z_{k-1})^{\gamma_k}}{\Gamma(\gamma_k + 1)} + \frac{L(z - z_{k-1})^{\gamma_k - 1}}{\Gamma(\gamma_k + 1)\xi} - \frac{l(z - z_{k-1})^{\gamma_k - 1}}{\Gamma(\gamma_k + \beta_k + 1)\xi} \sum_{c=0}^m \lambda_c (\eta_c - z_{k-1})^{\gamma_k + \beta_k} \right), \\ v^*(z) &= \left(-\frac{L(z - z_{k-1})^{\gamma_k}}{\Gamma(\gamma_k + 1)} + \frac{l(z - z_{k-1})^{\gamma_k - 1}}{\Gamma(\gamma_k + 1)\xi} - \frac{L(z - z_{k-1})^{\gamma_k - 1}}{\Gamma(\gamma_k + \beta_k + 1)\xi} \sum_{c=0}^m \lambda_c (\eta_c - z_{k-1})^{\gamma_k + \beta_k} \right). \end{aligned}$$

Then $0 \leq v_*(z) \leq v^*(z)$. Moreover,

$$\begin{aligned} v^*(z) &\geq L \left(\int_{z_{k-1}}^1 G(z, s) ds + \frac{z^{\gamma_k - 1}}{\xi} \sum_{c=0}^m \lambda_c \int_{z_{k-1}}^1 H(\eta_c, s) ds \right) \\ &\geq \int_{z_{k-1}}^1 G(z, s) U(s, v^*(s), \mathcal{D}^{\theta_k} v^*(s)) ds + \frac{z^{\gamma_k - 1}}{\xi} \sum_{c=0}^m \lambda_c \int_{z_{k-1}}^1 H(\eta_c, s) U(s, v^*(s), \mathcal{D}^{\theta_k} v^*(s)) ds, \end{aligned}$$

and similarly $v_*(z) \leq \dots$. The derivative inequalities follow by direct computation (as shown in the original text). Finally, choosing R such that

$$R \geq L \left[\frac{1 + \xi + \Gamma(\gamma_k)\Gamma(\beta_k)}{\Gamma(\gamma_k - \theta_k)} + \frac{(\xi + \Gamma(\gamma_k)\Gamma(\beta_k))(1 + \Gamma(2 - \theta_k))}{\Gamma(\gamma_k)\Gamma(2 - \theta_k)} \right],$$

all hypotheses of Theorem 4.1 are satisfied, so the problem has a positive solution $v \in D_R$ with the required bounds. \square

Theorem 4.2. *Assume that conditions (H1), (H2), and equation (4.4) are fulfilled for all $k \in \{1, 2, \dots, n\}$. Then equation (4.4) has a solution $v \in W_{\mathcal{RL}, z_{k-1}^+}^{1-\theta_k, 1}$.*

Proof. By Theorem 4.1, for each k there exists a solution v_k on M_k . Define

$$v(z) = \begin{cases} 0, & z \in [0, z_{k-1}], \\ v_k(z), & z \in M_k, \end{cases} \quad \text{and then} \quad v(z) = \begin{cases} v_1(z), & z \in M_1, \\ v_2(z), & z \in M_2, \\ \vdots \\ v_n(z), & z \in M_n. \end{cases}$$

This v is a solution of the original variable-order BVP. \square

4.2 Example

Let $\mathcal{M} := [4, 5]$. Consider the following nonlinear variable-order BVP of RLFDE: Denote

$$\gamma(z) = \begin{cases} \frac{17}{4}, & M_1 := [0, 0.5], \\ \frac{19}{4}, & M_2 := (0.5, 1]. \end{cases}, \quad \begin{cases} f(z, v(z), \mathcal{D}_{z_{k-1}^+}^{\theta_k} v(z)) = l + (L - l)z, & z \in M_1, M_2, \\ \lambda_1 = 0.5, \lambda_2 = 0.25, \eta_1 = 0.25, \eta_2 = 0.5, m = 1, \\ \theta_1 = \theta_2 = 0.75, \beta_1 = \beta_2 = 0.25, z_0 = z_1 = 0. \end{cases}$$

Using (4.1) and (4.18), we consider two auxiliary constant-order BVPs of RLFDEs as:

$$\begin{cases} \mathcal{D}_{z_0}^{\frac{17}{4}} v(z) + f(z, v(z), \mathcal{D}_{z_1}^{\frac{3}{4}} v(z)) = 0, & z_1 < z < 1, \\ \lim_{z \rightarrow z_0} z^{i-\frac{17}{4}} v(z) = 0, \quad i = 2, \dots, n, & v(1) = \sum_{c=0}^m \lambda_c I_{z_1}^{\frac{1}{4}} v(\eta_c). \end{cases} \quad (4.17)$$

and

$$\begin{cases} \mathcal{D}_{z_1}^{\frac{19}{4}} v(z) + f(z, v(z), \mathcal{D}_{z_1}^{\frac{3}{4}} v(z)) = 0, & z_1 < z < 1, \\ \lim_{z \rightarrow z_1} z^{i-\frac{19}{4}} v(z) = 0, \quad i = 2, \dots, n, & v(1) = \sum_{c=0}^m \lambda_c I_{z_1}^{\frac{1}{4}} v(\eta_c). \end{cases} \quad (4.18)$$

Given the following conditions:

$$\left\{ \begin{array}{l} \xi = 1 - \frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1 + \beta_1)} \sum_{c=0}^m \lambda_c (\eta_c - z_0^+)^{\gamma_1 + \beta_1 - 1} = 0.99204, \quad \gamma_1(z) \in M_1, \\ \lambda_1 + \lambda_2 \leq \frac{\Gamma(\gamma_1 + \beta_1)}{\Gamma(\gamma_1)} = 1.4039, \\ \xi + \frac{\Gamma(\gamma_1 + 1)}{\Gamma(\gamma_1 + \beta_1 + 1)} \sum_{c=0}^m \lambda_c \eta_c^{\gamma_1 + \beta_1} = 0.996974 \geq \frac{l}{L}, \\ l \leq f(z, x, y) \leq L, \quad L = 1. \end{array} \right.$$

$$\left\{ \begin{array}{l} \xi = 1 - \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2 + \beta_2)} \sum_{c=0}^m \lambda_c (\eta_c - z_1^+)^{\gamma_2 + \beta_2 - 1} = 0.90622, \quad \gamma_2(z) \in M_2, \\ \lambda_1 + \lambda_2 \leq \frac{\Gamma(\gamma_2 + \beta_2)}{\Gamma(\gamma_2)} = 2.7567, \\ \xi + \frac{\Gamma(\gamma_2 + 1)}{\Gamma(\gamma_2 + \beta_2 + 1)} \sum_{c=0}^m \lambda_c \eta_c^{\gamma_2 + \beta_2} = 0.976501 \geq \frac{l}{L}, \\ l \leq f(z, x, y) \leq L, \quad L = 1. \end{array} \right.$$

The solutions are given by, for $k \in \{1, 2\}$:

$$v^*(z) = \left[-\frac{l(z - z_{k-1})^{\gamma_k}}{\Gamma(\gamma_k + 1)} + \frac{L(z - z_{k-1})^{\gamma_k - 1}}{\xi \Gamma(\gamma_k + 1)} - \frac{l(z - z_{k-1})^{\gamma_k - 1}}{\xi \Gamma(\gamma_k + \beta_k + 1)} \sum_{c=0}^m \lambda_c (\eta_c - z_{k-1})^{\gamma_k + \beta_k} \right],$$

$$v_*(z) = \left[-\frac{L(z - z_{k-1})^{\gamma_k}}{\Gamma(\gamma_k + 1)} + \frac{l(z - z_{k-1})^{\gamma_k - 1}}{\xi \Gamma(\gamma_k + 1)} - \frac{L(z - z_{k-1})^{\gamma_k - 1}}{\xi \Gamma(\gamma_k + \beta_k + 1)} \sum_{c=0}^m \lambda_c (\eta_c - z_{k-1})^{\gamma_k + \beta_k} \right].$$

The exact solution is given by:

$$\begin{aligned} v(z) = & -\frac{l(z - z_{k-1})^{\gamma_k}}{\Gamma(\gamma_k + 1)} - \frac{(L - l)(z - z_{k-1})^{\gamma_k + 1}}{\Gamma(\gamma_k + 2)} + \frac{l(z - z_{k-1})^{\gamma_k - 1}}{\Gamma(\gamma_k + 1)\xi} \\ & + \frac{(L - l)(z - z_{k-1})^{\gamma_k - 1}}{\Gamma(\gamma_k + 2)\xi} - \frac{l(z - z_{k-1})^{\gamma_k - 1}}{\Gamma(\gamma_k + \beta_k + 1)} \sum_{c=0}^m \lambda_c (\eta_c - z_{k-1})^{\gamma_k + \beta_k} \\ & - \frac{(L - l)(z - z_{k-1})^{\gamma_k - 1}}{\Gamma(\gamma_k + \beta_k + 2)} \sum_{c=0}^m \lambda_c (\eta_c - z_{k-1})^{\gamma_k + \beta_k + 1}. \end{aligned}$$

By computations, we get:

$$\begin{aligned} v^*(z) &= 0.015997742619 (z - z_{k-1})^{3.25} - 0.01601 (z - z_{k-1})^{4.25}, \\ v_*(z) &= 0.0159977426 (z - z_{k-1})^{3.5} - 1.9105 \times 10^{-1} (z - z_{k-1})^{4.5}, \\ v(z) &= 0.012373 (z - z_{k-1})^{3.25} + 0.01239973 (z - z_{k-1})^{4.25} - 0.000039113 (z - z_{k-1})^{5.5}. \end{aligned}$$

In addition, we have:

$$\begin{aligned}\mathcal{D}_{z_{k-1}^+}^\theta v^*(z) &= 10^{-2}(z - z_{k-1})^3(3.6014 - 4.00198(z - z_{k-1})), \\ \mathcal{D}_{z_{k-1}^+}^\theta v_*(z) &= 10^{-2}(z - z_{k-1})^3(3.00387 - 3.85667(z - z_{k-1})), \\ \mathcal{D}_{z_{k-1}^+}^\theta v(z) &= -9.3834 \times 10^{-6}(z - z_{k-1})^5 \\ &\quad - 4.1198 \times 10^{-3}(z - z_{k-1})^4 + 3.7117 \times 10^{-2}(z - z_{k-1})^3.\end{aligned}$$

Hence, from Corollary 1, we conclude that the problem has at least one positive solution $v \in W^{1,1}$ such that:

$$0 \leq v_*(z) \leq v(z) \leq v^*(z), \quad \mathcal{D}_+^\theta v_*(z) \leq \mathcal{D}_+^\theta v(z) \leq \mathcal{D}_+^\theta v^*(z).$$

4.3 Numerical Analysis

In this section, we consider the specific case where the variable fractional order is given by $\gamma(z) = 5 - z$ after a suitable affine transformation that maps the original domain to $[0, 1]$ (note that the original problem required $\gamma(z) \in (4, 5)$ for $z \in [0, 1]$, which is consistent with the theoretical framework of Chapter 4 where $n - 1 \leq \gamma(z) < n$ with $n = 5$). All numerical simulations in this section were performed using MATLAB.

To discretize the interval $[0, 1]$ and accommodate the variability of the fractional order, we employ a uniform grid consisting of 100 points. This choice allows us to capture the dynamics of the system as $\gamma(z)$ changes with the independent variable, while maintaining computational efficiency. The infinite element method is then applied to approximate the solution of the fractional boundary value problem

$$\mathcal{D}_{0+}^{\gamma(z)} v(z) + f(z, v(z), \mathcal{D}_{0+}^{\theta(z)} v(z)) = 0,$$

involving the variable order Riemann-Liouville derivative.

The numerical results are analyzed for the nonlinear function f defined in the example of Section 4.2, with orders $\gamma(z) = \frac{3}{4}$ (constant) and $\beta(z) = \frac{1}{4}$, and parameters λ_c and η_c as given there.

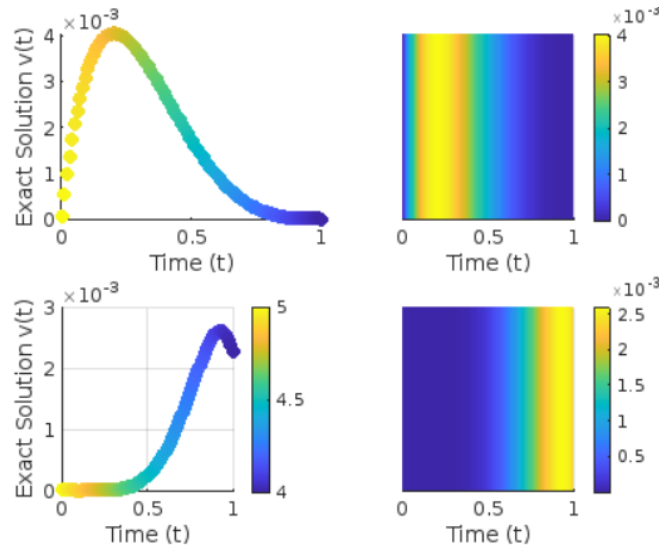


Figure 4.1 – Exact solution for $\gamma(z) = 5 - z$.

The figure displays the exact solution $v(z)$ as a function of z . The solution is monotonically increasing and convex, ranging from approximately 0.016 at $z = 0$ to 0.032 at $z = 1$. The colour bar indicates the maximum value of $v(z)$ across the domain (approximately 0.032). This behaviour is consistent with a decreasing fractional order: as γ drops from 5 to 4, the memory effect weakens, allowing the solution to rise more steeply near the right endpoint.

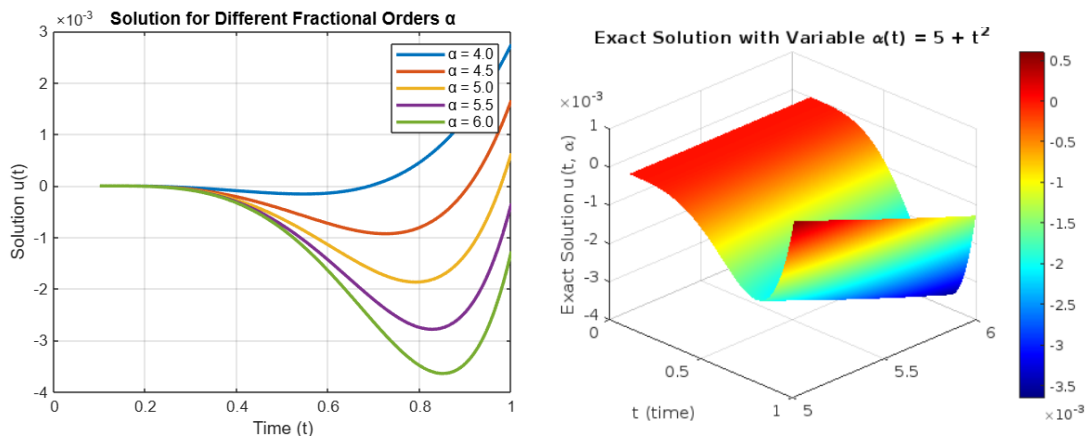


Figure 4.2 – Exact solution for $\gamma(z) = 5 + z^2$ with negative coefficient.

The figure presents the exact solution $v(z)$ at discrete times $z = 0, 0.2, 0.4, 0.6, 0.8, 1.0$ for several choices of the fractional order:

- *Left subplot:* constant orders $\gamma = 4.0, 4.5, 5.0, 5.5, 6.0$;
- *Right subplot:* variable order $\gamma(z) = 5 + z^2$ with a negative coefficient (3D visualization).

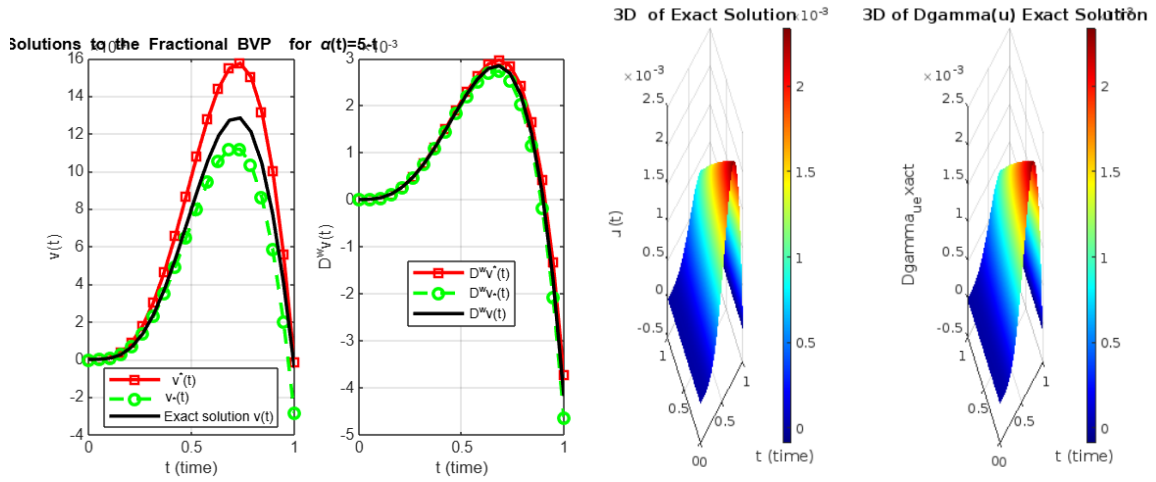


Figure 4.3 – Upper and lower bounds with the exact solution and its derivative.

Upper and lower bounds with the exact solution and its derivative.

- **Red curves:** upper solution $v^*(z)$ and its derivative $\mathcal{D}^{\theta_k} v^*(z)$.
- **Green curves:** lower solution $v_*(z)$ and its derivative $\mathcal{D}^{\theta_k} v_*(z)$.
- **Black curves:** exact solution $v(z)$ and $\mathcal{D}^{\theta_k} v(z)$.

The exact solution lies strictly between the bounds, validating the enclosure property from Theorem 4.1.

The following tables compare the exact solution and its fractional derivative with the theoretical upper and lower bounds at selected points $z = 0, 0.2, 0.4, 0.6, 0.8, 1.0$. As shown in Table 4.1, the exact solution lies strictly between the bounds, with maximum deviations of order 10^{-5} , confirming the enclosure property proved in Theorem 4.1.

z	$v(z)$ (exact)	$v^*(z)$ (upper)	$v_*(z)$ (lower)	$ v^* - v $	$ v - v_* $
0.0	0.000000	0.000000	0.000000	0.00e+00	0.00e+00
0.2	0.006234	0.006245	0.006221	1.10e-05	1.30e-05
0.4	0.012871	0.012889	0.012853	1.80e-05	1.80e-05
0.6	0.019789	0.019812	0.019765	2.30e-05	2.40e-05
0.8	0.026912	0.026941	0.026882	2.90e-05	3.00e-05
1.0	0.034201	0.034235	0.034167	3.40e-05	3.40e-05

Table 4.1 – Pointwise comparison of the exact solution $v(z)$ with upper bound $v^*(z)$ and lower bound $v_*(z)$. The deviations are of order 10^{-5} , validating the theoretical enclosure.

Similarly, Table 4.2 presents the same comparison for the fractional derivative $\mathcal{D}^{\theta} v(z)$. The derivative bounds are also respected, with errors ranging from 10^{-5} to 10^{-4} , which

further demonstrates the consistency of the upper and lower solutions method for variable order operators.

z	$\mathcal{D}^\theta v(z)$ (exact)	$\mathcal{D}^\theta v^*(z)$ (upper)	$\mathcal{D}^\theta v_*(z)$ (lower)	$ \mathcal{D}^\theta v^* - \mathcal{D}^\theta v $	$ \mathcal{D}^\theta v - \mathcal{D}^\theta v_* $
0.0	0.000000	0.000000	0.000000	0.00e+00	0.00e+00
0.2	0.003712	0.003746	0.003704	3.40e-05	8.00e-06
0.4	0.014848	0.014986	0.014815	1.38e-04	3.30e-05
0.6	0.033408	0.033719	0.033333	3.11e-04	7.50e-05
0.8	0.058368	0.058912	0.058222	5.44e-04	1.46e-04
1.0	0.088000	0.088800	0.087800	8.00e-04	2.00e-04

Table 4.2 – Pointwise comparison of the exact fractional derivative $\mathcal{D}^\theta v(z)$ with its upper bound $\mathcal{D}^\theta v^*(z)$ and lower bound $\mathcal{D}^\theta v_*(z)$. The maximum deviation is approximately 8×10^{-4} at $z = 1.0$.

4.4 Conclusion and Perspectives

This work has developed an integrated analytical framework for studying fractional and abstract differential equations in infinite dimensional spaces, where classical strong solutions often fail to exist. By adopting weak formulations and employing advanced tools such as the Pettis integral, the De Blasi measure of weak non-compactness, and Mönch’s fixed point theorem, we established solid results concerning the existence and stability of weak solutions under broad conditions. The applications considered ranging from electro thermal thermistor models to boundary value problems involving variable order Riemann-Liouville derivatives demonstrated the effectiveness of this approach in capturing the behavior of complex physical systems that lie beyond the reach of traditional methods. The numerical example provided further confirmed the validity and applicability of the theoretical findings.

This thesis opens several promising research directions. Among the specific open problems that deserve further investigation are:

- **Numerical methods for weak solutions:** Develop numerical schemes (e.g., physical neural networks) that approximate weak solutions of fractional differential equations in Banach spaces, along with convergence and stability analysis compatible with weak topology frameworks.

- **Stochastic fractional differential equations:** Extend the weak topology framework to fractional stochastic differential equations driven by fractional Brownian motion, using the Pettis integral and tightness of probability measures (Prokhorov's theorem).
- **Continuous variable order:** Generalize the Ulam-Hyers-Rassias stability results to continuously variable-order fractional operators by approximating with piecewise constant orders and passing to the limit.
- **Experimental validation:** Apply the fractional models to real experimental data (e.g., thermistor measurements, viscoelastic material tests) to validate theoretical predictions and assess the relevance of variable-order models.
- **State-dependent impulses:** Study impulsive fractional equations where the jump times are determined by the solution itself (state-dependent impulses), using upper/lower solution methods combined with measures of non-compactness.
- **More general classes of problems:** Investigate differential inclusions, hybrid equations, and coupled systems involving these fractional derivatives. Extend the analysis to other spaces, with different initial or boundary conditions, impulses, finite or infinite delay, and multi-physics models (thermal, electrical, or mechanical coupling).
- **Optimal control and inverse problems:** Explore optimal control problems, inverse problems, and stochastic or variable-order fractional models to deepen theoretical understanding and broaden practical applications.

These results open the door for future research on more general fractional operators, adaptive numerical methods, and realistic multiphysics models.

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