Ministry of higher education and scientific research
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COMPLEX ANALYSIS

Course intended primarily for students of "License L2"

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License L2

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Introduction

In This course we introduce the concept of a differentiable function of a complex variable, study the main properties of these functions and some of their applications (calculations of certain generalized integrals and summation of series).

In the following we give an outline of our organization of this course, which consists of six chapters.

Chapter one we give reviews of topology in complex plane, chapter two and tree, treat notion of complex function and elementary complex function respectively. Then we discuss the concept of complex integration in chapter four. Taylor and Laurent series developments are study in five. chapter Residue and its application and discussed in last chapter.

Chapter 1

Topology in the complex plane

Chapter contents

1.1 Complex numbers.

Definition 1.1.1 Complex number

A ***complex number*** is an expression of the form

$$z = x + iy \quad \text{(called } \textbf{\textit{algebraic form}}\text{)}, \tag{1.1}$$

where $i^2 = -1$ and $x, y \in \mathbb{R}$.
The set of all complex numbers is denoted by $\mathbb{C}$. We call x the real part of z and write $x = Re(z)$ and y the imaginary part of z and write$y = Im(z)$. (Note: the imaginary part of $z = x + iy$ is y, and not iy.).
For all $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}$, we have the the following properties

(1) $z_1 = z_2 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$.

(2) z is real $\Leftrightarrow Im(z) = 0$.

(3) z is pure imaginary $\Leftrightarrow Re(z) = 0$.

(4) $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.

(5) $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$.

(6) $z_1 \times z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$ for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

(7) Note that the binomial formula involving real numbers remains valid with complex numbers. That is, if z_1 and z_2 are any two nonzero complex numbers, then

$$(z_1 + z_2)^n = \sum_{k=0}^{n} C_n^k z_1^k z_2^{n-k} \quad (k = 1, 2, ..n.) \tag{1.2}$$

where

$$C_n^k = \frac{n!}{(n-k)!k!} \quad (k = 0, 1, 2, ..). \tag{1.3}$$

Remark 1.1.2.

(a) Many of the properties of the real number system $\mathbb{R}$ hold in the complex number system $\mathbb{C}$, but there are some remarkable differences as well. For example, the concept of order in $\mathbb{R}$ does not carry over to $\mathbb{C}$. In other words, we cannot compare two non-real complex numbers.

(b) On the other hand, some things that are impossible in real analysis, such as $e^x = -1$ and $\cos x = 10$ if x is a real variable, are perfectly correct, but it is possible in complex analysis when the symbol x is interpreted as a complex variable z.

(c) It is a well-known fact that the set $\mathbb{C}$ is the smallest field containing the field $\mathbb{R}$ and the roots of $x^2 + 1 = 0$.

Definition 1.1.3 Geometrically representation

Geometrically, we represent the complex number $z = x + iy$ in a two-dimensional coordinate system called the complex plane where real numbers lie on the horizontal axis and pure imaginary numbers on the vertical axis (see Figure 1.1.1).
It is useful to introduce another representation of complex numbers, namely polar coordinates (r, θ). Indeed, the complex number $z \in \mathbb{C}^*$ can be written in the ***polar form*** as :

$$z = x + iy = r(\cos\theta + i\sin\theta). \tag{1.4}$$

where

$$x = r\cos\theta \qquad y = r\sin\theta. \tag{1.5}$$

The number $r > 0$ is denoted by

$$r = \sqrt{x^2 + y^2} = |z|. \tag{1.6}$$

where the value $|z|$ is called the ***modulus*** of z. The angle θ is called the ***argument*** of z and is written $arg(z)$ and which can obtained by solve the following system of equations

$$\begin{cases} \cos(\theta) = \dfrac{x}{r} \\ \\ \sin(\theta) = \dfrac{y}{r} \end{cases} \tag{1.7}$$

Also, when $z \neq 0$, the values of θ can be found from Eq. (1.5) via standard trigonometry:

$$\tan(\theta) = \frac{y}{x}$$

where the quadrant in which x, y lie is understood as given. The symbol $e^{i\theta}$, or $exp(i\theta)$, is defined by means of ***Euler's formula*** as

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.8}$$

where θ is to be measured in radians. It enables us to write the polar form (1) more compactly in which called ***exponential form*** as

$$z = re^{i\theta} \qquad (r > 0). \tag{1.9}$$

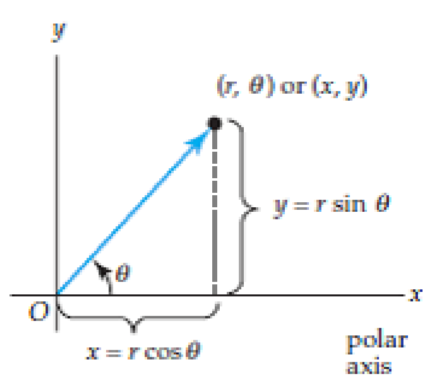

FIG.1.1

Note that the argument of a complex number is only defined up to the addition of integer multiples of 2π. In other words,there infinity of argument for non-zero complex number, so in this situation, we can define the principal value of the argument denoted by Arg to take values in the interval $(-\pi;\pi]$; that is, for any complex number z, one has

$$-\pi < Arg(z) \leq \pi. \tag{1.10}$$

We can see that

$$arg(z) = Arg(z) + 2k\pi \quad (k \in \mathbb{Z}). \tag{1.11}$$

Definition 1.1.4 Properties of modulus

The modulus of complex number satisfying the following properties for all $z_1, z_2 \in \mathbb{C}$:

(1) $|z| = 0$ if and only if $z = 0$;

(2) $|z_1 \times z_2| = |z_1| \times |z_1|$;

(3) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad (z_2 \neq 0)$.

(4) $|z^n| = |z|^n$;

(5) Complex numbers obey a version of the triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|. \tag{1.12}$$

An immediate consequence of the triangle inequality is the fact that

$$|z_1 + z_2| \geq ||z_1| - |z_2||\,. \tag{1.13}$$

(6) The real numbers $|z|$, $Rez = x$, and $Imz = y$ are related by the equation

$$|z|^2 = (Rez)^2 + (Imz)^2. \tag{1.14}$$

Thus

$$Rez \leq |Rez| \leq |z| \quad \text{and} \quad Imz \leq |Imz| \leq |z| \tag{1.15}$$

(7) $|z| = 0 \Leftrightarrow z = 0$.

The following properties for argument of non-zero complex numbers hold. Indeed, For all $z, z_1, z_2 \in \mathbb{C}^*$.

(1) $arg(z_1 \times z_2) = arg(z_1) + arg(z_2)$.

(2) $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2) \quad (z_2 \neq 0)$.

(3) $arg(z^n) = narg(z)$.

Definition 1.1.5 Complex conjugate

The complex conjugate of z is defined as

$$\bar{z} = x - iy = \cos\theta - i\sin\theta,$$

satisfying the following properties for all $z_1, z_2 \in \mathbb{C}$:

(1) $\bar{\bar{z}} = z$

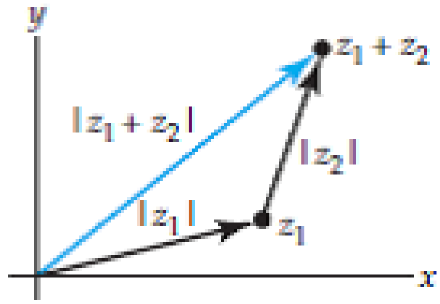

FIG.1.2

(2) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

(3) $\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$.

(4) $\overline{(z^n)} = (\bar{z})^n$.

(5) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$.

(6) $z \times \bar{z} = |z|^2 = (Rez)^2 + (Imz)^2$.

(7) $|\bar{z}| = |z|$.

(8) $arg(\bar{z}) = -arg(z)$.

Definition 1.1.6 De Moivre's formula

If n is an integer and θ is a real number, then we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{1.16}$$

Example 1.1.7

Calculate $z = (1 + i\sqrt{3})^9$

$$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Using Moivre's formula, we obtain :

$$\begin{aligned}(1 + i\sqrt{3})^6 &= 2^9\left(\cos\frac{9\pi}{3} + i\sin\frac{9\pi}{3}\right)\\ &= 2^9(\cos 3\pi + i\sin 3\pi)\\ &= -2^9 = -512.\end{aligned}$$

Definition 1.1.8 nth root of a complex number

A number w is called the nth root of a complex number z if only if

$$z^n = w. \tag{1.17}$$

Indeed, if $w = \rho e^{i\alpha}$ non-zero complex number, then for all $z = re^{i\theta}$, we find that

$$z_k = \rho^{\frac{1}{n}} \left(\cos\left(\frac{\alpha + 2\pi k}{n}\right) + i \sin\left(\frac{\alpha + 2\pi k}{n}\right)\right), k = 0, 2, \ldots n-1. \tag{1.18}$$

Remark 1.1.9.

1. If in the equation (1.17), we take $n = 2$ then z is called and we can find the value of z by two method

   - <u>The first one</u>: By using the equality (1.18), we find that

   $$z_k = \rho^{\frac{1}{2}} \left(\cos\left(\frac{\alpha + 2\pi k}{2}\right) + i \sin\left(\frac{\alpha + 2\pi k}{2}\right)\right), k = 0, 1.$$

   - <u>The second one</u>: By utilizing the algebraic form $z = x + iy$. ineeded, if $w = \alpha + i\beta$ then,

   $$(x + iy)^2 = \alpha + i\beta \Longleftrightarrow x^2 - y^2 = \alpha \,\text{and}\, 2xy = \beta$$

   with the help of the equality $|z|^2 = \alpha^2 + \beta^2$, we get square roots $z = x + iy$ by formula

   $$x = \pm\sqrt{\frac{\alpha^2 + \beta^2 + a}{2}}, \quad y = \pm\sqrt{\frac{\alpha^2 + \beta^2 - a}{2}}$$

2. If in the equation (1.17), we take $n = 1$, then z called unit root of and

$$z_k = \left(\cos\left(\frac{\alpha + 2\pi k}{n}\right) + i \sin\left(\frac{\alpha + 2\pi k}{n}\right)\right), k = 0, 2, \ldots n-1. \tag{1.19}$$

Example 1.1.10

Find the square roots of $w_1 = -5 - 12i$ and the cubic root of $w_2 = -5 - 12i$

1. Let $z = x + iy$, so to find the square roots of w_1, we solve the equation

   $$(x + iy)^2 = -5 - 12i$$

   which implies the system of equations

   $$\begin{cases} x^2 - y^2 = -5 \\ 2xy = -12 \\ x^2 + y^2 = 13. \end{cases} \tag{1.20}$$

   But $xy < 0$, then the square roots of w_1 are $z_1 = 2 - 3i$, $z_2 = -2 + 3i$

2. $$z^3 = -8i$$

So, if $-64i$ in exponential form :

$$-64i = 64e^{i\left(-\frac{\pi}{2}+2k\pi\right)}, \quad k \in \mathbb{Z}.$$

and $z = re^{i\theta}$, then

$$r^3 = 64 \quad \Rightarrow \quad r = 4,$$

then

$$3\theta = -\frac{\pi}{2} + 2k\pi \quad \Rightarrow \quad \theta = -\frac{\pi}{6} + \frac{2k\pi}{3}, \quad k = 0, 1, 2.$$

Therefore, the three roots are :

$$z_k = 4e^{i\left(-\frac{\pi}{6}+\frac{2k\pi}{3}\right)}, \quad k = 0, 1, 2.$$

The cubic roots z_k are:

$$\begin{aligned} z_1 &= 4\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 2\sqrt{3} - 2i, \\ z_2 &= 4\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 4i, \\ z_3 &= 4\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right) = -2\sqrt{3} - 2i. \end{aligned}$$

1.2 Topological properties.

In Section 1.2 we saw that the complex numbers $\mathbb{C}$, which were initially defined algebraically, can be identified with the points in the Euclidean plane $\mathbb{R}^2$. In this section we collect some definitions and results concerning the topology of the plane.

Definition 1.2.1 Euclidean distance

Let $z_1 = x_1 + iy_1$ et $z_2 = x_2 + iy_2$ be a complex numbers. We define the Euclidean distance by:

$$d(z_1, z_2) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = |z_1 - z_2|$$

Definition 1.2.2 Point sets

1. **Neighborhood** : the set defined as follows

$$\mathrm{V}(z_0, \epsilon) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\},$$

for $\epsilon > 0$ is called ϵ Neighborhood of $z_0 \in \mathbb{C}$.

2. **Interior, exterior and boundary points**

- It is said that z_0 is an interior point of S, if there is a neighborhood $\mathrm{V}(z_0, \epsilon) \subset \mathrm{S}$.
- It is said that $z_1$is an exterior point of S, if z_1 is an interior point of $\mathbb{C} \backslash S$.
- It is said that $z_2 \in \partial\ \mathrm{S}$ is a boundary point of S, if all neighborhood of z_2 Intersects with S and $\mathbb{C} \backslash S$.

3. **Open, closed, bounded and compact sets**
   - S is said be open $\Longleftrightarrow \forall z \in S, \exists \text{V}(z,\epsilon)$ such that $:\text{V}(z,\epsilon) \subset S$
   - S is said be closed $\Longleftrightarrow S^c$ is open set.
   - S is said be bounded, if $\exists M > 0$, such that: $\forall z \in S : |z| \leq M$.
   - The subset $B \subset \mathbb{C}$ said compact set, if

$$\forall\,(z_n)_{n\in\mathbb{N}} \subset B,\ \exists\,(z_{n_j})_{j\in\mathbb{N}} \subset (z_n)_{n\in\mathbb{N}} \text{ t.q. } \exists\, z^* = \lim_{j\longrightarrow\infty} z_{n_j} \text{ et } z^* \in B.$$

4. **Arc-connected set** : An open set $S \subset \mathbb{C}$ is said to be arc-connected if any two points can be connected by a path that lies entirely in S.
5. **Connected set**: An open set $S \subset \mathbb{C}$ is said to be connected if it cannot be written as the union of two non-empty disjoint open sets.
6. **Domaine** : A set $D \subset \mathbb{C}$ that is nonempty, open, and connected is called a domain or an open region.

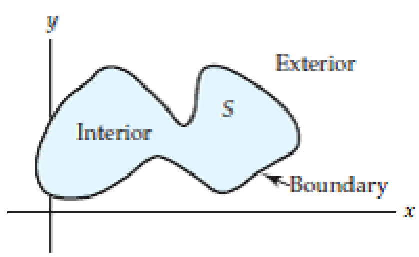

FIG.1.3

Proposition 1.2.3

Any part of $\mathbb{C}$ connected by arcs is connected

Example 1.2.4

The following sets are open

(1) The open disk of center z_0 and radius r,

$$D(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\}$$

(2) The open annulus (or ring-shaped region) centered at z_0 with inner radius r_1 and outer radius r_2

$$D(z_0; r_1, r_2) = \{\, z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2 \,\}.$$

(3) The set A of complex plane

$$A = \{\, z \in \mathbb{C},\ a < Re(z) < b \,\}.$$

where, $a, b \in \mathbb{R}$

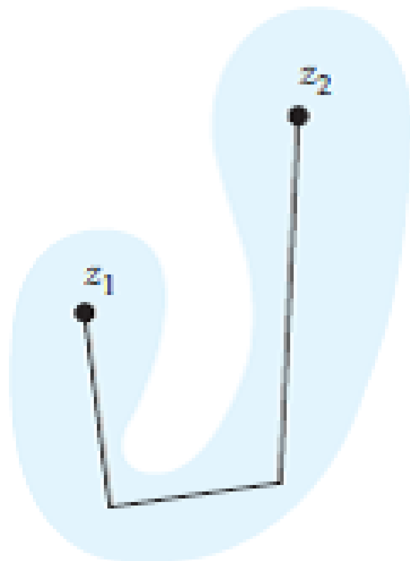

FIG.1.4

Example 1.2.5

The following sets are closed

(a) The closed disk of center z_0 and radius r,

$$D(z_0, r) = \{z \in \mathbb{C}, \, |z - z_0| \leq r\}$$

(b) The closed annulus centered at z_0 with inner radius r_1 and outer radius r_2

$$D(z_0; r_1, r_2) = \{\, z \in \mathbb{C}, \, r_1 \leq |z - z_0| \leq r_2 \,\}.$$

(c) The set B of complex plane

$$A = \{\, z \in \mathbb{C}, a \leq Re(z) \leq b \,\}.$$

where, $a, b \in \mathbb{R}$ and $a < b$.

1.3 Infinity in complex analysis

(1) Introduction: It is often useful to add the **point at infinity** (usually denoted by ∞ or z_∞) to our, so far open, complex plane. As opposed to a finite point where the neighborhood of z_0, say, is defined by $\mathrm{V}(z_0, \epsilon)$, here the neighborhood of z_∞ is defined by those points satisfying $|z| > \frac{1}{\epsilon}$ for all (sufficiently small) $\epsilon > 0$. One convenient way to define the point at infinity is to let $z = 1/t$ and then to say that $t = 0$ corresponds to the point z_∞. An unbounded region $\mathcal{R}$ contains the point z_∞. Similarly, we say a function has values at infinity if it is defined in a neighborhood of z_∞. The complex plane with the point z_∞ included is referred to as the **extended complex plane**.

(2) Stereographic Projection: Consider a unit sphere sitting on top of the complex plane with the south pole of the sphere located at the origin of the z plane (see Figure 1.1). In this subsection we show how the extended complex plane can be mapped onto the surface of a sphere whose south pole corresponds to the origin and whose north pole to the point z_∞. All other points of the complex plane can be mapped in a one-to-one fashion to points on the surface of the sphere by using the following construction. Connect the point z in the plane with the north pole using a straight line. This line intersects the sphere at the point P. In this way each point $(z = x + iy)$ on the complex plane corresponds uniquely to a point P on the surface of the sphere. This construction is called the stereographic projection and is diagrammatically illustrated in Figure 1.1 The extended complex plane.

is sometimes referred to as the ***compactified*** (closed) complex plane. It is often useful to view the complex plane in this way, and knowledge of the construction of the stereographic projection is valuable in certain advanced treatments. So, more concretely, the point $P : (X, Y, Z)$ on the sphere is put into correspondence with the point $z = x + iy$ in the complex plane by finding on the surface of the sphere, (X, Y, Z), the point of intersection of the line from the north pole of the sphere, $N : (0, 0, 2)$, to the point $z = x+iy$ on the plane. The construction is as follows. We consider three points in the three-dimensional setup:
$N = (0, 0, 2)$: north pole
$P = (X, Y, Z)$: point on the sphere
$C = (x, y, 0)$: point in the complex plane.
The stereographic projection maps any locus of points in the complex plane onto a corresponding locus of points on the sphere and vice versa.

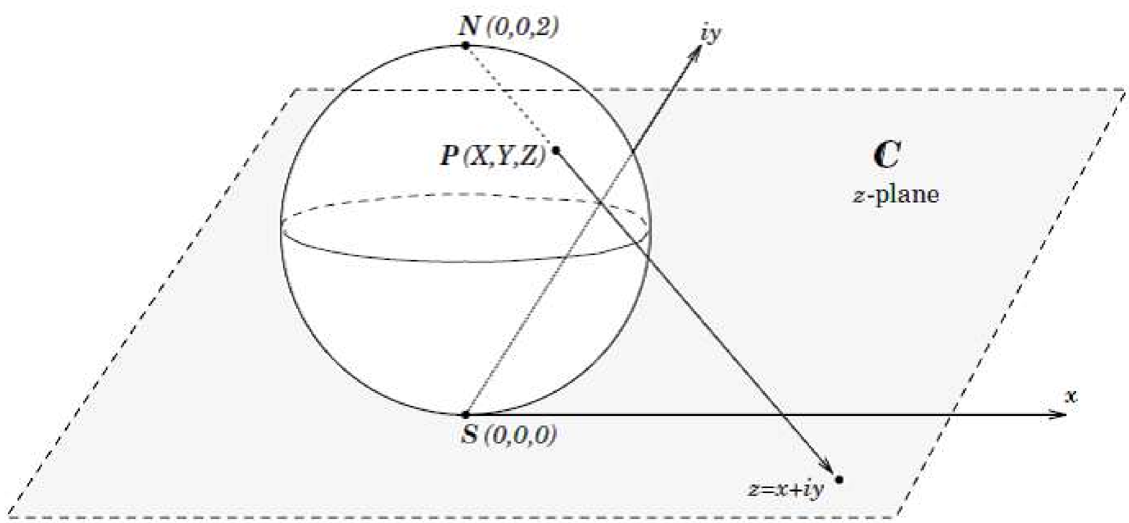

Figure 1.1: (Stereographic projection).

1.4 Exercises set

Exercise 1.4.1

Express in the form $x + iy$, $x, y \in \mathbb{R}$

(a) $\dfrac{i}{1-i} + \dfrac{1-i}{i}$.

(b) all cube roots of $-8i$

(c) $\left(\dfrac{1+i}{\sqrt{2}}\right)^{1337}$.

Solution. (a)

$$\frac{i}{1-i} = \frac{i(1+i)}{(1-i)(1+i)} = \frac{i+i^2}{2} = \frac{i-1}{2}, \qquad \frac{1-i}{i} = \frac{1}{i} - 1 = -i - 1.$$

Hence

$$\frac{i}{1-i} + \frac{1-i}{i} = \frac{i-1}{2} - i - 1 = -\frac{3}{2} - \frac{1}{2}i.$$

(b) Write $-8i = 8e^{-i\pi/2}$. The cube roots have modulus 2 and arguments

$$\theta_k = \frac{-\pi/2 + 2\pi k}{3} = -\frac{\pi}{6} + \frac{2\pi k}{3}, \quad k = 0, 1, 2.$$

Thus

$$z_k = 2(\cos\theta_k + i\sin\theta_k), \qquad \boxed{\sqrt{3} - i,\ 2i,\ -\sqrt{3} - i}.$$

(c)

$$\begin{aligned}\left(\frac{i+1}{\sqrt{2}}\right)^{1337} &= \left(\exp\left(i\frac{\pi}{4}\right)\right)^{1337} \\ &= \exp\left(i\frac{1337\pi}{4}\right) \\ &= \exp\left(i\left(167 \cdot 2\pi + \frac{\pi}{4}\right)\right) \\ &= \exp\left(i\frac{\pi}{4}\right) \\ &= \frac{1+i}{\sqrt{2}}.\end{aligned}$$

Exercise 1.4.2

(a) Use exponential (Euler) form to compute

(i) $(1 + \sqrt{3}\,i)^{2011}$;

(ii) $(1 + \sqrt{3}\,i)^{-2011}$.

(b) Prove that

$$\sum_{m=0}^{1005} \binom{2011}{2m} (-3)^m = 2^{2010} \qquad \text{and} \qquad \sum_{m=0}^{1005} \binom{2011}{2m+1} (-3)^m = 2^{2010}.$$

Solution.

$$\text{Since } 1+\sqrt{3}i = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2\exp\left(i\frac{\pi}{3}\right),$$

$$\begin{aligned}(1+\sqrt{3}i)^{2011} &= 2^{2011}\exp\left(i\frac{2011\pi}{3}\right)\\ &= 2^{2011}\exp\left(i\left(670\pi+\frac{\pi}{3}\right)\right)\\ &= 2^{2011}\exp\left(i\frac{\pi}{3}\right)\\ &= 2^{2011}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\\ &= 2^{2010}(1+\sqrt{3}i).\end{aligned}$$

Similarly

$$(1+\sqrt{3}i)^{-2011} = 2^{-2013}(1-\sqrt{3}i).$$

we have

$$2^{2010}(1+\sqrt{3}i) = (1+\sqrt{3}i)^{2011} = \sum_{m=0}^{1005}\binom{2011}{2m}(-3)^m + i\sum_{m=0}^{1005}\binom{2011}{2m+1}(-3)^m\sqrt{3}.$$

It follows that

$$\sum_{m=0}^{1005}\binom{2011}{2m}(-3)^m = \sum_{m=0}^{1005}\binom{2011}{2m+1}(-3)^m = 2^{2010}.$$

Exercise 1.4.3

(i) Establish the identity

$$1+z+z^2+\cdots+z^n = \frac{1-z^{n+1}}{1-z}, \quad (z\neq 1).$$

(ii) By using the above result, derive Lagrange's trigonometric identity:

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta = \frac{1}{2}+\frac{\sin\left((2n+1)\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}, \quad 0<\theta<2\pi.$$

Solution. Let

$$S = 1+z+z^2+\cdots+z^n.$$

Then

$$S - zS = (1+z+z^2+\cdots+z^n)-(z+z^2+\cdots+z^{n+1}) = 1-z^{n+1}.$$

Thus

$$S = \frac{1-z^{n+1}}{1-z}, \quad z\neq 1.$$

That is,

$$1+z+z^2+\cdots+z^n = \frac{1-z^{n+1}}{1-z}, \quad z\neq 1.$$

Putting $z=e^{i\theta}$ with $0<\theta<2\pi$ in this identity, we obtain

$$1+e^{i\theta}+e^{2i\theta}+\cdots+e^{in\theta} = \frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}.$$

Now the real part of the left-hand side is evidently

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta.$$

In order to find the real part of the right-hand side, we write it in the form

$$\frac{1-\exp\big[i(n+1)\theta\big]}{1-\exp(i\theta)} \cdot \exp\left(-\frac{i\theta}{2}\right) \cdot \exp\left(-\frac{i\theta}{2}\right) = \exp\left(-\frac{i\theta}{2}\right) \frac{\exp\left(-\frac{i\theta}{2}\right) - \exp\left[i\frac{(2n+1)\theta}{2}\right]}{\exp\left(-\frac{i\theta}{2}\right) - \exp\left(\frac{i\theta}{2}\right)}.$$

This becomes

$$\frac{\cos\left(-\frac{\theta}{2}\right) - i\sin\left(-\frac{\theta}{2}\right) - \cos\left(\frac{(2n+1)\theta}{2}\right) - i\sin\left(\frac{(2n+1)\theta}{2}\right)}{-2i\sin\left(-\frac{\theta}{2}\right)}.$$

Or, equivalently,

$$\frac{\sin\left(-\frac{\theta}{2}\right) + \sin\left(\frac{(2n+1)\theta}{2}\right) + i\left[\cos\left(-\frac{\theta}{2}\right) - \cos\left(\frac{(2n+1)\theta}{2}\right)\right]}{2\sin\left(-\frac{\theta}{2}\right)}.$$

The real part of this is clearly

$$\frac{1}{2} + \frac{\sin\left(\frac{(2n+1)\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}.$$

Thus we arrive at Lagrange's trigonometric identity:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left(\frac{(2n+1)\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}, \quad 0 < \theta < 2\pi.$$

Exercise 1.4.4

Prove that

$$\text{If } (|z| = 1 \text{ and } z \neq 1), \quad i\left(\frac{z+1}{z-1}\right) \in \mathbb{R}.$$

Solution. We shall prove that: $\overline{w} = w$ Since $|z| = 1$ we have $\overline{z} = 1/z$.
Consider

$$w := i\frac{z+1}{z-1}.$$

Compute the complex conjugate, yield

$$\overline{w} = -i\frac{\overline{z}+1}{\overline{z}-1} = -i\frac{\frac{1}{z}+1}{\frac{1}{z}-1} = -i\frac{1+z}{1-z} = i\frac{z+1}{z-1} = w.$$

Thus $\overline{w} = w$, so w is real.

Exercise 1.4.5

Let z be complex number, verify $|z| = 1$. Find that

$$Arg\left(\frac{z-1}{z+1}\right) = \begin{cases} \frac{\pi}{2}, & \text{si } Im(z) > 0, \\ -\frac{\pi}{2}, & \text{si } Im(z) < 0. \end{cases}$$

Solution. We pose $z = x + iy$ pour $x, y \in \mathbb{R}$. we get

$$\frac{z-1}{z+1} = \frac{x-1+iy}{x+1+iy} = \frac{(x-1+iy)(x+1-iy)}{(x+1+iy)(x+1-iy)}.$$

After calculation, we find

$$\frac{z-1}{z+1} = \frac{x^2+y^2-1}{(x+1)^2+y^2} + i\,\frac{2y}{(x+1)^2+y^2}.$$

Using the fact that $|z| = 1$, on a $x^2 + y^2 = 1$. Therefore,

$$\frac{z-1}{z+1} = i\,\frac{2y}{(x+1)^2+y^2}.$$

So this complex is purely imaginary. It is therefore clear that

$$Arg\left(\frac{z-1}{z+1}\right) = \begin{cases} \frac{\pi}{2}, & \text{if } Im(z) = y > 0, \\ -\frac{\pi}{2}, & \text{if } Im(z) = y < 0. \end{cases}$$

Exercise 1.4.6

Show that, for all $z \in \mathbb{C}^*$,

$$\overline{\left(\frac{1}{z}\right)} = \frac{1}{\overline{z}}.$$

Solution. For all $z \in \mathbb{C}^*$,we have

$$\overline{\left(\frac{1}{z}\right)} = \overline{\left(\frac{\overline{z}}{z\overline{z}}\right)}.$$

Since $z\overline{z} = |z|^2$, then

$$\overline{\left(\frac{1}{z}\right)} = \overline{\left(\frac{\overline{z}}{|z|^2}\right)} = \frac{1}{|z|^2}\,z = \frac{1}{\overline{z}}.$$

Therefore

$$\overline{\left(\frac{1}{z}\right)} = \frac{1}{\overline{z}}.$$

Exercise 1.4.7

Find the set of points z of the complex plane verifying:

(1) $|z-1| = |z+i|$,

(2) $2|z| = |z-2|$,

(3) $\{z \in \mathbb{C} : \arg(z-2i) = \frac{\pi}{4}\}$.

(4) $|z-2i| + |z+4i| = 10$

(5) $\left\{z \in \mathbb{C} : |z| < 1 - \frac{1}{2i}(z-\overline{z})\right\}$.

(6) $\left\{z \in \mathbb{C} : \left|\frac{z}{z+3i}\right| < 1\right\}$.

Solution. (1) Let $z = x + iy$ with $x \in \mathbb{R}$ and $y \in \mathbb{R}$. We have :

$$\begin{aligned}|z-1| = |z+i| &\iff |(x-1)+yi| = |x+(y+1)i| \\ &\iff (x-1)^2 + y^2 = x^2 + (y+1)^2. \\ &\iff x^2 - 2x + 1 + y^2 = x^2 + y^2 + 2y + 1 \\ &\iff -2x = 2y \\ &\iff x + y = 0.\end{aligned}$$

Thus, the set of points is the line with equation $y = -x$.

(2) Let $z = x + iy$ with $x \in \mathbb{R}$ and $y \in \mathbb{R}$. We have

$$\begin{aligned}2|z| = |z-2| &\iff 2\sqrt{x^2+y^2} = \sqrt{(x-2)^2+y^2}. \\ &\iff 4(x^2+y^2) = (x-2)^2 + y^2. \\ &\iff 4x^2 + 4y^2 = x^2 - 4x + 4 + y^2 \\ &\iff 3x^2 + 3y^2 + 4x - 4 = 0. \\ &\iff 3\left[\left(x + \frac{2}{3}\right)^2 - \frac{4}{9}\right] + 3y^2 = 4..\end{aligned}$$

Therefore, the point set is the circle with center $\left(-\frac{2}{3}, 0\right)$ and radius $\frac{4}{3}$.

(3) Using exponential form and for all $z \neq 2i$ we get

$$z = 2i + t\, e^{i\pi/4},\ t > 0$$

. then, the set of points is Half-line from $2i$with angle $\pi/4$: equation $y - 2 = x$.

(4) $\{z \in \mathbb{C} : |z - 2i| + |z + 4i| = 10\}$. Let $z = x + iy$, then

$$|z-2i| + |z+4i| = 10 \iff \frac{x^2}{4^2} + \frac{(y+1)^2}{5^2} = 1$$

Thus, the set of the points is Ellipse of foci:$2i$, $-4i$, center $(0, -1)$,$(a = 5,\ c = 3,\ b = 4)$

(5) Let $z = x + iy$, we have

$$\begin{aligned}|z| < 1 - \frac{1}{2i}(z - \bar{z}) &\iff |z| < 1 - \Im z \\ &\iff x^2 + y^2 < (1-y)^2,\ (y < 1) \\ &\iff y < \frac{1-x^2}{2}.\end{aligned}$$

The required set is all the points of the plane lies under the parabola with equation $y = \frac{1-x^2}{2}$.

(6) Let $z = x + iy$, then, we get

$$\begin{aligned}\left|\frac{z}{z+3i}\right| < 1 &\iff |z| < |z+3i|\ (z \neq -3i) \\ &\iff y > -\frac{3}{2}.\end{aligned}$$

Then the set of points is Half-plane above the line $y = -\frac{3}{2}$with $z \neq -3i$.

Exercise 1.4.8

1. Find all $z \in \mathbb{C}$ such that
$$|z - 3i| = 2.$$

2. Solve for $z \in \mathbb{C}$:
$$z^2 + (2 - i)z + 1 - 3i = 0.$$

3. Determine the set of points $z \in \mathbb{C}$ satisfying
$$\operatorname{Re}(z) > \operatorname{Im}(z).$$

4. Let $z = re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leq \pi$. Compute z^4 in Cartesian form.

5. Sketch in the complex plane:
$$\{z \in \mathbb{C} : |z - 1| + |z + 1| = 4\}.$$

6. Show that for any $z \in \mathbb{C}$:
$$|z + 1|^2 - |z - 1|^2 = 4 \operatorname{Re}(z).$$

7. Find all cube roots of $8(\cos \pi + i \sin \pi)$.

8. Determine the set:
$$\{z \in \mathbb{C} : \arg(z - i) = \tfrac{\pi}{3}\}.$$

9. For $z = x + iy$, show that:
$$|z|^2 = z\overline{z}.$$

10. Let $u = 1 + i$ and $v = 2 - 3i$. Compute:
$$\frac{u}{v}, \quad u \cdot \overline{v}, \quad |u|, \quad |v|, \quad u^{1446}, \quad v^{2026}.$$

Chapter

2

Function of complex variable

Chapter contents

In real analysis, differentiation is one of the central ideas: it allows us to study how functions change and to approximate them locally by linear functions. A natural question arises when we move to the setting of complex numbers: can we define and study derivatives of functions of a complex variable in a similar way ?

In this chapter, we take up this question. Our aim is to develop a systematic theory of complex differentiation, which not only parallels but also goes far beyond the real case. Indeed, as we shall see, requiring differentiability in the complex sense imposes very strong conditions on a function, leading to remarkable consequences such as the Cauchy–Riemann equations, analyticity, and the rich theory of holomorphic functions.

2.1 Definition of the function of the complex variable

Definition 2.1.1 Function of the complex variable

A complex-valued function

$$f : \mathbb{C} \longrightarrow \mathbb{C},$$

$$z \longmapsto w = f(z).$$

is a rule that associated to each complex number $z \in \mathbb{C}$ a complex number w. called the value (image) of f at the point z, denoted by $w = f(z)$.

(i) The set $S_1 = \{z \in \mathbb{C},\ f(z)\text{exist}\}$ is called **domain of definition** of the function of f, denoted by D_f.

(ii) $S_1 = \{f(z) \in \mathbb{C},\ z \in D_f\}$ is called **range** of $f(z)$.

(iii) If the function f, for any z associate **unique** value of w, then f is called **single-valued function**.

(iv) If the function f, for any z associate more one value of w, then this function is called **multivalues function** .

Example 2.1.2

Some examples of complex functions and their domains.

(1) $f_1(z) = 3z^3 + 3z - 32, \qquad Df_1 = \mathbb{C}.$

(2) $f_2(z) = |z|^2, \qquad Df_2 = \mathbb{C}.$

(3) $f_3(z) = \frac{iz^3}{(z-i)(z+2)}, \qquad Df_3 = \mathbb{C} - \{i, -2\}$

(4) $f_4(z) = arg(z), \qquad Df_4 = \mathbb{C}^*.$

Definition 2.1.3 Real Part and Imaginary Part of a Complex Function

Let $w = u + iv$ is the value of f at point z

1. If $z = x + iy$, then
$$w = f(z) = f(x+iy) = u(x,y) + iv(x,y).$$
Thus
$$Re[f(z)] = u(x,y) \quad \text{and} \quad Im[f(z)] = v(x,y)$$
2. If $z = re^{i\theta}$, then
$$Re[f(z)] = u(r\cos\theta, r\sin\theta) \quad \text{and} \quad Im[f(z)] = v(r\cos\theta, r\sin\theta)$$

Example 2.1.4

If $f(z) = z|z|^2$, then
$$f(x+iy) = (x+iy)|(x^2+y^2) = (x^3+xy^2) + i(x^2y+y^3).$$
Thus,
$$u(x,y) = x^3 + xy^2 = Re(f) \quad \text{and} \quad x^2y + y^3 = Im(f).$$
If we use the polar form of z, we obtain
$$f(re^{i\theta}) = (re^{i\theta})r^2 = r^3e^{i\theta} = r^3\cos(\theta) + ir^3\sin(\theta).$$
Thus,
$$u(r,\theta) = r^3\cos(\theta) = Re(f) \quad \text{and} \quad v(r,\theta) = r^3\sin(\theta) = Im(f).$$

Remark 2.1.5. A complex function f is called also transformation From $z-$ plane to $w-$ plane.

Example 2.1.6

The function $w = z^2$ maps the upper half z-plane, including the real axis ($\operatorname{Im} z \geq 0$), to the entire w-plane (see Figure 2.1). This is particularly clear when we use the polar representation $z = re^{i\theta}$. In the z-plane, θ lies inside $0 \leq \theta < \pi$, whereas in the w-plane,

$$w = r^2 e^{2i\theta} = Re^{i\varphi}, \quad R = r^2, \quad \varphi = 2\theta,$$

and φ lies in $0 \leq \varphi < 2\pi$.

Example 2.1.7

The function $w = \overline{z}$ maps the upper half z-plane ($\operatorname{Im} z > 0$) into the lower half w-plane (see Figure 2.2). Namely, if $z = x + iy$ with $y > 0$, then

$$w = \overline{z} = x - iy.$$

Thus, if $w = u + iv$, then

$$u = x, \qquad v = -y.$$

So, the study and understanding of complex mappings is very important, and we will see that there are many applications

2.2 Limits, Continuity and Complex Differentiation.

Definition 2.2.1 Limit of a Complex Function

Consider a function $w = f(z)$ defined at all points in some neighborhood of $z = z_0$, except possibly for z_0 itself. We say $f(z)$ has the limit ℓ if as z approaches z_0, $f(z)$, approaches ℓ. Mathematically, we say

$$\lim_{z \longrightarrow z_0} f(z) = \ell \Longleftrightarrow (\forall \epsilon > 0)(\exists \delta > 0) : 0 < |z - z_0| < \delta \quad \Longrightarrow \quad |f(z) - \ell| < \varepsilon.$$

Example 2.2.2

Show that

$$\lim_{z \longrightarrow i} 2\left(\frac{z^2 + iz + 2}{z - i}\right) = 6i.$$

Given $\varepsilon > 0$, we must find $\delta > 0$ such that

$$\left| 2\left(\frac{z^2 + iz + 2}{z - i}\right) - 6i \right| < \varepsilon \tag{2.1}$$

whenever,

$$0 < |z - i| < \delta.$$

Factor the numerator:

$$z^2 + iz + 2 = (z - i)(z + 2i),$$

so for $z \neq i$,

$$2\left(\frac{z^2 + iz + 2}{z - i}\right) = 2(z + 2i).$$

Hence

$$\left|2\left(\frac{z^2 + iz + 2}{z - i}\right) - 6i\right| = \left|2(z + 2i) - 6i\right| = \left|2z - 2i\right| = 2|z - i| < \delta.$$

Therefore, if we choose $\delta = \frac{\varepsilon}{2}$, then (2.1) will be satisfied.

Definition 2.2.4 Continuity at z_0

A function f is said to be continuous at $z_0 \in D$ if

$$\lim_{z \longrightarrow z_0} f(z) = f(z_0).$$

Definition 2.2.5 Continuity on S

A function f is said to be continuous on S if, it is continuous at all points $z \in S$.

Theorem 2.2.6 Properties of Continuous Functions

Suppose that $f, g : D \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ are continuous functions and $c \in \mathbb{C}$. Then:

(a) The functions

$$Ref(z), \quad Imf(z), \quad cf(z), \quad |f(z)|, \quad f(z) + g(z), \quad f(z) \cdot g(z)$$

are all continuous.

(b) The quotient

$$\frac{f(z)}{g(z)},$$

is defined and continuous at every point $a \in D$ such that $g(a) \neq 0$.

Furthermore, the composition of continuous functions is also continuous. More explicitly: if $f : D \longrightarrow E$ and $g : E \longrightarrow \mathbb{C}$ are continuous at $a \in D$ and at $f(a) \in E$ respectively, then the function

$$(g \circ f)(z) = g(f(z)), \qquad z \in D,$$

is continuous at a.

2.3 Analytic functions.

Definition 2.3.1 Complex Differentiation

Let $D \subset \mathbb{C}$ be an open set and let $f : D \longrightarrow \mathbb{C}$ be a complex function. We say that f is $\mathbb{C}$–**differentiable** (or simply *differentiable*) at a point $z_0 \in D$ if the following limit exists:

$$f'(z_0) = \lim_{z \longrightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

If this limit exists, it is called the **derivative** of f at z_0. Moreover, if we set

$$h = z - z_0, \quad z = z_0 + h,$$

then f is $\mathbb{C}$–differentiable at z_0 if and only if

$$f'(z_0) = \lim_{h \longrightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

Definition 2.3.2 Complex Differentiation on S

A function f is said Differentiable on S if, it is Differentiable at all points $z \in S$

Theorem 2.3.3

If f is differentiable at z_0 then, it is continuous at z_0.

Proof. The result of theorem follows from

$$\lim_{z \longrightarrow z_0} \big(f(z) - f(z_0)\big) = \lim_{z \longrightarrow z_0} \left(\frac{f(z) - f(z_0)}{z - z_0}\right)(z - z_0).$$

Since f is differentiable at z_0, then

$$\lim_{z \longrightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0).$$

Moreover,

$$\lim_{z \longrightarrow z_0} (z - z_0) = 0.$$

Therefore,

$$\lim_{z \longrightarrow z_0} \big(f(z) - f(z_0)\big) = f'(z_0) \cdot 0 = 0.$$

□

Proposition 2.3.4 Derivative rules

Let $f, g : D \longrightarrow \mathbb{C}$ be differentiable functions on an open set $D \subset \mathbb{C}$, and let $c \in \mathbb{C}$ be a constant. Then:

1. $(c)' = 0$,
2. $(cf)' = cf'$,
3. $(f+g)' = f' + g'$,
4. $(fg)' = f'g + fg'$,
5. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \quad \text{when } g(z) \neq 0.$
6. $$(f \circ g)'(z) = f'(g(z)) \cdot g'(z).$$

2.4 Cauchy-Riemann Conditions

In the following theorem, we will give a necessary condition for the differentiability of a function $f(z)$ at a point z_0.

Theorem 2.4.1

Let $f : D \longrightarrow \mathbb{C}$ and write $f(x + iy) = u(x, y) + iv(x, y)$ where $u, v : D \longrightarrow \mathbb{R}$. Suppose that f is differentiable at $z_0 = x_0 + iy_0$. Then:

1. The partial derivatives
$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}$$
exist at (x_0, y_0),
2. The following relations hold:
$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0),$$
$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$
Which called **Cauchy-Riemann conditions** (Or Cauchy-Riemann Equations).
3. $f'(z_0)$ is presented by the formula:
$$f(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

Proof. By assumption, f is differentiable at z_0, hence $f'(z_0)$ exists. To establish the Cauchy–Riemann conditions, we proceed as follows:

$$f'(z_0) = \lim_{z \longrightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x,y) \longrightarrow (x_0,y_0)} \frac{u(x, y) - u(x_0, y_0)}{(x - x_0) + i(y - y_0)} + i\frac{v(x, y) - v(x_0, y_0)}{(x - x_0) + i(y - y_0)}.$$

- Case 1: Fix $y = y_0$, then as $x \longrightarrow x_0$,
$$f'(z_0) = \lim_{x \longrightarrow x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i\frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}.$$
Thus
$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$
- Case 2: Fix $x = x_0$, Then as $y \longrightarrow y_0$,
$$f'(z_0) = \lim_{y \longrightarrow y_0} \frac{u(x_0, y) - u(x_0, y_0)}{i(y - y_0)} + i\frac{v(x_0, y) - v(x_0, y_0)}{i(y - y_0)}.$$
Hence
$$f'(z_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0).$$

Since the derivative $f'(z_0)$ must be the same along both directions, we equate the two expressions and obtain the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \qquad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

□

Example 2.4.2

Show that the Cauchy–Riemann equations are satisfied for

$$f(z) = z^2 + z + 3$$

at every point of the complex plane $\mathbb{C}$.
Write $z = x + iy$. Then

$$f(z) = z^2 + z + 3 = (x^2 - y^2 + x + 3) + i(2xy + y).$$

Hence

$$u(x,y) = x^2 - y^2 + x + 3, \qquad v(x,y) = 2xy + y.$$

Compute the partial derivatives:

$$\frac{\partial u}{\partial x} = 2x + 1, \qquad \frac{\partial u}{\partial y} = -2y,$$

$$\frac{\partial v}{\partial x} = 2y, \qquad \frac{\partial v}{\partial y} = 2x + 1.$$

The Cauchy–Riemann equations require

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Here we have

$$\frac{\partial u}{\partial x} = 2x + 1 = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x},$$

so the Cauchy–Riemann equations hold at every point $(x,y) \in \mathbb{R}^2$.

Example 2.4.3

We can use the Cauchy–Riemann equations to examine whether the function

$$f(z) = \overline{z}$$

is differentiable on $\mathbb{C}$.
Writing $z = x + iy$, we have

$$f(z) = \overline{z} = x - iy.$$

Thus

$$f(z) = u(x,y) + iv(x,y), \quad \text{where } u(x,y) = x,\ v(x,y) = -y.$$

Now, compute the partial derivatives:

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} = -1,$$

so the first equation is never satisfied.
Therefore, there are no points in $\mathbb{C}$ where $f(z) = \overline{z}$ is differentiable.

Remark 2.4.4. The Cauchy-Riemann conditions are necessary but not sufficient conditions.

Example 2.4.5

Let $f(z) = |z|^2 = x^2 + y^2$ and $f(z) = u(x, y) + iv(x, y)$ with $(u(x, y) = x^2 + y^2$, $(v(x, y) = 0$. Compute partials:

$$u_x = 2x, \quad u_y = 2y, \quad v_x = 0, \quad v_y = 0.$$

At the origin $(0, 0)$ we have

$$u_x(0, 0) = 0 = v_y(0, 0), \qquad u_y(0, 0) = 0 = -v_x(0, 0),$$

so the Cauchy–Riemann equations hold at $z = 0$. Compute the derivative at $z = 0$:

$$f'(0) = \lim_{z \longrightarrow 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \longrightarrow 0} \frac{|z|^2}{z} = \lim_{z \longrightarrow 0} \overline{z} = 0.$$

(1). If we consider the direction $x \longrightarrow 0$ and $y = 0$, then the limit equal 1.
(2). If we consider the direction $x \longrightarrow 0$ and $y = 0$, then the limit equal -1.

Proposition 2.4.6 Converse to the Cauchy–Riemann Theorem

Let $f : D \longrightarrow \mathbb{C}$ be a continuous function, and write

$$f(x + iy) = u(x, y) + iv(x, y).$$

Let $z_0 = x_0 + iy_0 \in D$. Suppose that the partial derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}$$

exist and are continuous at (x_0, y_0), and further suppose that the **Cauchy–Riemann equations**

$$u_x(x_0, y_0) = v_y(x_0, y_0), \qquad u_y(x_0, y_0) = -v_x(x_0, y_0)$$

hold at (x_0, y_0).
Then f is differentiable at z_0.

The following theorem is a direct consequence of the Cauchy-Riemann equations

Theorem 2.4.7 Constant Functions

Suppose the function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D.

(i) If $|f(z)|$ is constant in D, then $f(z)$ itself is constant in D.

(ii) If $f'(z) = 0$ in D, then $f(z) = c$ in D, where c is a constant.

Proposition 2.4.8 Cauchy–Riemann equations in polar coordinates

Let f be analytic at z_0. In polar coordinates $z = re^{i\theta}$, the Cauchy–Riemann equations take the form:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Corollary 2.4.9

If $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$, the polar form of $f'(z)$ becomes

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r}(r,\theta) + i\,\frac{\partial v}{\partial r}(r,\theta)\right).$$

Definition 2.4.10 Analyticity of a Function

A complex-valued function $f : D \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ is said to be analytic at z if f is differentiable in a neighborhood of z. Similarly, f is **analytic** on $S \subseteq \mathbb{C}$ if it is differentiable at all points of some open set containing S. Moreover if $S = D$ then f is called An ***entire*** function.

Remark 2.4.11.

- (a) It should be noted that analyticity is a property that a function is defined on an open set (a neighborhood of a particular point), while differentiability is a property of a function that occurs at a particular point only.
- (b) A point z at which a complex function f fails to be analytic is called a singular point. For example, $z = i$ is a singular point of the rational function $f(z) = \frac{z^2}{z-i}$.

Theorem 2.4.12

If $f = u(x,y) + iv(x,y)$ is analytic at z, Iff

- **(1)** The functions $u, v : \mathbb{R}^2 \longrightarrow \mathbb{R}$ have continuous first partial derivatives with respect to x and y.
- **(2)** The functions u and v satisfy the Cauchy-Riemann Equations, i.e.,

$$\begin{cases} \frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \\ \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) \end{cases}$$

Remark 2.4.13. Let $U \subset \mathbb{R}^2$ $\big(u, v \in C^1(U)\big)$ *if and only if* (the first partial derivatives of u, v with respect to x and y exist and continuous)

Definition 2.4.14 The operator $\bar{\partial}$

Let $U \subset \mathbb{R}^2$ an open set , $f = u+iv$ and the functions (u, v) be C^1. We define the differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

For $u + iv$ we set

$$\frac{\partial}{\partial z}(u+iv) = \frac{\partial u}{\partial z} + i\frac{\partial v}{\partial z}, \qquad \frac{\partial}{\partial \bar{z}}(u+iv) = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}}.$$

Remark 2.4.15. If $f = u + iv$ is Analytic on U (so that u, v satisfy the Cauchy–Riemann equations), then

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\Big) + \frac{i}{2}\Big(\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\Big).$$

Using $u_x = v_y$ and $u_y = -v_x$ (Cauchy–Riemann equations), we get

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\big(u_x - v_y\big) + \frac{i}{2}\big(u_y + v_x\big) = 0 + i0 = 0.$$

Similarly,

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial z} + i\frac{\partial v}{\partial z} = \frac{1}{2}\Big(u_x - iu_y\Big) + \frac{i}{2}\Big(v_x - iv_y\Big).$$

Rearranging and again using the Cauchy–Riemann equations, we obtain

$$\frac{\partial f}{\partial z} = \frac{1}{2}\big(u_x + v_y\big) + \frac{i}{2}\big(-u_y + v_x\big) = u_x + iv_x = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

But this is exactly the derivative $f'(z)$. Hence

$$\frac{\partial f}{\partial z} = f'(z).$$

Theorem 2.4.16

Let $U \subset \mathbb{C}$ be an open set. If $f : U \longrightarrow \mathbb{C}$ is Analytic, then

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad \text{and} \quad \frac{\partial f}{\partial z} = f'(z).$$

Remark 2.4.17. Analytic functions can be thought of as functions which are independent of $\bar{z}$ (in the sense that their $\bar{\partial}$-derivative vanishes).

Definition 2.4.18 functions of class C^2

Let $D \subset \mathbb{R}^2$ and $\varphi : D \longrightarrow \mathbb{R}$. The function φ is said to be of class C^2 on D (we write $\varphi \in C^2(D, \mathbb{R}$ if the partial derivatives

$$\frac{\partial \varphi}{\partial x}, \quad \frac{\partial \varphi}{\partial y}, \quad \frac{\partial^2 \varphi}{\partial x \partial y}, \quad \frac{\partial^2 \varphi}{\partial x^2}, \quad \frac{\partial^2 \varphi}{\partial y^2}$$

exist and are continuous for all $(x, y) \in D$.

Definition 2.4.19 Harmonic functions

Let $D \subset \mathbb{R}^2$ be an open set and let $\varphi \in C^2(D, \mathbb{R})$. We say that φ is **harmonic** on D if it satisfies *Laplace's equation*:

$$\Delta f(x, y) \;=\; \frac{\partial^2 \varphi}{\partial x^2}(x, y) \;+\; \frac{\partial^2 \varphi}{\partial y^2}(x, y) \;=\; 0, \qquad \forall (x, y) \in D.$$

Example 2.4.20

Consider the function

$$f : \mathbb{C} \longrightarrow \mathbb{C}, \qquad f(z) = z^3,$$

where

$$u(x, y) = x^3 - 3xy^2, \qquad v(x, y) = 3x^2 y - y^3.$$

Clearly, $u, v \in C^1(\mathbb{R}^2, \mathbb{R})$. We compute the partial derivatives:

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \qquad \frac{\partial^2 u}{\partial x^2} = 6x,$$

$$\frac{\partial u}{\partial y} = -6xy, \qquad \frac{\partial^2 u}{\partial y^2} = -6x,$$

and

$$\frac{\partial v}{\partial x} = 6xy, \qquad \frac{\partial^2 v}{\partial x^2} = 6y,$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2, \qquad \frac{\partial^2 v}{\partial y^2} = -6y.$$

Hence, the Laplacians are

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0,$$

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6y - 6y = 0.$$

Thus both functions u and v are harmonic on $\mathbb{R}^2$.

Definition 2.4.21

Let u be a harmonic function on $E \subset \mathbb{R}^2$. A function v is called a **harmonic conjugate** of u if the pair (u, v) satisfies the Cauchy-Riemann equations.

Proposition 2.4.22

et u be a harmonic function on $E \subset \mathbb{R}^2$. Then there exists an analytic function $f : E \subset \mathbb{C} \longrightarrow \mathbb{C}$ such that

$$Ref = u.$$

Remark 2.4.23. Every harmonic function $u(.,.)$ is the real part of some analytic function f, uniquely determined up to an additive purely imaginary constant.

2.5 Exercises set

Exercise 2.5.1

Suppose that

$$f(z) = x^2 - y^2 - 2y + i(2x - 2xy), \qquad z = x + iy.$$

Find f in the terms of z.

Solution.

$$x = \frac{z + \bar{z}}{2}, \qquad y = \frac{z - \bar{z}}{2i},$$

so we have

$$f(z) = \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 - 2\left(\frac{z-\bar{z}}{2i}\right) + i\left[2\left(\frac{z+\bar{z}}{2}\right) - 2\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)\right].$$

$$f(z) = \frac{z^2 + 2z\bar{z} + \bar{z}^2}{4} + \frac{z^2 - 2z\bar{z} + \bar{z}^2}{4} + i(z - \bar{z}) + i\left(z + \bar{z} - \frac{z^2 - \bar{z}^2}{2i}\right).$$

$$f(z) = \frac{z^2 + \bar{z}^2}{2} + 2iz - \frac{z^2 - \bar{z}^2}{2} = \bar{z}^2 + 2iz.$$

$$\boxed{f(z) = \bar{z}^2 + 2iz}$$

Exercise 2.5.2

Let $f(z) = z^5 + 4z^2 - 6$, let Express f in the polar coordinate form $u(r, \theta) + iv(r, \theta)$.

Solution. Let $z = r(\cos\theta + i\sin\theta)$. By (De Moivre's Theorem), we have

$$\begin{aligned} f(z) &= r^5(\cos 5\theta + i\sin 5\theta) + 4r^2(\cos 2\theta + i\sin 2\theta) - 6 \\ &= \left(r^5\cos 5\theta + 4r^2\cos 2\theta - 6\right) + i\left(r^5\sin 5\theta + 4r^2\sin 2\theta\right). \end{aligned}$$

Therefore, we see that

$$u(r,\theta) = r^5\cos 5\theta + 4r^2\cos 2\theta - 6, \quad v(r,\theta) = r^5\sin 5\theta + 4r^2\sin 2\theta.$$

Exercise 2.5.3

Let Let

$$f(z) = \begin{cases} \dfrac{z^2}{|z|}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Prove that f is continuous on $\mathbb{C}$ but not analytic at any point on $\mathbb{C}$.

Solution. Consider the function

$$f(z) = \begin{cases} \dfrac{z^2}{|z|}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Continuity. Since both z and $|z|$ are continuous on $\mathbb{C} \setminus \{0\}$, their quotient $\dfrac{z^2}{|z|}$ is continuous on $\mathbb{C} \setminus \{0\}$. To check continuity at 0, note that

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{z^2}{|z|} = \lim_{z \to 0} \frac{|z|^2}{|z|} = \lim_{z \to 0} |z| = 0 = f(0).$$

Hence f is continuous everywhere on $\mathbb{C}$.

Non-analyticity. To see that f is not analytic anywhere, we check the Cauchy–Riemann equations. For $z = x + iy \neq 0$, we can write

$$f(z) = \frac{z^2}{|z|} = \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} + i\,\frac{2xy}{\sqrt{x^2 + y^2}}.$$

Define

$$U(x,y) = \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}, \qquad V(x,y) = \frac{2xy}{\sqrt{x^2 + y^2}}.$$

We compute the partial derivatives:

$$\begin{aligned} U_x &= \frac{2x\sqrt{x^2+y^2} - (x^2 - y^2)\frac{x}{\sqrt{x^2+y^2}}}{x^2 + y^2}, \\ V_y &= \frac{2x\sqrt{x^2+y^2} - 2xy^2/\sqrt{x^2+y^2}}{x^2 + y^2}, \\ U_y &= \frac{-2y\sqrt{x^2+y^2} - (x^2 - y^2)\frac{y}{\sqrt{x^2+y^2}}}{x^2 + y^2}, \\ V_x &= \frac{2y\sqrt{x^2+y^2} - 2x^2y/\sqrt{x^2+y^2}}{x^2 + y^2}. \end{aligned}$$

For all $(x, y) \neq (0, 0)$ we have

$$U_x \neq V_y, \qquad U_y \neq -V_x.$$

Thus the Cauchy–Riemann equations are not satisfied at any nonzero point.

At $z = 0$, the derivative would be

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{f(z)}{z}.$$

Along the real axis $z = t$, we have $\frac{f(t)}{t} = 1$. Along the imaginary axis $z = it$, we obtain $\frac{f(it)}{it} = -1$. Since the limit depends on the direction, it does not exist. Hence f is not differentiable at 0.

Hence, the function f is continuous everywhere on $\mathbb{C}$, but is nowhere analytic.

Exercise 2.5.4

1) Show that the function U

$$U(x, y) = 2x^3 - 6xy^2 + x^2 - y^2 - y$$

is harmonic.

2) Let $z = x + iy$. Find all functions $V(x, y)$ such that the complex function $f(z) = U(x, y) + iV(x, y)$ is analytic.

Solution. 1. U is harmonic. Compute partial derivatives:

$$U_x = 6x^2 - 6y^2 + 2x, \qquad U_{xx} = 12x + 2,$$

$$U_y = -12xy - 2y - 1, \qquad U_{yy} = -12x - 2.$$

Thus

$$U_{xx} + U_{yy} = (12x + 2) + (-12x - 2) = 0,$$

so U is harmonic.

2. Find V so that $f = U + iV$ is analyic. The Cauchy–Riemann equations are

$$U_x = V_y, \qquad U_y = -V_x.$$

From $U_x = 6x^2 - 6y^2 + 2x$ we integrate with respect to y:

$$V(x, y) = \int U_x \, dy = 6x^2y - 2y^3 + 2xy + g(x),$$

where g is a function of x alone. Differentiating with respect to x gives

$$V_x = 12xy + 2y + g'(x).$$

But $V_x = -U_y = 12xy + 2y + 1$, hence $g'(x) = 1$ and $g(x) = x + C$ for some constant $C \in \mathbb{R}$.

Therefore the general harmonic conjugate is

$$V(x, y) = 6x^2y - 2y^3 + 2xy + x + C,$$

and all analytic functions of the form $f = U + iV$ are

$$f(z) = \left(2x^3 - 6xy^2 + x^2 - y^2 - y\right) + i\left(6x^2y - 2y^3 + 2xy + x + C\right),$$

with C an arbitrary real constant.

Exercise 2.5.5

1) Show that the function U

$$U(x,y) = 2x^3 - 6xy^2 + x^2 - y^2 - y$$

is harmonic.
2) Let $z = x + iy$. Find all functions $V(x,y)$ such that the complex function $f(z) = U(x,y) + iV(x,y)$ is analytic

Solution. 1. U is harmonic. Compute partial derivatives:

$$U_x = 6x^2 - 6y^2 + 2x, \qquad U_{xx} = 12x + 2,$$
$$U_y = -12xy - 2y - 1, \qquad U_{yy} = -12x - 2.$$

Thus

$$U_{xx} + U_{yy} = (12x + 2) + (-12x - 2) = 0,$$

so U is harmonic.

2. Find V so that $f = U + iV$ is analytic. The Cauchy–Riemann equations are

$$U_x = V_y, \qquad U_y = -V_x.$$

From $U_x = 6x^2 - 6y^2 + 2x$ we integrate with respect to y:

$$V(x,y) = \int U_x \, dy = 6x^2y - 2y^3 + 2xy + g(x),$$

where g is a function of x alone. Differentiating with respect to x gives

$$V_x = 12xy + 2y + g'(x).$$

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Therefore the general harmonic conjugate is

$$V(x,y) = 6x^2y - 2y^3 + 2xy + x + C,$$

and all analytic functions of the form $f = U + iV$ are

$$f(z) = \big(2x^3 - 6xy^2 + x^2 - y^2 - y\big) + i\big(6x^2y - 2y^3 + 2xy + x + C\big),$$

with C an arbitrary real constant.

Exercise 2.5.6

Find a real function v to make $f(z) = e^{-x}(x \sin y - y \cos y)$ analytic.

Solution. Using the Cauchy-Riemann equations, we obtain

$$v_y = u_x = e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y \tag{1}$$
$$v_x = -u_y = e^{-x} \cos y - xe^{-x} \cos y - ye^{-x} \sin y. \tag{2}$$

Integrating equation (1) with respect to y, keeping x constant, we obtain

$$v(x,y) = ye^{-x} \sin y + xe^{-x} \cos y + F(x),$$

for some real function $F(x)$. Putting this into equation (2) and after simplification, we get

$$F'(x) = 0,$$

which implies $F(x) = c$ for some constant c.Taking $c = 0$, one possible function is

$$v(x,y) = ye^{-x} \sin y + xe^{-x} \cos y.$$

Exercise 2.5.7

Assume that f is analytic in a domain D with the condition

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{in } D.$$

where u is the real part of$f$ and v is the imaginary part of f. Show that f' is constant in D.

Solution. Suppose $f = u + iv$ is analytic on a domain D, with the condition

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{in } D.$$

From the Cauchy–Riemann equations we have

$$u_x = v_y, \qquad u_y = -v_x.$$

Substituting $u_x = v_y$ into the given condition, we get

$$u_x + v_y = 2u_x = 0 \quad \text{in } D,$$

hence

$$u_x = 0 \quad \text{and} \quad v_y = 0 \quad \text{in } D.$$

Since u is harmonic (because f is holomorphic), we have

$$u_{xx} + u_{yy} = 0.$$

But $u_x = 0$ implies $u_{xx} = 0$, so $u_{yy} = 0$ in D. Therefore u_y is constant in D; let $u_y = k \in \mathbb{R}$.

By the Cauchy–Riemann equations, $v_x = -u_y = -k$ is also constant. Thus, the complex derivative is

$$f'(z) = u_x + iv_x = 0 + i(-k) = -ik,$$

which is constant in D.

$$\boxed{f'(z) \text{ is constant in } D}$$

Exercise 2.5.8

Suppose that f is holomorphic in an open set D. Show that in each of the following cases:

(a) $\text{Re}(f)$ is constant,

(b) $\text{Im}(f)$ is constant,

(c) $|f|$ is constant,

one can conclude that f is constant.

Solution. Let

$$f(z) = f(x, y) = u(x, y) + iv(x, y), \quad z = x + iy.$$

(a) Since $\text{Re}(f)$ is constant, we have

$$\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial y} = 0.$$

Using the Cauchy–Riemann equations,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0.$$

Therefore, in D,

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + 0 = 0.$$

Thus, $f(z)$ is constant.

(b) Since $\text{Im}(f)$ is constant, we have

$$\frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} = 0.$$

Using the Cauchy–Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

Therefore, in D,

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + 0 = 0.$$

Thus, $f(z)$ is constant.

(c) Since $|f| = \sqrt{u^2 + v^2}$ is constant, we have

$$0 = \frac{\partial}{\partial x}(u^2 + v^2) = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x},$$

$$0 = \frac{\partial}{\partial y}(u^2 + v^2) = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}.$$

Using the Cauchy–Riemann equations, this becomes

$$u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial x} = 0 \quad (1),$$

$$-u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0 \quad (2).$$

Multiplying (1) by u and (2) by v, we obtain

$$u^2\frac{\partial v}{\partial y} + uv\frac{\partial v}{\partial x} = 0,$$

$$-uv\frac{\partial v}{\partial x} + v^2\frac{\partial v}{\partial y} = 0.$$

Adding these two equations gives

$$(u^2 + v^2)\frac{\partial v}{\partial y} = 0.$$

If $u^2 + v^2 = 0$, then $u = v = 0$, hence $f = 0$ which is constant. If $u^2 + v^2 \neq 0$, then $\frac{\partial v}{\partial y} = 0$. From (1) and (2), it follows that $\frac{\partial v}{\partial x} = 0$, and by the Cauchy–Riemann equations, $\frac{\partial u}{\partial x} = 0$. Therefore,

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0.$$

Hence, $f(z)$ is constant.

Exercise 2.5.9

Let Ω be an open convex subset of $\mathbb{C}$ and let $f \in H(\Omega)$. If $a, b \in \Omega$ are two distinct points such that $f(a) = f(b) = 0$, prove that there exist z_1 and z_2 lying on the segment joining a and b such that

$$\operatorname{Re}\big(f'(z_1)\big) = 0 \quad \text{and} \quad \operatorname{Im}\big(f'(z_2)\big) = 0.$$

Solution. Denote

$$\alpha_1 = \operatorname{Re}(a), \quad \alpha_2 = \operatorname{Im}(a), \quad \beta_1 = \operatorname{Re}(b), \quad \beta_2 = \operatorname{Im}(b),$$

and let $u = \operatorname{Re}(f)$, $v = \operatorname{Im}(f)$.

Define the real-valued function $\phi : [0,1] \longrightarrow \mathbb{R}$ by

$$\phi(t) = (\beta_1 - \alpha_1)\, u\big(a + t(b-a)\big) + (\beta_2 - \alpha_2)\, v\big(a + t(b-a)\big).$$

This function is differentiable. By hypothesis,

$$u(a) = u(b) = v(a) = v(b) = 0,$$

which implies

$$\phi(0) = (\beta_1 - \alpha_1)u(a) + (\beta_2 - \alpha_2)v(a) = 0,$$
$$\phi(1) = (\beta_1 - \alpha_1)u(b) + (\beta_2 - \alpha_2)v(b) = 0.$$

By Rolle's Theorem, there exists $t_1 \in (0,1)$ such that $\phi'(t_1) = 0$. Let

$$z_1 = a + t_1(b-a),$$

which lies on the segment joining a and b. Then

$$\phi'(t_1) = \frac{d}{dt}\phi(t)\Big|_{t=t_1} = (\beta_1 - \alpha_1)\frac{d}{dt}u\big(a + t(b-a)\big)\Big|_{t=t_1} + (\beta_2 - \alpha_2)\frac{d}{dt}v\big(a + t(b-a)\big)\Big|_{t=t_1}.$$

Since

$$x = \operatorname{Re}(a) + t\, \operatorname{Re}(b-a), \qquad y = \operatorname{Im}(a) + t\, \operatorname{Im}(b-a),$$

we compute

$$\phi'(t_1) = (\beta_1 - \alpha_1)\left(u_x(z_1)\frac{\partial x}{\partial t} + u_y(z_1)\frac{\partial y}{\partial t}\right) + (\beta_2 - \alpha_2)\left(v_x(z_1)\frac{\partial x}{\partial t} + v_y(z_1)\frac{\partial y}{\partial t}\right).$$

Simplifying,

$$0 = (\beta_1 - \alpha_1)\big((\beta_1 - \alpha_1)u_x(z_1) + (\beta_2 - \alpha_2)u_y(z_1)\big) + (\beta_2 - \alpha_2)\big((\beta_1 - \alpha_1)v_x(z_1) + (\beta_2 - \alpha_2)v_y(z_1)\big).$$

That is,

$$0 = (\beta_1 - \alpha_1)^2 u_x(z_1) + (\beta_1 - \alpha_1)(\beta_2 - \alpha_2)[u_y(z_1) + v_x(z_1)] + (\beta_2 - \alpha_2)^2 v_y(z_1). \qquad (*)$$

By the Cauchy–Riemann equations, the middle term vanishes, so

$$\big[(\beta_1 - \alpha_1)^2 + (\beta_2 - \alpha_2)^2\big]u_x(z_1) = 0.$$

Since $a \neq b$, the prefactor is nonzero, hence

$$u_x(z_1) = 0.$$

Therefore,

$$\operatorname{Re}\big(f'(z_1)\big) = u_x(z_1) = 0.$$

Now apply the same argument to the analytic function $g = -if$. Then there exists $t_2 \in (0,1)$ such that $z_2 = a + t_2(b-a)$ satisfies

$$0 = \operatorname{Re}\big(g'(z_2)\big) = v_x(z_2) = -u_y(z_2) = \operatorname{Im}\big(f'(z_2)\big).$$

Hence the proof is complete.

Chapter 3

Elementary Complex Functions

Chapter contents

Complex functions are a natural extension of real functions on the plane of complex numbers $\mathbb{C}$. Through such an extension, these functions are enriched with new properties. For example, the exponential function of a complex variable z becomes periodic, the functions $sinz$ and $\cos z$ cease to be bounded, and the logarithm of negative numbers (and, in general, of any non-zero complex number) becomes meaningful. In this chapter, we will study the main properties of elementary complex functions, their domains of analysis, and their derivatives.

3.1 Polynomial Functions

Definition 3.1.1

Polynomial functions $P : \mathbb{C} \longrightarrow \mathbb{C}$ are defined by:

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n,$$

where $a_0 \neq 0$, $a_1, \ldots, a_n$ are complex constants, and n is a positive integer called the degree of the polynomial P.

3.2 Rational Functions

Definition 3.2.1

Rational functions $f : \mathbb{C} \longrightarrow \mathbb{C}$ are defined by:

$$f(z) = \frac{P(z)}{Q(z)}, \quad (Q(z) \neq 0) \tag{3.1}$$

where $P(z)$ and $Q(z)$ are polynomials.

The special case: for $ad - bc \neq 0$

$$f(z) = \frac{az+b}{cz+d}, \quad (cz + d \neq 0) \tag{3.2}$$

is called a **homographic transformation** (or Möbius transformation).

3.3 The Complex Exponential Function

One of the most important functions in all branches of mathematics is the complex exponential function, which is defined as follows:

Definition 3.3.1 The Complex Exponential Function

Let $z = x + iy$, where $x, y \in \mathbb{R}$. The function e^z is defined by

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y). \tag{3.3}$$

Therefore, the real and imaginary parts of e^z are

$$u(x, y) = e^x \cos y, \qquad v(x, y) = e^x \sin y.$$

3.4 Properties of the Complex Exponential Function

Let $z, w \in \mathbb{C}$. The exponential function e^z satisfies the following properties:

1. **(Addition, inverse, division and power) Law:**

$$e^{z+w} = e^z \cdot e^w \qquad e^{-z} = \frac{1}{e^z} \qquad \frac{e^z}{e^w} = e^{z-w}, \qquad (e^z)^n = e^{nz}.$$

2. **values:** we have

$$e^0 = 1, \qquad e^z \neq 0 \text{ for all } z \in \mathbb{C}.$$

Moreover:$e^z \in \mathbb{C}^*$, for example

$$e^{i\pi} = -1 \qquad e^{i\frac{\pi}{2}} = i$$

3. **Periodicity:**

$$e^{z+2\pi i} = e^z.$$

Hence, the function is periodic in the imaginary direction with period $2\pi i$.

4. **Modulus:**

$$|e^{x+iy}| = |e^x(\cos y + i \sin y)| = e^x.$$

5. **Real and Imaginary Parts:**

$$Re(e^{x+iy}) = e^x \cos y, \qquad Im(e^{x+iy}) = e^x \sin y.$$

6. **Euler's Formula:** For purely imaginary $z = iy$,

$$e^{iy} = \cos y + i \sin y.$$

7. **Derivative:**

$$\frac{d}{dz}(e^z) = e^z, \qquad \forall z \in \mathbb{C}.$$

8. **Entire Function:** The function e^z is Analytic on $\mathbb{C}$ and has no singularities.

9. **Series Representation:**

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \forall z \in \mathbb{C}.$$

10. **Geometric Interpretation:** For $z = iy$, the exponential lies on the unit circle:

$$e^{iy} = \cos y + i \sin y, \qquad |e^{iy}| = 1.$$

As $y \in [0, 2\pi]$, the point e^{iy} traces the unit circle in the complex plane counterclockwise.

3.5 Trigonometric (Circular) Functions

Definition 3.5.1

The trigonometric (or circular) functions are defined in terms of the exponential function as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2},$$

$$\sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}}, \qquad \csc z = \frac{1}{\sin z} = \frac{2i}{e^{iz} - e^{-iz}},$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)}, \qquad \cot z = \frac{\cos z}{\sin z} = i\,\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}.$$

Most of the properties of the real trigonometric functions remain valid in the complex case. For example:

$$\sin^2 z + \cos^2 z = 1, \qquad 1 + \tan^2 z = \sec^2 z, \qquad 1 + \cot^2 z = \csc^2 z,$$

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2,$$

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2,$$

$$\tan(z_1 + z_2) = \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2},$$

$$\sin(-z) = -\sin z, \qquad \cos(-z) = \cos z, \qquad \tan(-z) = -\tan z.$$

3.6 Hyperbolic Functions

The hyperbolic functions are defined in terms of the exponential function as follows:

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2},$$
$$z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}}, \qquad z = \frac{1}{\sinh z} = \frac{2}{e^z - e^{-z}},$$
$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}}, \qquad \coth z = \frac{\cosh z}{\sinh z} = \frac{e^z + e^{-z}}{e^z - e^{-z}}.$$

The hyperbolic functions satisfy identities similar to the trigonometric functions. For example:

$$\cosh^2 z - \sinh^2 z = 1, \qquad 1 - \tanh^2 z =^2 z, \qquad \coth^2 z - 1 =^2 z,$$

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2,$$
$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2,$$
$$\tanh(z_1 + z_2) = \frac{\tanh z_1 + \tanh z_2}{1 + \tanh z_1 \tanh z_2},$$
$$\sinh(-z) = -\sinh z, \qquad \cosh(-z) = \cosh z, \qquad \tanh(-z) = -\tanh z.$$

3.7 Logarithmic Function

3.7.1 Branches of Multi-valued Functions

Let f be a multi-valued function. A branch of f is a *continuous function* Fon a domain D such that it assigns exactly one of the multiple values of f to each $z \in D$. A branch cut for a branch $F$of a multi-valued function f is a line or a curve γ that is deleted from the domain D of f so that $F$is analytic in $D - \gamma$. A branch point is any point common to all branch cuts. It should be noted that it is possible to consider a multi-valued function as being single-valued on a certain surface. Such a surface is called as a Riemann surface which consists of a finite or an infinite number of sheets and these sheets are connected by branch cuts. The end points of a branch cut are called the branch points. The manner in which the sheets are connected depends on the particular multi-valued function.

3.7.2 The Logarithm Function

Definition 3.7.1

The inverse of the exponential function is the logarithm function. For a nonzero complex number z, we define $\log z$ to be any complex number such that

$$e^w = z. \tag{3.4}$$

To determine w in terms of z, we write

$$w = u + iv \qquad \text{and} \qquad z = re^{i\theta},$$

where $r > 0$ and $\theta = \arg z$.

Then the equation

$$e^{u+iv} = re^{i\theta} \tag{3.5}$$

becomes

$$e^u = r \qquad \text{and} \qquad e^{iv} = e^{i\theta}.$$

The first part gives

$$u = \log r,$$

which is the usual natural logarithm of the positive number r.

Since the exponential function e^z is not injective, the second part implies

$$v = \theta + 2k\pi, \qquad k \in \mathbb{Z}.$$

Therefore, the logarithm of a complex number is given by

$$\log z = \log |z| + i \arg z = \log r + i(\theta + 2k\pi), \qquad k \in \mathbb{Z}. \tag{3.6}$$

Proposition 3.7.2

The logarithm of a complex variable z has the following properties:

1. $\log(z_1 z_2) = \log(z_1) + \log(z_2)$
2. $\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2)$
3. $e^{\log z} = z$
4. $\log(e^z) = z + 2i\pi k, \quad k \in \mathbb{Z}$

Example 3.7.3

Let calculate following numbers:

$$\log(5), \quad \log(-1),, \quad \log(\sqrt{3} + i)$$

we have

1. $$\log(5) = \ln(5) + i \arg(5) = \ln(5) + 2k, \quad k \in \mathbb{Z}.$$
2. $$\log(-1) = \ln(1) + i \arg(-1) = (2k + 1), \quad k \in \mathbb{Z}.$$
3. $$\log(\sqrt{3} + i) = \ln(2) + \left(\frac{\pi}{6} + 2k\pi\right) i, \quad k \in \mathbb{Z}$$

Remark 3.7.4. Unlike the real logarithm, this formula (3.6) shows that $\log z$ is multi-valued (in fact, infinitely many-valued) because $argz$ takes multiple values. Now we define a principal value of $\log z$ that satisfies the formula (3.4), by taking the principal value of the argument $Argz$ as $(-\pi, \pi]$, we can make the logarithm single-valued.

Definition 3.7.5 The principal value of the Logarithm

The principal value of the complex logarithm of $z \neq 0$ is defined by

$$\operatorname{Log} z = \log |z| + i \operatorname{Arg} z, \quad -\pi < \operatorname{Arg} z \leq \pi \tag{3.7}$$

Furthermore, the general logarithm is given by

$$\log z = \operatorname{Log} z + 2k\pi i, \qquad k \in \mathbb{Z}.$$

The principal value reduces to the usual real logarithm when z is positive.

Definition 3.7.6 the cut plane

The complex plane with the negative real-axis (including 0) removed is called the **cut plane**. See Figure

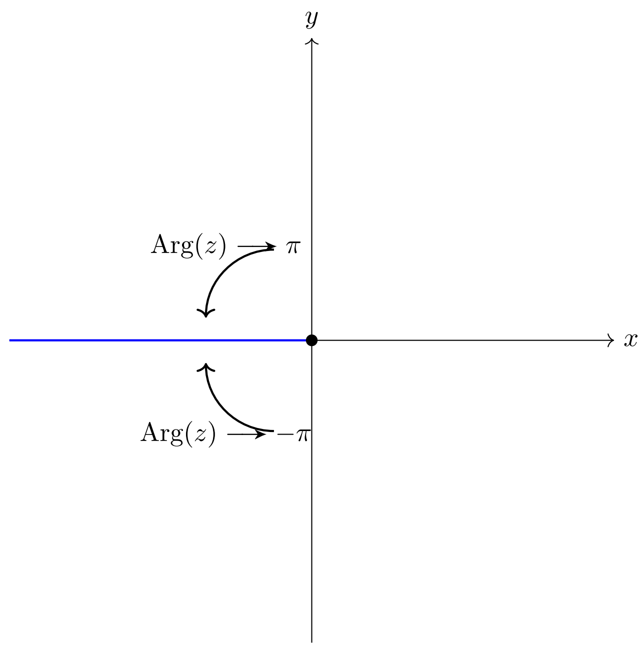

Figure 3.4.1: The cut plane: this is the complex plane with the negative real axis removed.

Proposition 3.7.7

The principal logarithm $Logz$ is continuous on the cut plane.

Proof. This follows from the fact that the principal value of the argument $Argz$ is continuous on the cut-plane. □

Definition 3.7.8 The principal branch of the Logarithm

The principal branch of the complex logarithm of $z \neq 0$ is defined by

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z, \quad -\pi < \operatorname{Arg} z < \pi \tag{3.8}$$

Which can de written for $z = re^{i\theta}$ as:

$$\log z = \ln|r| + i\theta. \quad (r > 0, -\pi < \theta < \pi), \tag{3.9}$$

with components

$$u(r,\theta) = \ln r \quad \text{and} \quad v(r,\theta) = \theta,$$

Note 3.7.9

Note that Special care must be taken in anticipating that familiar properties of $\ln x$ in calculus carry over to be properties of $logz$ and $Logz$ se for example

- $$\log(i^2) \neq 2\log i$$

  because

  $$\ln(i^2) = \log(-1) = (2k+1)\pi i, \qquad k = 0, \pm 1, \pm 2, \dots,$$

  On the other hand,

  $$2\log i = 2\left(\ln 1 + i\left(\tfrac{\pi}{2} + 2k\pi\right)\right) = (4k+1)\pi i, \qquad k = 0, \pm 1, \pm 2, \dots$$

- In general,

  $$Log(z_1 z_2) \neq Log(z_1) + Log(z_2).$$

  For instance, take $z_1 = z_2 = -1$. Then

  $$Log(z_1 z_2) = Log(1) = 0,$$

  while

  $$Log(z_1) + Log(z_2) = Log(-1) + Log(-1) = 2\, Log(-1) = 2\pi i$$

Remark 3.7.10. 1. The function defined in (5.14) is single-valued and continuous in the stated domain (see Fig.4)

$$\mathcal{D} = \{\, z \in \mathbb{C} : |z| > 0,\ -\pi < Argz < \pi \,\}$$

2. Note that if the function were to be defined on the ray $\theta = \alpha$, it would not be continuous there. For if z is a point on that ray, there are points arbitrarily close to z at which the values of v in Eq. (3.9) are near α and also points such that the values of v are near $(\alpha + 2\pi)$ (see**Fig.4**).

3. If we let α denote any real number and restrict the value of θ in expression (3.6) so that

$$\alpha < \theta < \alpha + 2\pi,$$

the function defined by

$$\log z = \ln r + i\theta, \qquad (r > 0,\ \alpha < \theta < \alpha + 2\pi), \tag{3.10}$$

defines also, a branch of the logarithm.

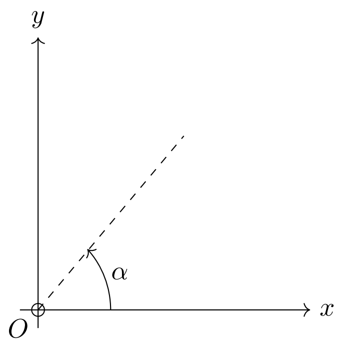

Fig.4

Having seen that the principal logarithm is continuous, we can go on to show that it is differentiable.

Theorem 3.7.11

The principal branch of logarithm $Logz$ is analytic on

$$\mathcal{D} = \{\, z \in \mathbb{C} : |z| > 0,\ -\pi < Argz < \pi \,\}$$

Moreover

$$\frac{d}{dz}\,Logz = \frac{1}{z}. \quad (z \in \mathcal{D}).$$

Proof. Let $z = re^{i\theta}$ be in the region defined in $\mathcal{D}$. Then $-\pi < \theta < \pi$ and

$$z = \left(re^{i\theta}\right) = u + iv = \ln r + i\theta.$$

We compute

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial v}{\partial \theta} = 1, \quad \frac{\partial v}{\partial r} = 0, \quad \frac{\partial u}{\partial \theta} = 0.$$

Thus, u and v satisfy the Cauchy–Riemann equations in polar coordinates:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

This shows that z is analytic in the given domain, and its derivative is

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{1}{e^{i\theta}} = \frac{1}{z}.$$

□

Note 3.7.12

Note that :

- The region

$$\{z \in \mathbb{C} : |z| > 0,\ -\pi < z < \pi\}$$

  is often called the *domain of analyticity* of $Logz$.

- The above region It is equivalent to

$$\mathbb{C} \setminus \{z = x + iy \in \mathbb{C} : Rez \leq 0,\ y = 0\}.$$

3.8 Circular (Trigonometric) Functions

Definition 3.8.1

The trigonometric functions extend naturally to the complex plane via the exponential function:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

They are entire functions and satisfy the identity

$$\sin^2 z + \cos^2 z = 1.$$

3.9 Hyperbolic Functions

Definition 3.9.1

The hyperbolic functions are defined in $\mathbb{C}$ by

$$\sinh z = \frac{e^{z} - e^{-z}}{2}, \qquad \cosh z = \frac{e^{z} + e^{-z}}{2}.$$

They satisfy the identity

$$\cosh^2 z - \sinh^2 z = 1.$$

Power Functions

Definition 3.9.2

For $z \in \mathbb{C} \setminus \{0\}$ and $\alpha \in \mathbb{C}$, the power function is defined by

$$z^{\alpha} = e^{\alpha \log z}, \tag{3.11}$$

where $\log z$ denotes the complex logarithm.

Note 3.9.3

Note that :

- Because of the logarithm, z^a is, in general, multiple-valued.

  $$\{z \in \mathbb{C} : |z| > 0,\ -\pi < Argz < \pi\}$$

  is often called the *domain of analyticity* of $Logz$.

- We mention here two other expected properties of the power function $z \longmapsto z^a$

  a)$\frac{1}{z^a} = z^{-a}$, indeed

  $$z^a = \exp(a \log z) = \exp(-a \log z) = z^{-a}.$$

  b)The property of differentiation rule for pricipal branch of function power $z \longmapsto z^a$. indeed, for

$z \in \mathcal{D} = \{z \in \mathbb{C} : |z| > 0,\ -\pi < z < \pi\}$ we have

$$\begin{aligned}\frac{d}{dz} z^a =& \frac{d}{dz} \exp(a \log z)\\ =& \frac{a}{z} \exp(a \log z)\\ =& \frac{a}{\exp(\log z)} \exp(a \log z), \quad (z = \exp(\log z)),\\ =& c \exp\big[(a-1)\log z\big]\\ =& c z^{a-1}.\end{aligned}$$

- The principal value of z^c occurs when $\log z$ is replaced by z in Definition (3.11):

$$\textbf{P.V.}\, z^a = e^{a\ Logz}.$$

Thus power functions are, in general, multi-valued in the complex plane. Indeed, by (3.11) we have

Example 3.9.4

Consider the power function

$$i^i = e^{i \log i}.$$

Since

$$\log i = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = \left(\frac{2n+1}{2}\right)\pi i, \qquad (n = 0, \pm 1, \pm 2, \dots),$$

we may write

$$i^i = \exp\left(i \cdot \frac{2n+1}{2}\pi i\right) = \exp\left(-\frac{2n+1}{2}\pi\right), \qquad (n = 0, \pm 1, \pm 2, \dots).$$

Thus the principal value is

$$\text{P. V.}(i^i) = \exp\left(-\tfrac{\pi}{2}\right).$$

Note that all values of i^i are real numbers.

3.10 Exercises set

Exercise 3.10.1

Prove that

$$\lim_{r\longrightarrow\infty,\, z=re^{i\theta}} \tan z = \begin{cases} i, & \text{if } 0<\theta<\pi, \\ -i, & \text{if } -\pi<\theta<0. \end{cases}$$

Solution. We have

$$\tan z = \frac{\sin z}{\cos z} = \frac{\frac{1}{2i}\left(e^{iz}-e^{-iz}\right)}{\frac{1}{2}\left(e^{iz}+e^{-iz}\right)} = -i\cdot\frac{e^{2iz}-1}{e^{2iz}+1}. \tag{4.29}$$

Note that

$$|e^{2iz}| = e^{-2r\sin\theta}.$$

Case 1: $0<\theta<\pi$. Here $\sin\theta>0$, hence

$$|e^{2iz}| = e^{-2r\sin\theta} \longrightarrow 0 \quad \text{as } r\longrightarrow\infty.$$

Therefore,

$$\lim_{r\longrightarrow\infty,\, z=re^{i\theta}} \tan z = i.$$

Case 2: $-\pi<\theta<0$. In this case we first rewrite (4.29) as

$$\tan z = -i\cdot\frac{1-e^{-2iz}}{1+e^{-2iz}}.$$

Since $\sin\theta<0$, we have

$$|e^{-2iz}| = e^{2r\sin\theta} \longrightarrow 0 \quad \text{as } r\longrightarrow\infty.$$

Thus,

$$\lim_{r\longrightarrow\infty,\, z=re^{i\theta}} \tan z = -i.$$

Hence, the desired result is proved.

Exercise 3.10.2

Let $f:\mathbb{C}\longrightarrow\mathbb{C}$ be an entire function such that

$$f(z_1+z_2) = f(z_1)f(z_2), \qquad \forall z_1,z_2\in\mathbb{C}, \tag{1}$$

and

$$f(x) = e^x, \qquad \forall x\in\mathbb{R}. \tag{2}$$

Prove that $f(z)=e^z$ for all $z\in\mathbb{C}$.

Solution. Let $z=x+iy$. From equations (1) and (2), we have

$$f(z) = f(x+iy) = f(x)f(iy) = e^x f(iy).$$

Write

$$f(iy) = U(y)+iV(y),$$

where U and V are real-valued functions. Then

$$f(z) = e^x U(y) + ie^x V(y).$$

Since f is entire, the Cauchy–Riemann equations give

$$U(y) = V'(y), \qquad U'(y) = -V(y).$$

Thus

$$U''(y) = -U(y),$$

which implies

$$U(y) = \alpha \cos y + \beta \sin y,$$

for some constants $\alpha, \beta \in \mathbb{C}$. Consequently,

$$V(y) = -\beta \cos y + \alpha \sin y.$$

From the condition (**??**), we have $U(0) = \alpha = 1$ and $V(0) = -\beta = 0$. Hence

$$f(z) = e^x \big(\cos y + i \sin y\big) = e^z.$$

This completes the proof.

Exercise 3.10.3

Solve the equation in $\mathbb{C}$

$$\cos z = 3 + 2e^{iz}.$$

Solution. We start with the equation

$$\cos z = 3 + 2e^{iz}.$$

Since

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$

we obtain

$$\frac{e^{iz} + e^{-iz}}{2} = 3 + 2e^{iz} \quad \Longrightarrow \quad e^{iz} + e^{-iz} = 6 + 4e^{iz}.$$

Multiplying through by e^{iz} gives

$$1 + e^{2iz} = 6e^{iz} + 4e^{2iz},$$

which simplifies to

$$3e^{2iz} + 6e^{iz} - 1 = 0.$$

Let $X = e^{iz}$. Then the quadratic equation is

$$3X^2 + 6X - 1 = 0.$$

Its solutions are

$$X_1 = \frac{-6 + \sqrt{36 - 4(3)(-1)}}{6} = -1 + \frac{\sqrt{2}}{3}, \quad X_2 = -1 - \frac{\sqrt{2}}{3}.$$

Thus,

$$e^{iz_1} = -1 + \tfrac{\sqrt{2}}{3}, \qquad e^{iz_2} = -1 - \tfrac{\sqrt{2}}{3}.$$

Hence,

$$iz_1 = \ln\Big(-1 + \tfrac{\sqrt{2}}{3}\Big), \qquad iz_2 = \ln\Big(-1 - \tfrac{\sqrt{2}}{3}\Big).$$

Taking into account the argument, we can write

$$iz_1 = \ln\Big|-1 + \tfrac{\sqrt{2}}{3}\Big| + i\pi + 2k\pi i, \qquad iz_2 = \ln\Big|-1 - \tfrac{\sqrt{2}}{3}\Big| + i\pi + 2k'\pi i, \quad k, k' \in \mathbb{Z}.$$

Exercise 3.10.4

Show that

$$|\cos(z)|^2 = (\cos x)^2 + (\sinh y)^2$$

for all $z \in \mathbb{C}$, where $x = \Re(z)$ and $y = \Im(z)$.

Solution.

$$\begin{aligned}
|\cos(z)|^2 &= \left|\cos(x+iy)\right|^2 \\
&= \left|\cos(x)\cos(iy) - \sin(x)\sin(iy)\right|^2 \\
&= \left|\cos(x)\cosh(y) - i\sin(x)\sinh(y)\right|^2 \\
&= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\
&= \cos^2 x\left(1+\sinh^2 y\right) + \sin^2 x \sinh^2 y \\
&= \cos^2 x + \left(\cos^2 x + \sin^2 x\right)\sinh^2 y \\
&= \cos^2 x + \sinh^2 y.
\end{aligned}$$

Thus,

$$|\cos(z)|^2 = (\cos x)^2 + (\sinh y)^2.$$

Exercise 3.10.5

Show that

$$\tan(z_1+z_2) = \frac{\tan z_1 + \tan z_2}{1-(\tan z_1)(\tan z_2)}$$

for all complex numbers z_1, z_2 such that

$$z_1,\ z_2,\ z_1+z_2 \neq \tfrac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}.$$

Solution. We compute

$$\begin{aligned}
\tan z_1 + \tan z_2 &= \frac{i\left(e^{-iz_1} - e^{iz_1}\right)}{e^{iz_1} + e^{-iz_1}} + \frac{i\left(e^{-iz_2} - e^{iz_2}\right)}{e^{iz_2} + e^{-iz_2}} \\
&= \frac{i\Big(\left(e^{-iz_1} - e^{iz_1}\right)\left(e^{iz_2} + e^{-iz_2}\right) + \left(e^{-iz_2} - e^{iz_2}\right)\left(e^{iz_1} + e^{-iz_1}\right)\Big)}{\left(e^{iz_1} + e^{-iz_1}\right)\left(e^{iz_2} + e^{-iz_2}\right)} \\
&= \frac{-2i\left(e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}\right)}{\left(e^{iz_1} + e^{-iz_1}\right)\left(e^{iz_2} + e^{-iz_2}\right)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
1-(\tan z_1)(\tan z_2) &= 1 - \frac{i\left(e^{-iz_1} - e^{iz_1}\right)}{e^{iz_1} + e^{-iz_1}} \cdot \frac{i\left(e^{-iz_2} - e^{iz_2}\right)}{e^{iz_2} + e^{-iz_2}} \\
&= \frac{\left(e^{-iz_1} + e^{iz_1}\right)\left(e^{-iz_2} + e^{iz_2}\right) + \left(e^{-iz_1} - e^{iz_1}\right)\left(e^{-iz_2} - e^{iz_2}\right)}{\left(e^{iz_1} + e^{-iz_1}\right)\left(e^{iz_2} + e^{-iz_2}\right)} \\
&= \frac{2\left(e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}\right)}{\left(e^{iz_1} + e^{-iz_1}\right)\left(e^{iz_2} + e^{-iz_2}\right)}.
\end{aligned}$$

Therefore,

$$\frac{\tan z_1 + \tan z_2}{1 - (\tan z_1)(\tan z_2)} = \frac{-i\left(e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}\right)}{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}$$
$$= \tan(z_1 + z_2).$$

This proves the identity.

Exercise 3.10.6

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2, \quad \forall z_1, z_2 \in \mathbb{C}.$$

Solution.

$$\begin{aligned}\cosh(z_1 + z_2) &= \cos(iz_1 + iz_2)\\ &= \cos(iz_1)\cos(iz_2) - \sin(iz_1)\sin(iz_2)\\ &= \cos(iz_1)\cos(iz_2) + (-i\sin(iz_1))(-i\sin(iz_2))\\ &= \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2.\end{aligned}$$

Exercise 3.10.7

Find all values of z such that

$$\text{(a) } e^z = -2; \qquad \text{(b) } e^z = 1 + i; \qquad \text{(c) } \exp(2z - 1) = 1.$$

Solution.

$$\text{(a) } z = \ln 2 + (2n+1)\pi i, \quad n \in \mathbb{Z};$$

$$\text{(b) } z = \tfrac{1}{2}\ln 2 + \tfrac{(2n+1)\pi}{4} i, \quad n \in \mathbb{Z};$$

$$\text{(c) } z = \tfrac{1}{2} + n\pi i, \quad n \in \mathbb{Z}.$$

Exercise 3.10.8

Show that $\overline{\sin z} = \sin(\bar{z})$.

Solution.

If $z = x + iy$, then we have

$$\overline{e^{iz}} = \overline{e^{i(x+iy)}} = e^{-y} \cdot \overline{e^{ix}} = e^{-y} \cdot e^{-ix} = e^{-i(x-iy)} = e^{-i\bar{z}},$$

and similarly,

$$\overline{e^{-iz}} = e^{i\bar{z}}.$$

By the equations (4.4), we see that

$$\overline{\sin z} = \overline{\tfrac{1}{2i}\left(e^{iz} - e^{-iz}\right)} = -\tfrac{1}{2i}\overline{\left(e^{iz} - e^{-iz}\right)} = -\tfrac{1}{2i}\left(e^{-i\bar{z}} - e^{i\bar{z}}\right) = \sin \bar{z}.$$

Exercise 3.10.9

Let $f(z) = u(x,y) + iv(x,y)$ be an entire function satisfying $u(x,y) \leq x$ for all $z = x + iy$. Show that $f(z)$ is a polynomial of degree at most one.

Solution.

$$\text{Let } g(z) = \exp(f(z) - z).$$
$$\text{Then } |g(z)| = \exp(u(x,y) - x).$$
$$\text{Since } u(x,y) \leq x, \quad |g(z)| \leq 1 \quad \text{for all } z.$$
$$\text{And since } g(z) \text{ is entire, } g(z) \text{ must be constant by Louisville's theorem.}$$
$$\text{Therefore, } g'(z) \equiv 0.$$
$$\text{That is, } (f'(z) - 1)\exp(f(z) - z) \equiv 0,$$
$$\text{and hence } f'(z) = 1 \quad \text{for all } z.$$
$$\text{So } f(z) \equiv z + c \quad \text{for some constant } c.$$

Exercise 3.10.10

$$\text{Let } z = x + iy, \quad x = Rez, \ y = Imz.$$

Show that

$$\left|e^{z^3+i} + e^{-iz^2}\right| \leq e^{x^3-3xy^2} + e^{2xy},$$

Solution. Note that $|e^z| = e^{Re(z)}$. Therefore

$$\left|e^{z^3+i} + e^{-iz^2}\right| \leq |e^{z^3+i}| + |e^{-iz^2}| = e^{Re(z^3+i)} + e^{Re(-iz^2)}.$$
$$= e^{Re\left((x^3-3xy^2)+i(3x^2y-y^3+1)\right)} + e^{Re\left(2xy-i(x^2-y^2)\right)}.$$
$$= e^{x^3-3xy^2} + e^{2xy}.$$

Chapter 4

Complex integration

Chapter contents

This chapter contains some of the most important results in complex analysis. These include the Cauchy-Goursat theorem and the Cauchy integral formula. A fascinating result derived from the Cauchy integral formula is that if a complex function is once differentiable at a point, then derivatives of any order exist, and these derivatives are themselves analytic. Other important theorems in this chapter are Gauss's mean value theorem, Liouville's theorem, and the maximum modulus theorem. Many properties of integrals of functions of a complex variable are very similar to those of integrals of functions of a real variable; for example, when the integral satisfies certain conditions, the integral can be calculated by finding the antiderivative of the function to be integrated and evaluating the anti- derivative at both endpoints. However, there are other properties that are unique to integration in the complex plane.

4.1 Curve integral curviligne.

Definition 4.1.1

The **curvilinear integral** is an integral where the function to be integrated is evaluated on a curve .

Definition 4.1.2 Curves, parametrization and terminology

Let $z : [a, b] \longrightarrow \mathbb{C}$ is an application defined by

$$z(t) = x(t) + iy(t), \quad a \leq t \leq b. \tag{4.1}$$

Then, the image points $z(t) = (x(t), y(t))$ is called "**Curve**" a parameterized by (4.1).

If the initial and terminal points of C, that is, $(x(a), y(a))$ and $(x(b), y(b))$, be denoted by the symbols A and B, respectively, we say that:

(i) C is a **smooth curve** if x' and y' are continuous on the closed interval $[a, b]$ and not simultaneously zero on the open interval (a, b).

(ii) C is a **piecewise smooth curve** if it consists of a finite number of smooth curves $C_1, C_2, \ldots, C_n$ joined end to end, that is, the terminal point of one curve C_k coinciding with the initial point of the next curve C_{k+1}.

(iii) C is a **simple curve** if the curve C does not cross itself except possibly at $t = a$ and $t = b$.

(iv) C is a **closed curve** if $A = B$.

(v) C is a **simple closed curve** if the curve C does not cross itself and $A = B$; that is, C is simple and closed.

(vi) A **contour** is smooth curve consisting of finite number of connected curves (see 4.1).

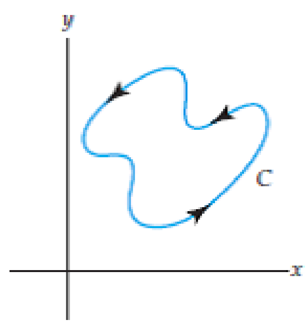

FIG4.1 Contour;

Example 4.1.3

1. **Line segment from** z_0 to z_1:

$$z(t) = (1 - t)z_0 + tz_1, \quad 0 \leq t \leq 1.$$

2. **Circle of radius** r centered at z_0:

$$z(t) = z_0 + re^{it}, \quad 0 \leq t < 2\pi.$$

3. **Semicircle of radius** r centered at z_0:

$$z(t) = z_0 + re^{it}, \quad 0 \leq t \leq \pi \quad \text{(upper semicircle)},$$

$$z(t) = z_0 + re^{it}, \quad \pi \leq t \leq 2\pi \quad \text{(lower semicircle)}.$$

4. **Arc of a circle of radius** r centered at z_0, from angle θ_1 to θ_2:

$$z(t) = z_0 + re^{it}, \quad \theta_1 \leq t \leq \theta_2.$$

Remark 4.1.4. The parametrization of a curve is not unique. Verify that the following equations are all parametrizations of the unit circle $|z| = 1$ oriented positively:

1. $\gamma(t) = e^{it}, \quad 0 \leq t \leq 2\pi.$
2. $\gamma(t) = e^{2\pi it}, \quad 0 \leq t \leq 1.$
3. $\gamma(t) = e^{i\pi t/2}, \quad 0 \leq t \leq 4.$

Definition 4.1.5 Length of a curve

Let $z : [a, b] \longrightarrow \mathbb{C}$ be a differentiable and γ curve prametrized by,

$$z(t) = x(t) + i\,y(t).$$

Then the length $L(\gamma)$ of the path γ is given by

$$L(\gamma) := \int_a^b |\gamma'(t)|\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_a^b \left|\frac{dz}{dt}\right|\,dt,$$

for any parameterization $\gamma(t)$ with $a \leq t \leq b$.

Theorem 4.1.6

If f is continuous on a smooth curve C given by the parameterization

$$z(t) = x(t) + iy(t), \quad a \leq t \leq b,$$

then

$$\int_C f(z)\,dz = \int_a^b f\big(z(t)\big)\,z'(t)\,dt.$$

Example 4.1.7

$$\text{Evaluate} \quad \int_C \overline{z}\,dz, \qquad C: \; x = 3t, \; y = t^2, \; -1 \leq t \leq 4.$$

First of all, the parameterization of C is

$$z(t) = 3t + it^2, \qquad -1 \leq t \leq 4.$$

Hence

$$\overline{z(t)} = 3t - it^2, \qquad z'(t) = 3 + 2it.$$

Therefore

$$\int_C \overline{z}\,dz = \int_{-1}^4 (3t - it^2)(3 + 2it)\,dt = \int_{-1}^4 \left(2t^3 + 9t + 3it^2\right)dt.$$

Compute the anti-derivative:

$$\int \left(2t^3 + 9t + 3it^2\right) dt = \frac{t^4}{2} + \frac{9}{2}t^2 + it^3.$$

Evaluate from $t = -1$ to $t = 4$:

$$\left[\tfrac{1}{2}t^4 + \tfrac{9}{2}t^2 + it^3\right]_{-1}^{4} = (128 + 72 + 64i) - \left(\tfrac{1}{2} + \tfrac{9}{2} - i\right).$$

$$= (200 + 64i) - (5 - i) = 195 + 65i.$$

Let

$$f(z) = z^2, \qquad g(z) = z, \qquad \gamma_1(t) = t + it, \qquad \gamma_2(t) = t + it^2, \quad 0 \le t \le 1.$$

Note that the two paths are different, but they have the same starting point $z = 0$ and the same endpoint $z = 1 + i$.

Example 4.1.8

(a)

$$\int_{\gamma_1} f(z)\, dz = \int_0^1 \left((t + it)^2\right)(1 + i)\, dt = \int_0^1 t^2(1 + i)^2(1 + i)\, dt = \tfrac{2}{3}(-1 + i).$$

(b)

$$\int_{\gamma_2} f(z)\, dz = \int_0^1 (t + it^2)^2(1 + 2it)\, dt = \tfrac{2}{3}(-1 + i).$$

(c)

$$\int_{\gamma_1} g(z)\, dz = \int_0^1 (t - it)(1 + i)\, dt = 1.$$

(d)

$$\int_{\gamma_2} g(z)\, dz = \int_0^1 (t - it^2)(1 + 2it)\, dt = 1 + \tfrac{i}{3}.$$

Remark 4.1.9. In above example, observe that

$$\int_{\gamma_1} f = \int_{\gamma_2} f \quad \text{but} \quad \int_{\gamma_1} g \neq \int_{\gamma_2} g.$$

We shall see later that the reason is that f is analytic, whereas g is not.
If f is analytic in a domain containing a curve C joining two points z_0 and z_1, then

$$\int_C f(z)\, dz$$

is independent of the choice of the curve C.
If f is not analytic, then

$$\int_C f(z)\, dz$$

does depend on the choice of the curve joining z_0 and z_1.

The following properties of contour integrals are analogous to the properties of real line integrals

Proposition 4.1.10 Properties of Complex Line Integrals

Suppose the functions f and g are continuous in a domain D, and let C be a smooth curve lying entirely in D. Then:

(i) $\int_C kf(z)\,dz = k\int_C f(z)\,dz, \quad k \in \mathbb{C}$ (a constant).

(ii) $\int_C \big(f(z)+g(z)\big)\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz.$

(iii) $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$, where C consists of the smooth curves C_1 and C_2 joined end to end.

(iv) $\int_{C^-} f(z)\,dz = -\int_C f(z)\,dz$, where $-C$ denotes the curve having the opposite orientation of C.

(v) If C consists of n connected contours with endpoints from z_1 to z_2 for C_1, from z_2 to z_3 for C_2, $\ldots$, from z_n to z_{n+1} for C_n, then we have

$$\int_C f(z)\,dz = \sum_{j=1}^{n}\int_{C_j} f(z)\,dz.$$

Theorem 4.1.11

If f is continuous on a smooth curve C and if

$$|f(z)| \leq M \quad \text{for all } z \in C,$$

then

$$\left|\int_C f(z)\,dz\right| \leq ML,$$

where L is the length of C.

Proof. We have

$$I = \left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right|. \tag{2.4.16}$$

From real variables we know that, for $a \leq t \leq b$,

$$\left|\int_a^b G(t)\,dt\right| \leq \int_a^b |G(t)|\,dt,$$

hence

$$I \leq \int_a^b \big|f(z(t))\big|\,\big|z'(t)\big|\,dt.$$

(This can be shown by using Eq. (2.4.19) below, with the triangle inequality.)

Since $|f|$ is bounded on C, i.e. $|f(z)| \leq M$ on C, where M is a constant, then

$$I \leq M \int_a^b |z'(t)|\, dt.$$

However, because

$$\begin{aligned} |z'(t)|\, dt &= |x'(t) + iy'(t)|\, dt \\ &= \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = ds, \end{aligned}$$

where s represents arc length along C. □

Example 4.1.12

We want to find an upper bound for

$$\int_C \frac{e^z}{z+1}\, dz,$$

where C is the circle $|z| = 4$ First, the length L (circumference) of the circle of radius 4 is

$$L = 8\pi.$$

It follows for all points z on the circle that

$$|z+1| \geq |z| - 1 = 4 - 1 = 3.$$

Thus

$$\left|\frac{e^z}{z+1}\right| \leq \frac{|e^z|}{|z|-1} = \frac{|e^z|}{3}. \tag{4.2}$$

In addition,

$$|e^z| = |e^x(\cos y + i \sin y)| = e^x.$$

For points on the circle $|z| = 4$, the maximum of $x = \Re(z)$ can be 4, and so (4.2) yields

$$\left|\frac{e^z}{z+1}\right| \leq \frac{e^4}{3}.$$

From the ML-inequality we have

$$\left|\int_C \frac{e^z}{z+1}\, dz\right| \leq 8\pi \cdot \frac{e^4}{3}.$$

$$\boxed{\left|\int_C \frac{e^z}{z+1}\, dz\right| \leq \frac{8\pi}{3} e^4}$$

Note 4.1.13

Note that :

(a) We define the positive direction(or positive orientation) on a curve C to be the direction on the curve corresponding to increasing values of the parameter t.

(b) If Csimple closed curve, then the positive orientation is corresponds to the counterclockwise direction or the direction that a person must walk on C in order to keep the interior of C to the left

(c) If C is a *closed contour* (that is, the endpoints of C coincide), then the contour integral is denoted by

$$\oint_C f(z)\,dz.$$

(d) If C has an orientation, the opposite curve, that is, a curve with opposite orientation, is denoted by C^-.

The fundamental theorem of calculus yields the following result.

Theorem 4.1.14 Fundamental Theorem of Calculus

Suppose $F(z)$ is an analytic function and that $f(z) = F'(z)$ is continuous in a domain D. Then for a contour C lying in D with endpoints z_1 and z_2,

$$\int_C f(z)\,dz = F(z_2) - F(z_1).$$

Proof. Using the definition of the integral, the chain rule, and assuming for simplicity that $z'(t)$ is continuous (otherwise add integrals separately over smooth arcs), we have

$$\int_C f(z)\,dz = \int_C F'(z)\,dz = \int_a^b F'(z(t))\,z'(t)\,dt = \int_a^b \frac{d}{dt}[F(z(t))]\,dt.$$

Thus,

$$\int_C f(z)\,dz = F(z(b)) - F(z(a)) = F(z_2) - F(z_1). \quad \square$$

As a consequence of Theorem 2.4.1, for closed curves we have

$$\oint_C f(z)\,dz = \oint_C F'(z)\,dz = 0 \tag{2.4.9}$$

where $\oint_C$ denotes a closed contour C (that is, the endpoints are equal). $\square$

Definition 4.1.15 Simply and Multiply Connected Domains

We say:

(a) That a domain D is **simply connected** if every simple closed contour C lying entirely in D can be shrunk to a point without leaving D, another way, a simply connected domain has no "holes" in it. The points within a circle, square, and polygon are examples of a simply connected domain.

(b) Domain that is not simply connected is called a **multiply connected domain**. An annulus (doughnut) is not simply connected.

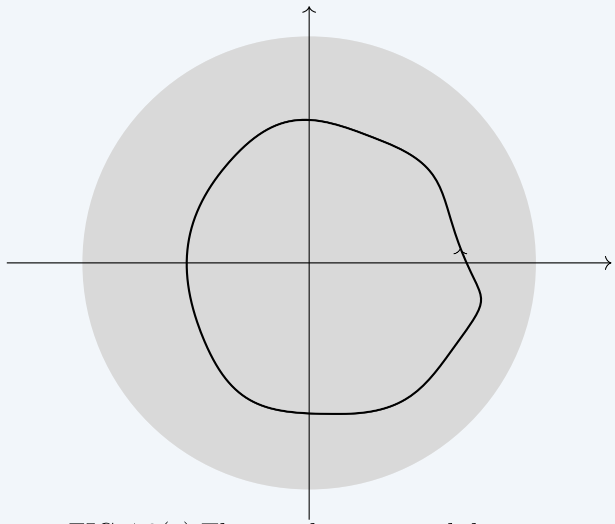

FIG 4.2(a):The simply connected domain

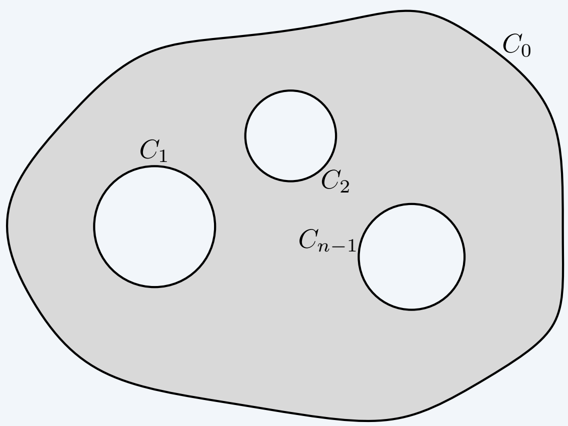

FIG4.3(b) :Multiple connect domain

4.2 Cauchy's theorem.

1825 the French mathematician Louis-Augustin Cauchy proved one the most important theorems in complex analysis.

Theorem 4.2.1 Cauchy Theorem

Suppose that a function f is analytic in a simply connected domain D and that f' is continuous in D. Then for every simple closed contour C in D,

$$\int_C f(z)\,dz = 0.$$

In 1883 the French mathematician Edouard Goursat proved that the assumption of continuity of f' is not necessary to reach the conclusion of Cauchy's theorem. The resulting modified version of Cauchy's theorem is known today as the **Cauchy-Goursat theorem**.

Theorem 4.2.2 Cauchy–Goursat Theorem

Suppose that a function f is analytic in a simply connected domain D. Then for every simple closed contour C in D,

$$\int_C f(z)\,dz = 0. \tag{4.3}$$

Now, we introduce Cauchy-Goursat Theorem for Multiply Connected Domains. To do this let consider two simple closed contours C_1 and C_2 such that all the points of C_2 lie interior to C_1 in domain D multiple connect where . To begin, that C_1 surrounds the "hole" in the domain and is interior to C see FIG. 4.5

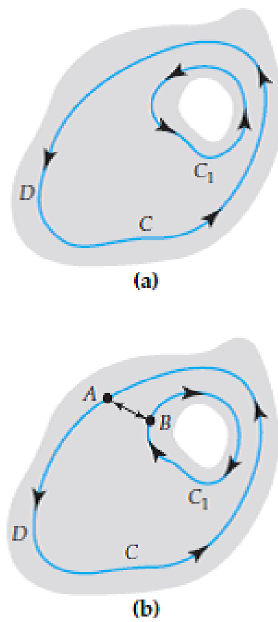

FIG.4.5 Deformation of Contour inMultiple Connect Domain.;

Theorem 4.2.3 Cauchy-Goursat Theorem for Multiply Connected Domains

Consider two simple closed contours C and C_1 such that all the points of C_1 lie interior to C. If a function f is analytic not only on C and C_1 but also at all points of the doubly connected domain D whose boundaries are C and C_1, then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz. \tag{4.4}$$

Proof. By introducing the crosscut AB shown in Figure5.(b), the region bounded between the curves is now simply connected. observe that the integral from A to B has the opposite value of the integral from B to A, and so from (4.3). we have

$$\int_C f(z)\,dz + \int_{AB} f(z)\,dz + \int_{-AB} f(z)\,dz + \int_{C_1} f(z)\,dz = 0$$

or

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz.$$

□

Example 4.2.4

(1) Find the value of the integral

$$\oint_C \frac{dz}{z-i},$$

where C is the contour shown in black in the FIG.4

(2) evaluated the following integral

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i, & n=1, \\ 0, & n \neq 1. \end{cases} \tag{4.5}$$

In view of (4.4), we choose the more convenient circular contour C_1 drawn in color in the figure. By taking the radius of the circle to be$r = 1$, we are guaranteed that $C1$ lies within C. In other words, C_1 is the circle On the circle $|z - i| = 1$, we can parameterize C_1 as

$$z = i + e^{it}, \quad 0 \leq t \leq 2\pi.$$

Then

$$z - i = e^{it}, \qquad dz = ie^{it}\,dt.$$

Hence

$$\oint_C \frac{dz}{z-i} = \oint_{C_1} \frac{dz}{z-i} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

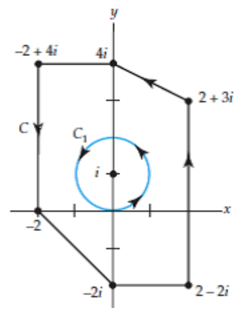

FIG.4.6

2. The fact that the integral in (4.5) is zero when $n \neq 1$ follows only partially from the Cauchy-Goursat theorem. when $n = 1$, let suppose that the circle γ of centerz_0 and radius R so parameterized by $z(t) = Re^{it}$, $0 \leq t \leq 2\pi$ and lying in C. then

$$\oint_C \frac{dz}{z-i} = \oint_\gamma \frac{dz}{z-i} = \int_0^{2\pi} \frac{iRe^{it}}{Re^{it}}\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

Thus

Theorem 4.2.5 Cauchy-Goursat Theorem for Multiply Connected Domains (generalized)

Suppose $C, C_1, \ldots, C_n$ are simple closed curves with a positive orientation such that $C_1, C_2, \ldots, C_n$ are interior to C but the regions interior to each C_k, $k = 1, 2, \ldots, n$, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k = 1, 2, \ldots, n$, then

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz.$$

Example 4.2.6

Evaluate

$$\oint_C \frac{dz}{z^2+1}, \qquad C: \ |z| = 4.$$

In this case the denominator of the integrand factors as

$$z^2 + 1 = (z-i)(z+i).$$

Consequently, the integrand $\dfrac{1}{z^2+1}$ is not analytic at $z = i$ and at $z = -i$. Both of these points lie within the contour C. Using partial fraction decomposition,

$$\frac{1}{z^2+1} = \frac{1}{2i}\frac{1}{z-i} - \frac{1}{2i}\frac{1}{z+i}.$$

We now surround the points $z = i$ and $z = -i$ by circular contours C_1 and C_2, respectively, that lie entirely within C. Specifically, the choice $|z-i| = \frac{1}{2}$ for C_1 and $|z+i| = \frac{1}{2}$ for C_2 will suffice (see Figure 4.7). From Theorem 4.2.5 we can write

$$\oint_C \frac{dz}{z^2+1} = \frac{1}{2i}\oint_{C_1}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)dz + \frac{1}{2i}\oint_{C_2}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)dz.$$

That is,

$$\oint_C \frac{dz}{z^2+1} = \frac{1}{2i}\oint_{C_1}\frac{dz}{z-i} - \frac{1}{2i}\oint_{C_1}\frac{dz}{z+i} + \frac{1}{2i}\oint_{C_2}\frac{dz}{z-i} - \frac{1}{2i}\oint_{C_2}\frac{dz}{z+i}. \tag{4.6}$$

Because $\frac{1}{z+i}$ is analytic on C_1 and at each point in its interior, and because $\frac{1}{z-i}$ is analytic on C_2 and at each point in its interior, it follows from (4.3) that the second and third integrals in (4.6) are zero. Moreover, it follows from (4.5) with $n = 1$ that

$$\oint_{C_1}\frac{dz}{z-i} = 2\pi i \qquad \text{and} \qquad \oint_{C_2}\frac{dz}{z+i} = 2\pi i.$$

Thus, equation (4.6) becomes

$$\oint_C \frac{dz}{z^2+1} = \pi - \pi = 0.$$

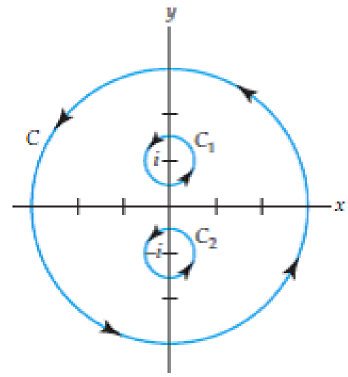

FIG.4.7

4.3 Cauchy's Integral Formulas for derivatives and Their Consequences.

In the last section, we saw the importance of the Cauchy-Goursat theorem in the evaluation of contour integrals. In this section we are going to examine several more consequences of the Cauchy-Goursat theorem. Unquestionably, the most significant of these is the following result:

(a) *The value of a analytic function f at any point z_0 in a simply connected domain can be represented by a contour integral.*

(b) *An analytic function f in a simply connected domain possesses derivatives of all orders.*

Theorem 4.3.1

Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then for any point z_0 within C,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz. \tag{4.7}$$

Example 4.3.2

Using Cauchy's Integral Formula Evaluate

$$\oint_C \frac{z^2 - 4z + 4}{z + i}\, dz,$$

where C is the circle $|z| = 2$.
First, we identify $f(z) = z^2 - 4z + 4$ and $z_0 = -i$ as a point within the circle C. Next, we observe that f is analytic at all points within and on the contour C. Thus, by the Cauchy integral formula, we obtain

$$\oint_C \frac{z^2 - 4z + 4}{z + i}\, dz = 2\pi i\, f(-i).$$

Since

$$f(-i) = (-i)^2 - 4(-i) + 4 = -1 + 4i + 4 = 3 + 4i,$$

we get

$$\oint_C \frac{z^2-4z+4}{z+i}\,dz = 2\pi i\,(3+4i).$$

Finally,

$$2\pi i\,(3+4i) = 6\pi i + 8\pi i^2 = 6\pi i - 8\pi = \pi(-8+6i).$$

Theorem 4.3.3 Mean Value Formula

Suppose f is analytic on a simply connected domain D and that the disk $C(z_0;r) \subset D$. Then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{it}\right)\,dt.$$

Proof. Let $\gamma(t) = z_0 + re^{it}$, with $0 \le t \le 2\pi$, be a parametrization of the circle $C(z_0;r)$. By Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_{C(z_0;r)} \frac{f(z)}{z-z_0}\,dz.$$

Substituting the parametrization $z = z_0 + re^{it}$ gives

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{it})}{z_0+re^{it}-z_0}\,(rie^{it}dt).$$

Simplifying,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})\,dt.$$

□

Next, we shall now build on Theorem 4.3.1 by using it to prove that the values of the derivatives $f^{(n)}(z_0), n = 1, 2, 3, \ldots$ of an analytic function are also given by a integral formula. This second integral formula is similar to (4.7) and is known by the name **Cauchy's integral formula for derivatives**.

Theorem 4.3.4 Cauchy's Integral Formula for Derivatives

Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then for any point z_0 within C, The n-th derivative of f at z_0 is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz. \tag{4.8}$$

Example 4.3.5

U sing Cauchy's Integral Formula for Derivatives
Evaluate

$$\oint_C \frac{z+1}{z^4+2iz^3}\,dz, \qquad C: \ |z| = 1.$$

Inspection of the integrand shows that it is not analytic at $z = 0$ and $z = -2i$, but only $z = 0$ lies within the closed contour. By writing the integrand as

$$\frac{z+1}{z^4+2iz^3} = \frac{z+1}{(z+2i)\,z^3},$$

we can identify $z_0 = 0$, $n = 2$, and

$$f(z) = \frac{z+1}{z+2i}.$$

The second derivative of f is

$$f''(z) = \frac{2-4i}{(z+2i)^3}.$$

Therefore,

$$f''(0) = \frac{2-4i}{(2i)^3} = \frac{2-4i}{-8i} = \frac{2i-1}{4i}.$$

By Cauchy's integral formula for derivatives,

$$\oint_C \frac{z+1}{z^4+2iz^3}\,dz = \frac{2\pi i}{2!} f''(0).$$

Thus,

$$\oint_C \frac{z+1}{z^4+2iz^3}\,dz = \pi i \cdot \frac{2i-1}{4i} = -\frac{\pi}{4} + \frac{\pi}{2} i.$$

Corollary 4.3.6

Suppose f is analytic in a simply connected domain $D \subset \mathbb{C}$. Then f possesses derivatives of all orders at every point $z \in D$. Moreover the derivatives $f', f'', f^{(3)}, \ldots$ are analytic functions on D.

4.4 Cauchy's inequality.

The following inequality derived from the Cauchy integral formula for derivatives.

Theorem 4.4.1 Cauchy's Inequality

Suppose that f is analytic in a simply connected domain D and let C be a circle defined by

$$|z - z_0| = r$$

that lies entirely in D. If $|f(z)| \leq M$ for all points z on C, then

$$\left|f^{(n)}(z_0)\right| \;\leq\; \frac{n!\,M}{r^n}.$$

Proof. Let f be analytic on an open set containing the closed disk $\{z : |z - z_0| \leq r\}$ and suppose $|f(z)| \leq M$ for $|z - z_0| = r$. By Cauchy's formula for derivatives,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz,$$

where C is the circle $|z - z_0| = r$ oriented positively. For $z \in C$, we have

$$\left|\frac{f(z)}{(z-z_0)^{n+1}}\right| = \frac{|f(z)|}{|z-z_0|^{n+1}} \leq \frac{M}{r^{n+1}}.$$

Applying the ML-inequality (since the length of C is $2\pi r$), it follows that

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} \cdot (2\pi r) = \frac{n!M}{r^n}.$$

□

4.5 Liouville's theorem-Morera's theorem

Next, using theorem4.4.1 we prove the next result. Although it bears the name "Liouville's theorem," it probably was first proved by Cauchy. The gist of the theoremis that an entire function f, one that is analytic for all z, cannot be bounded unless f itself is a constant.

Theorem 4.5.1 Théorème de Liouville

Every entire and bounded function is constant.

Proof. Let $z_0 \in \mathbb{C}$ be arbitrary. By Theorem 4.3.4 with $n = 1$, we have

$$|f'(z_0)| \; \leq \; \frac{M}{r}$$

for every $r > 0$. By taking $r \longrightarrow \infty$, it follows that $|f'(z_0)| = 0$. Hence $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$. Therefore, f is constant by Theorem 2.4.7. □

Next, theorem 4.5.1 enables us to establish a result usually learned but never proved in elementary algebra.

Theorem 4.5.2 Fundamental Theorem of Algebra

Every non-constant polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \qquad n > 0,$$

with complex coefficients has at least one zero in $\mathbb{C}$.

Proof. Suppose, for the sake of contradiction, that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then the reciprocal

$$f(z) = \frac{1}{p(z)}$$

is an entire function.

For large $|z|$ we can factor:

$$|f(z)| = \frac{1}{|a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0|} = \frac{1}{|z|^n \left|a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right|}.$$

As $|z| \longrightarrow \infty$, the denominator grows like $|a_n||z|^n$, so $|f(z)| \longrightarrow 0$. Thus f is bounded on $\mathbb{C}$.

By Liouville's Theorem, f must be constant. Hence $p(z) = 1/f(z)$ is constant, which contradicts the assumption that p is a non-constant polynomial.

Therefore, our assumption was false, and there must exist at least one $z \in \mathbb{C}$ such that $p(z) = 0$. □

Note 4.5.3

if $p(z)$ is a nonconstant polynomial of degree$n$, then $p(z) = 0$ has exactly n roots (counting multiple roots).

Corollary 4.5.4

Let

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \qquad n \geq 1,$$

be a polynomial with coefficients $a_j \in \mathbb{C}$. Then there exist numbers $z_1, z_2, \ldots, z_n \in \mathbb{C}$ such that

$$P(z) = (z - z_1)(z - z_2)\cdots(z - z_n).$$

the next theorem enshrined the name of the Italian mathematician Giacinto Morera forever in texts on complex analysis. Morera's theorem, which gives a sufficient condition for analyticity, is often taken to be the converse of the Cauchy-Goursat theorem.

Theorem 4.5.5 Morera's Theorem

If f is continuous in a simply connected domain D and

$$\int_C f(z)\,dz = 0$$

for every closed contour C in D, then f is analytic in D.

Proof. By the hypotheses of continuity of f and the fact that

$$\int_C f(z)\,dz = 0$$

for every closed contour C in D, we conclude that the integral

$$\int_C f(z)\,dz$$

is independent of the path between two points of D. we saw that the function F defined by

$$F(z) = \int_{z_0}^{z} f(s)\,ds,$$

where s denotes a complex variable, z_0 is a fixed point in D, and z is any point in D, satisfies,

$$F'(z) = f(z). \tag{4.9}$$

Hence F is analytic in D. In addition, by corollary 4.3.6, $F'(z)$ is analytic. Since, in view of Eq.(4.9) we conclude that f is analytic in D. □

Theorem 4.5.6 Maximum modulus principle

If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point $z_0$in the domain such that $|f(z)| \leq |f(z_0)|$ for all points z in it.

Corollary 4.5.7 Maximum Modulus Principle

Let $D \subset \mathbb{C}$ be a bounded domain (open and connected), and let $f : D \longrightarrow \mathbb{C}$ be continuous on $\overline{D}$ and analytic on D. Then

$$\max_{z \in D} |f(z)| \; = \; \max_{z \in \partial D} |f(z)|.$$

In other words, $|f|$ attains its maximum on the boundary ∂D and nowhere else inside D.

Example 4.5.8

Find the maximum of $|f(z)|$ for $f(z) = 2z + 5i$ on the disk $\{|z| \leq 2\}$. By the Maximum Modulus Principle, the maximum of $|f|$ on $\{|z| \leq 2\}$ is attained on the boundary $|z| = 2$. For $z = x + iy$, we compute

$$|2z + 5i|^2 = (2z + 5i)(2\overline{z} - 5i) = 4|z|^2 + 10i(\overline{z} - z) + 25 = 4|z|^2 + 20\,\Im z + 25.$$

By the Maximum Modulus Principle we have

$$\max_{|z| \leq 2} |2z + 5i| = \max_{|z|=2} |2z + 5i| = \max_{|z|=2} \sqrt{4|z|^2 + 20\,\Im z + 25}\,.$$

Since $|z| = 2$, this gives

$$\sqrt{4 \cdot 2^2 + 20 \cdot 2 + 25} = \sqrt{81} = 9.$$

4.6 Exercises Set

Exercise 4.6.1

Evaluate the complex integral:

$$\int_{\Gamma} (z^2 + 3z)\, dz$$

for the following curves:.

(1) The segment from $(0,0)$ to $(0,1)$

(2) The quarter circle from $(2,0)$ to $(0,-2)$ centered at the origin.

Solution. **(1)** The segment from $(0,0)$ to $(0,1)$

Parameterize the path as:

$$z(t) = it, \quad t \in [0,1], \quad dz = i\, dt$$

Then:

$$z^2 + 3z = (it)^2 + 3(it) = -t^2 + 3it$$

So the integral becomes:

$$\int_0^1 (-t^2 + 3it) \cdot i\, dt = \int_0^1 (-it^2 - 3t)\, dt$$

$$= -i \int_0^1 t^2\, dt - 3\int_0^1 t\, dt = -i \cdot \frac{1}{3} - 3 \cdot \frac{1}{2} = -\frac{i}{3} - \frac{3}{2}$$

$$\boxed{\int_{\Gamma_1} (z^2 + 3z)\, dz = -\frac{3}{2} - \frac{i}{3}}$$

(2) The quarter circle from $(2,0)$ to $(0,-2)$ centered at the origin. Parameterize the quarter-circle in the complex plane as:

$$z(t) = 2e^{it}, \quad t \in \left[0, \frac{3\pi}{2}\right], \quad dz = 2ie^{it} dt$$

Then:

$$z^2 + 3z = (2e^{it})^2 + 3(2e^{it}) = 4e^{2it} + 6e^{it}$$

The integral becomes:

$$\int_{\Gamma_2} (z^2 + 3z)\, dz = \int_0^{\frac{3\pi}{2}} (4e^{2it} + 6e^{it}) \cdot 2ie^{it} dt = 2i \int_0^{\frac{3\pi}{2}} (4e^{3it} + 6e^{2it})\, dt$$

Compute each integral:

$$\int_0^{\frac{3\pi}{2}} e^{3it} dt = \frac{e^{\frac{9\pi i}{2}} - 1}{3i} = \frac{e^{\frac{\pi i}{2}} - 1}{3i} = \frac{i-1}{3i}$$

$$\int_0^{\frac{3\pi}{2}} e^{2it} dt = \frac{e^{3\pi i} - 1}{2i} = \frac{-1-1}{2i} = \frac{-2}{2i} = \frac{-1}{i} = i$$

So:

$$2i\left[4 \cdot \frac{i-1}{3i} + 6 \cdot i\right] = 2i\left(\frac{4(i-1)}{3i} + 6i\right) = 2i\left(\frac{4}{3} + \frac{4}{3}i + 6i\right) = 2i\left(\frac{4}{3} + \frac{22}{3}i\right)$$

Now multiply:

$$2i \cdot \left(\frac{4}{3} + \frac{22}{3}i\right) = \frac{8i}{3} + \frac{44i^2}{3} = \frac{8i}{3} - \frac{44}{3}$$

$$\boxed{\int_{\Gamma_2} (z^2 + 3z)\, dz = -\frac{44}{3} + \frac{8i}{3}}$$

Exercise 4.6.2

Evaluate the following complex line integral:

$$\int_{\Gamma} (2z + 3|z|^2)\, dz$$

where Γ is curves indected in above Figure.

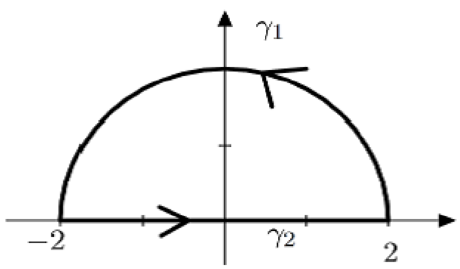

FIG.curve of integration;

Solution. Recall that Γ_1 is the semicircle centered at $z_0 = 0$ with radius $r = 2$, and Γ_2 is the line segment joining -2 to 2.

Thus, we have the following parametrizations (other parametrizations are possible):

$$\Gamma_1(t) = 2e^{it}, \quad 0 \le t \le \pi, \qquad \Gamma_2(t) = t, \quad -2 \le t \le 2.$$

Applying the definition of complex integrals, we get

$$\int_{\Gamma} (2z + 3|z|^2)\, dz = \int_{\Gamma_1} (2z + 3|z|^2)\, dz + \int_{\Gamma_2} (2z + 3|z|^2)\, dz.$$

For Γ_1:

$$\int_{\Gamma_1} (2z + 3|z|^2)\, dz = \int_0^{\pi} \left(4e^{it} + 12\right)\left(2ie^{it}\right) dt = 4i \int_0^{\pi} \left(2 + 6e^{it}\right) dt.$$

For Γ_2:

$$\int_{\Gamma_2} (2z + 3|z|^2)\, dz = \int_{-2}^{2} \left(2t + 3t^2\right)(1)\, dt = 16.$$

Hence,

$$\int_{\Gamma} (2z + 3|z|^2)\, dz = 4i \int_0^{\pi} \left(2 + 6e^{it}\right) dt + 16 = 8\pi i - 32.$$

Exercise 4.6.3

Compute the contour integral

$$\oint_C \frac{z^3+3}{z\,(z-i)^2}\,dz,$$

where C is the curves shown in the figure below.

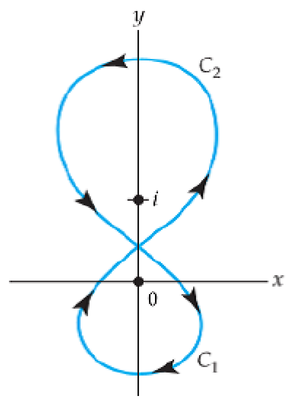

FIG.curve of integration;

Solution. We have

$$\oint_C \frac{z^3+3}{z(z-i)^2}\,dz = -\oint_{C_1} \frac{z^3+3}{(z-i)^2 z}\,dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}\,dz = -I_1 + I_2.$$

To compute I_1, we choose

$$f_1(z) = \frac{z^3+3}{(z-i)^2}, \quad z_0 = 0,$$

and we obtain

$$I_1 = \oint_{C_1} \frac{z^3+3}{(z-i)^2 z}\,dz = \oint_{C_1} \frac{f_1(z)}{z}\,dz = 2\pi i\, f_1(0) = -6\pi i.$$

To compute I_2, we choose

$$f_2(z) = \frac{z^3+3}{z}, \quad z_0 = i,$$

and we find

$$I_2 = \oint_{C_2} \frac{z^3+3}{z(z-i)^2}\,dz = \oint_{C_2} \frac{f_2(z)}{(z-i)^2}\,dz = 2\pi i\, f_2'(i) = 2\pi i(3+2i) = -4\pi + 6\pi i.$$

Finally, we obtain

$$\oint_C \frac{z^3+3}{z(z-i)^2}\,dz = -I_1 + I_2 = -4\pi + 12\pi i.$$

Exercise 4.6.4

Compute the following contour integrals using the Cauchy integral formulas. Provide only the final value of each integral (no intermediate steps required).

1.
$$\oint_{|z|=4} \frac{e^z}{z^2\left(z^2+\pi^2\right)}\,dz.$$

2.
$$\oint_{|z+1|=1} \frac{e^{3z}}{z^2+1}\,dz.$$

3.
$$\oint_{|z|=3} \frac{z^2+4}{z^3+2z^2+2z}\,dz.$$

Solution. 1) We notice that the function
$$\frac{e^z}{z^2(z^2+\pi^2)}$$
has three singularities: 0, $i\pi$, and $-i\pi$.

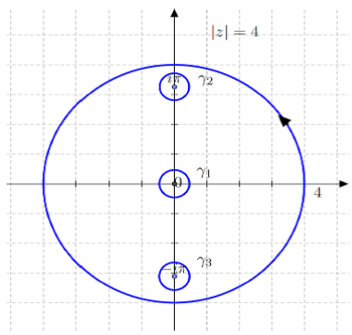

FIG.curve of integration;

Thus,
$$\oint_{|z|=4} \frac{e^z}{z^2(z^2+\pi^2)}\,dz = \oint_{\Gamma_1} \frac{e^z}{z^2(z^2+\pi^2)}\,dz + \oint_{\Gamma_2} \frac{e^z}{z^2(z^2+\pi^2)}\,dz + \oint_{\Gamma_3} \frac{e^z}{z^2(z^2+\pi^2)}\,dz.$$

That is,
$$\oint_{|z|=4} \frac{e^z}{z^2(z^2+\pi^2)}\,dz = \oint_{\Gamma_1} \frac{f(z)}{z^2}\,dz + \oint_{\Gamma_2} \frac{g(z)}{z-i\pi}\,dz + \oint_{\Gamma_3} \frac{h(z)}{z+i\pi}\,dz,$$
with
$$f(z) = \frac{e^z}{z^2+\pi^2}, \quad g(z) = \frac{e^z}{z^2(z+i\pi)}, \quad h(z) = \frac{e^z}{z^2(z-i\pi)}.$$

The functions f, g, and h are holomorphic inside $\Gamma_1, \Gamma_2, \Gamma_3$, respectively. Therefore, by Cauchy's integral formulas, we obtain
$$\oint_{|z|=4} \frac{e^z}{z^2(z^2+\pi^2)}\,dz = 2\pi i f'(0) + 2\pi i g(i\pi) + 2\pi i h(-i\pi),$$
$$= 2\pi i\left(\frac{1}{\pi^2}\right) + 2\pi i\left(\frac{1}{2\pi^3 i}\right) + 2\pi i\left(\frac{-1}{2\pi^3 i}\right) = \frac{2i}{\pi}.$$

2) We notice that the function
$$\frac{e^{3z}}{z^2+1}$$

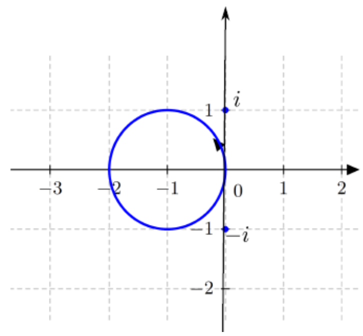

FIG.curve of integration;

is analytic inside the contour $|z+1|=1$.

Hence, by Cauchy's theorem,

$$\oint_{|z+1|=1} \frac{e^{3z}}{z^2+1}\,dz = 0.$$

3) The function

$$\frac{z^2+4}{z^3+2z^2+2z}$$

has three singularities: 0, $z_1=-1+i$, and $z_2=-1-i$.

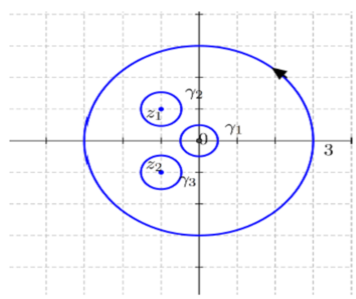

FIG.curve of integration;

Thus,

$$\oint_{|z|=3} \frac{z^2+4}{z^3+2z^2+2z}\,dz = \oint_{\Gamma_1} \frac{f(z)}{z}\,dz + \oint_{\Gamma_2} \frac{g(z)}{z-z_1}\,dz + \oint_{\Gamma_3} \frac{h(z)}{z-z_2}\,dz,$$

with

$$f(z) = \frac{z^2+4}{z^2+2z+2}, \quad g(z) = \frac{z^2+4}{z(z-z_2)}, \quad h(z) = \frac{z^2+4}{z(z-z_1)}.$$

The functions f, g, and h are holomorphic inside $\Gamma_1, \Gamma_2, \Gamma_3$, respectively. Thus, by Cauchy's integral formulas, we obtain

$$\oint_{|z|=3} \frac{z^2+4}{z^3+2z^2+2z}\,dz = 2\pi i\,(2) + 2\pi i\left(\frac{-i+2}{-i-1}\right) + 2\pi i\left(\frac{-i+2}{i-1}\right).$$

Hence,

$$\oint_{|z|=3} \frac{z^2+4}{z^3+2z^2+2z}\,dz = 2\pi i.$$

Exercise 4.6.5 Estimate of the integral (no evaluation)

Without evaluating the integral, show that

$$\left|\int_C \frac{dz}{z^2+z+1}\right| \le \frac{9\pi}{16},$$

where C is the arc of the circle $|z| = 3$ from $z = 3$ to $z = 3i$ lying in the first quadrant.

Solution. Let

$$I = \int_C \frac{dz}{z^2+z+1},$$

where C is the arc of the circle $|z| = 3$ from $z = 3$ to $z = 3i$ in the first quadrant. By the ML-estimate we have

$$|I| \le M \cdot L,$$

where L is the length of C and $M = \max_{z\in C}\left|\frac{1}{z^2+z+1}\right|$.

The arc C corresponds to the angle interval $[0, \frac{\pi}{2}]$ on the circle of radius 3, so its length is

$$L = 3 \cdot \frac{\pi}{2} = \frac{3\pi}{2}.$$

For every $z \in C$ we have $|z| = 3$, hence

$$|z^2 + z + 1| \ge |z|^2 - |z| - 1 = 9 - 3 - 1 = 5.$$

Therefore

$$M = \max_{z\in C} \frac{1}{|z^2+z+1|} \le \frac{1}{5}.$$

Combining these estimates gives

$$|I| \le \frac{1}{5} \cdot \frac{3\pi}{2} = \frac{3\pi}{10}.$$

Since $\frac{3\pi}{10} < \frac{9\pi}{16}$, we conclude

$$\left|\int_C \frac{dz}{z^2+z+1}\right| \le \frac{9\pi}{16},$$

as required.

Exercise 4.6.6

Let $f(z)$ be an entire function satisfying

$$|f(z)| \le |z|^2, \quad \text{for all } z \in \mathbb{C}.$$

Show that

$$f(z) = az^2$$

for some constant a with $|a| \le 1$.

Solution. Let f be entire and satisfy $|f(z)| \le |z|^2$ for all $z \in \mathbb{C}$. Then $f(z) = az^2$ for some constant a with $|a| \le 1$.

First of all, fix $z_0 \in \mathbb{C}$ and $R > 0$. By Cauchy's formula for derivatives,

$$f^{(3)}(z_0) = \frac{3!}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^4}\, dz.$$

For z on the circle $|z - z_0| = R$ we have $|z| \leq R + |z_0|$, hence

$$\left|\frac{f(z)}{(z-z_0)^4}\right| \leq \frac{|f(z)|}{R^4} \leq \frac{|z|^2}{R^4} \leq \frac{(R+|z_0|)^2}{R^4}.$$

Therefore the integral is bounded by

$$|f^{(3)}(z_0)| \leq \frac{3!}{2\pi} \cdot 2\pi R \cdot \frac{(R+|z_0|)^2}{R^4} = 6\frac{(R+|z_0|)^2}{R^3}.$$

Letting $R \longrightarrow \infty$ gives

$$|f^{(3)}(z_0)| \leq \lim_{R\longrightarrow\infty} 6\frac{(R+|z_0|)^2}{R^3} = 0,$$

so $f^{(3)}(z_0) = 0$. Since z_0 was arbitrary, $f^{(3)} \equiv 0$, hence f is a polynomial of degree at most 2. Thus

$$f(z) = az^2 + bz + c$$

for some constants $a, b, c \in \mathbb{C}$.

Using the growth condition $|f(z)| \leq |z|^2$ for all z, evaluate at $z = 0$ to get $|c| \leq 0$, so $c = 0$. Thus

$$|az^2 + bz| \leq |z|^2 \qquad \text{for all } z.$$

Divide by $|z|$ for $z \neq 0$ to obtain

$$|az + b| \leq |z| \qquad (z \neq 0).$$

Letting $z \longrightarrow 0$ yields $|b| \leq 0$, so $b = 0$. Hence $f(z) = az^2$. Finally, for any z

$$|a||z|^2 = |f(z)| \leq |z|^2,$$

so $|a| \leq 1$.

This completes the proof.

Exercise 4.6.7

Suppose that f analytic in $\big(D(0;R)\big)$ where $R > 1$. Prove that

$$\int_{-\pi}^{\pi} f(e^{it}) \cos^2\frac{t}{2}\, dt = \frac{\pi}{2}\big(2f(0) + f'(0)\big).$$

Solution. By Cauchy's Theorem in a Disc, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})e^{it}\, dt = \frac{1}{2\pi i}\int_{C(0;1)} f(z)\, dz = 0. \tag{1}$$

Combining (The Cauchy Integral Formula) and The Cauchy Integral Formula for Derivatives, we get

$$f(0) = \frac{1}{2\pi i}\int_{C(0;1)} \frac{f(z)}{z-0}\, dz = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\, dt, \tag{2}$$

$$f'(0) = \frac{1}{2\pi i}\int_{C(0;1)} \frac{f(z)}{(z-0)^2}\, dz = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})e^{-it}\, dt. \tag{3}$$

Hence we apply the identity

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \quad \text{and} \quad \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right),$$

to the sum of the expressions (1), (2) and (3) to conclude that

$$2f(0) + f'(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\,(2 + e^{it} + e^{-it})\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\left(\frac{e^{it}}{2} + \frac{e^{-it}}{2}\right)^2 dt = \frac{2}{\pi}\int_{-\pi}^{\pi} f(e^{it})\cos^2\frac{t}{2}\,dt.$$

This certainly implies our desired result and we complete the proof of the problem.

Chapter 5

Taylor series and Laurent series development

Chapter contents

5.1 Sequences and Series

In this section we explore the definitions of convergence and divergence for complex sequences and complex infinite series. In addition, we give some tests for convergence of infinite series

Definition 5.1.1 sequence

A sequence $z_{n n\in\mathbb{N}}$is a function whose domain is the set of positive integers and whose range is a subset of the complex numbers $\mathbb{C}$

Definition 5.1.2 convergence

If

$$\lim_{n\longrightarrow\infty} z_n = \ell,$$

we say that the sequence $\{z_n\}$ is **convergent**. In other words, z_n converges to the number ℓ if for each positive real number ϵ an N can be found such that $|z_n - \ell| < \epsilon$ whenever $n > N$.

Remark 5.1.3.

(a) If the limit exists, it is unique.

(b) Geometrically, this means that the open disk

$$D(\ell;\epsilon) = \{\, z \in \mathbb{C} : |z-\ell| < \epsilon \,\}$$

contains infinitely many elements of the sequence (z_n), while its complement contains only finitely many elements of the sequence (z_n).

(c) A sequence that is not convergent is said to be **divergent**.

Proposition 5.1.4

A sequence z_n converges to a complex number $\ell = \alpha + i\beta$ if and only if $Re(z_n)$ converges to $Re(\ell) = \alpha$ and $Im(z_n)$ converges to $Im(\ell) = \beta$.

Proof. By virtue of the inequalities

$$\sup\{\,|\operatorname{Re}(z_n - z)|,\ |\operatorname{Im}(z_n - z)|\,\} \ \le\ |z_n - z| \ \le\ |\operatorname{Re}(z_n - z)| + |\operatorname{Im}(z_n - z)|,$$

it is clear that $z_n \longrightarrow z$ if and only if $x_n \longrightarrow x$ and $y_n \longrightarrow y$. □

Example 5.1.5

Consider the sequence

$$z_n = \frac{3+ni}{n+2ni}.$$

From

$$z_n = \frac{3+ni}{n+2ni} = \frac{(3+ni)(n-2ni)}{n^2+4n^2} = \frac{2n^2+3n}{5n^2} + i\,\frac{n^2-6n}{5n^2},$$

we see that

$$\operatorname{Re}(z_n) = \frac{2n^2+3n}{5n^2} = \frac{2}{5} + \frac{3}{5n} \longrightarrow \frac{2}{5},$$

and

$$\operatorname{Im}(z_n) = \frac{n^2-6n}{5n^2} = \frac{1}{5} - \frac{6}{5n} \longrightarrow \frac{1}{5},$$

as $n \longrightarrow \infty$.
From Theorem 6.1, these results are sufficient for us to conclude that the given sequence converges to

$$a + ib = \frac{2}{5} + \frac{1}{5}i.$$

Definition 5.1.6 Series

1. The serie $\sum_{k=1}^{\infty} z_k$ converge if the sequence (S_n) converge and we write

$$\sum_{k=1}^{\infty} z_k := \lim_{n\longrightarrow\infty} S_n := L.$$

and we say that the series converges to L or that the sum of the series is L.

2. The series$\sum_{k=1}^{\infty} z_k$ diverges if the sequence (S_n) diverges.
3. The series $\sum_{k=1}^{\infty} z_k$ converges absolutely if the series of real numbers

$$\sum_{k=1}^{\infty} |z_k| \text{ converge}$$

Proposition 5.1.7 Geometric Series

Consider the geometric series

$$\sum_{k=0}^{\infty} z^k. \tag{5.1}$$

1. The series converges absolutely for $|z| < 1$ to the function

$$f(z) = \frac{1}{1-z}.$$

2. The series converges uniformly for $|z| \leq r < 1$ to the function

$$f(z) = \frac{1}{1-z}.$$

3. The series diverges for $|z| \geq 1$.

Theorem 5.1.9 A Necessary Condition for Convergence

If
$$\left(\sum_{k=1}^{\infty} z_k \text{ converges}\right) \Longrightarrow \left(\lim_{n\longrightarrow\infty} z_n = 0\right).$$

Theorem 5.1.10 A Test for Divergence

$$\lim_{n\longrightarrow\infty} z_n \neq 0 \Longrightarrow \sum_{k=1}^{\infty} z_k \text{ diverges.}$$

Example 5.1.11

The series
$$\sum_{k=1}^{\infty} \frac{ik+5}{k}$$
diverges, since
$$z_n = \frac{in+5}{n} \longrightarrow i \neq 0 \quad \text{as } n \longrightarrow \infty.$$
The geometric series
$$\sum_{n=0}^{\infty} z^n$$
diverges if $|z| \geq 1$, because even in the case when $\lim_{n\longrightarrow\infty} |z^n|$ exists, the limit is not zero.

In the next theorems, three of the most frequently used tests for convergence of infinite series are given.

Theorem 5.1.12 Comparison Test

If $|z_k| \leq M_k$ and the series $\sum_{k=1}^{\infty} M_k$ converges, then the series
$$\sum_{k=1}^{\infty} z_k$$
converges absolutely.

Theorem 5.1.13 Ratio Test

Suppose
$$\sum_{k=1}^{\infty} z_k$$

is a series of nonzero complex terms such that

$$\lim_{n\longrightarrow\infty}\left|\frac{z_{n+1}}{z_n}\right| = L.$$

(i) If $L<1$, then the series converges absolutely.

(ii) If $L>1$ or $L=\infty$, then the series diverges.

(iii) If $L=1$, the test is inconclusive.

Theorem 5.1.14 Root Test

Suppose

$$\sum_{k=1}^{\infty} z_k$$

is a series of complex terms such that

$$\lim_{n\longrightarrow\infty}\sqrt[n]{|z_n|} = L.$$

(i) If $L<1$, then the series converges absolutely.

(ii) If $L>1$ or $L=\infty$, then the series diverges.

(iii) If $L=1$, the test is inconclusive.

Example 5.1.15

Study the convergence of the following series:

(a) $\displaystyle\sum_{k=1}^{\infty}\frac{3^k}{k!}$ (b) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2+ik^{1/k}}$ (c) $\displaystyle\sum_{k=1}^{\infty}\frac{e^{ik}}{k^2}$ (d) $\displaystyle\sum_{k=1}^{\infty}\frac{3+2i}{(k+1)k}$.

we have

(a)

$$\left|\frac{z_{k+1}}{z_k}\right| = \frac{3^{k+1}}{(k+1)!}\cdot\frac{k!}{3^k} = \frac{3}{k+1} \longrightarrow 0,$$

hence the series converges absolutely.

(b) Since $k^{1/k}\longrightarrow 1$, we have

$$\frac{1}{k^2+ik^{1/k}} \longrightarrow \frac{1}{i} = -i \neq 0,$$

hence the series diverges.

(c)

$$\left|\frac{e^{ik}}{k^2}\right| = \frac{1}{k^2}.$$

Since $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges, the series

$$\sum_{k=1}^{\infty}\frac{e^{ik}}{k^2}$$

converges absolutely.

(d)

$$\frac{3+2i}{(k+1)k} \sim \frac{3+2i}{k^2}.$$

Since $\sum \frac{1}{k^2}$ converges, the series converges absolutely.

5.2 Complex sequence and convergence of complex series.

Before involving ourselves in complex series and the question of their convegence, we must examine something more elementary: the notion of a sequence of complex functions and the possible limit of such a sequence.

Definition 5.2.1 Convergence of a Complex Sequence

We say the sequence $f_n(z)$ converges to $f(z)$ on $\mathcal{R} \subset \mathbb{C}$ if

$$\lim_{n \longrightarrow \infty} f_n(z) = f(z).\text{(exists and is finite)} \tag{5.2}$$

This means that for each z, given $\varepsilon > 0$ there is an N depending on ε and z, such that whenever $n > N$ we have

$$|f_n(z) - f(z)| < \varepsilon. \tag{5.3}$$

If the limit does not exist (or is infinite), we say the sequence diverges for those values of z.

Definition 5.2.2 Convergence uniform of a Complex Sequence

We say that the sequence of functions $f_n(z)$, defined for z in a region R, *converges uniformly in* $\mathcal{R}$ if it is possible to choose N depending on ε only (and not on z): $N = N(\varepsilon)$ such that

$$|f_n(z) - f(z)| < \varepsilon. \tag{5.4}$$

whenever $n > N$.

Example 5.2.3

consider the sequence of functions

$$f_n(z) = \frac{1}{nz}, \quad n = 1, 2, \ldots \tag{5.5}$$

In the annular region $1 \leq |z| \leq 2$, the sequence of functions $\{f_n\}$ converges uniformly to zero. Namely, given $\varepsilon > 0$, for n sufficiently large we have

$$\begin{aligned} |f_n(z) - f(z)| &= \left|\frac{1}{nz} - 0\right| \\ &= \frac{1}{n|z|} < \varepsilon. \end{aligned}$$

Thus, the estimate $\frac{1}{n|z|} \leq \frac{1}{n}$ holds in the region $1 \leq |z| \leq 2$ for the first integer n such that $n > N(\varepsilon) = \frac{1}{\varepsilon}$. Therefore, the sequence is uniformly convergent to zero.

Definition 5.2.4 Convergence of Series

$\sum_{n=1}^{\infty} f_n(z)$ be an infinite series of an infinite sequence $\{f_n(z)\}$,and noting that a sequence of partial sums may be formed by

$$S_n(z) = \sum_{k=1}^{n} f_k(z). \tag{5.6}$$

then

$$\sum_{n=1}^{\infty} f_n(z) \text{ converge to } S(z) \Longleftrightarrow \lim_{n\longrightarrow\infty} S_n(z) = S(z) \tag{5.7}$$

Theorem 5.2.5 nth Term Test

The series $\sum_{k=1}^{\infty} f_k(z)$ diverges if

$$\lim_{k\longrightarrow\infty} f_k(z) \neq 0,$$

or equivalently, if

$$\lim_{k\longrightarrow\infty} |f_k(z)| \neq 0.$$

Remark 5.2.6. When we have uniform convergence, then

$$\lim_{n\longrightarrow\infty} \lim_{z\longrightarrow z_0} f_n(z) = \lim_{z\longrightarrow z_0} \lim_{n\longrightarrow\infty} f_n(z).$$

Theorem 5.2.7

Let the sequence of functions $f_n(z)$ be continuous for each integer n and let $f_n(z)$ converge to $f(z)$ uniformly in a region $\mathcal{R}$. Then $f(z)$ is continuous, and for any finite contour C inside $\mathcal{R}$

$$\lim_{n\longrightarrow\infty} \int_C f_n(z)\, dz = \int_C f(z)\, dz. \tag{5.8}$$

Proof. **(a)** First we prove the continuity of $f(z)$. For $z, z_0 \in R$, we write

$$f(z) - f(z_0) = f_n(z) - f_n(z_0) + f_n(z_0) - f(z_0) + f(z) - f_n(z),$$

and hence

$$|f(z) - f(z_0)| \leq |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)| + |f(z) - f_n(z)|.$$

Uniform convergence of $\{f_n(z)\}$ allows us to choose an N independent of z such that for $n > N$

$$|f_n(z_0) - f(z_0)| < \varepsilon/3 \quad \text{and} \quad |f(z) - f_n(z)| < \varepsilon/3.$$

Continuity of $f_n(z)$ allows us to choose $\delta > 0$ such that

$$|f_n(z) - f_n(z_0)| < \varepsilon/3 \quad \text{for } |z - z_0| < \delta.$$

Thus, for $n > N$ and $|z - z_0| < \delta$,

$$|f(z) - f(z_0)| < \varepsilon,$$

which establishes the continuity of $f(z)$.

(b) Because the function $f(z)$ is continuous, so it can be integrated. Given the continuity of $f(z)$ we shall prove Eq.(5.8) namely, for $\varepsilon > 0$ we must find N such that when $n > N$

$$\left| \int_C f_n(z)\, dz - \int_C f(z)\, dz \right| < \varepsilon. \tag{3.1.8}$$

□

An immediate consequence of Theorem 5.2.7 applies to series expansions.

Corollary 5.2.8

If the sequence of continuous partial sums converges uniformly, then we may integrate term by term. That is, for $f_n(z)$ continuous,

$$\sum_{n=1}^{\infty} \left(\int_C f_n(z)\, dz \right) = \int_C \left(\sum_{n=1}^{\infty} f_n(z) \right) dz.$$

Theorem 5.2.9

Suppose that (f_n) is a sequence of analytic functions on a domain D and that (f_n) converges uniformly to f on D. Then f is analytic on D.

Proof. Let γ be a closed curve in D. By Theorem 5.2.7, we have

$$\lim_{n \longrightarrow \infty} \int_\gamma f_n(z)\, dz = \int_\gamma f(z)\, dz.$$

But

$$\int_\gamma f_n(z)\, dz = 0 \quad \text{for every } n,$$

since each f_n is analytic. Hence

$$\int_\gamma f(z)\, dz = 0.$$

By Morera's theorem, it follows that f is analytic on D. □

Theorem 5.2.10 Weierstrass M-Test

Let $|f_k(z)| \le M_k$ in a region $\mathcal{R}$, with M_k constant. If the series $\sum_{k=1}^{\infty} M_k$ converges, then the series

$$S(z) = \sum_{k=1}^{\infty} f_k(z)$$

converges uniformly in $\mathcal{R}$.

Next, we define the notion of a power series is important in the study of analytic functions

Definition 5.2.11

An infinite series of the form

$$\sum_{k=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots, \tag{5.9}$$

where the coefficients a_n are complex constants, is called a **power series** in $z - z_0$. The power series (5.9) is said to be centered at z_0; the complex point z_0 is referred to as the *center of the series*. In (5.9) it is also convenient to define

$$(z - z_0)^0 = 1 \quad \text{even when } z = z_0.$$

Remark 5.2.12. Every complex power series (5.9) has a *radius of convergence*. Analogous to the concept of an interval of convergence for real power series, a complex power series (5.9) has a *circle of convergence*, which is the circle centered at z_0 of largest radius $R > 0$ for which (5.9) converges at every point within the circle $|z - z_0| = R$.
A power series converges absolutely at all points z within its circle of convergence, that is, for all z satisfying $|z - z_0| < R$, and diverges at all points z exterior to the circle, that is, for all z satisfying $|z - z_0| > R$.
The radius of convergence can be:

(*i*) $R = 0$ (in which case (5.9) converges only at its center $z = z_0$),

(*ii*) R a finite positive number (in which case (5.9) converges at all interior points of the circle $|z - z_0| = R$), or

(*iii*) $R = \infty$ (in which case (5.9) converges for all z).

A power series may converge at some, all, or at none of the points on the actual circle of convergence. See Figure 4.8

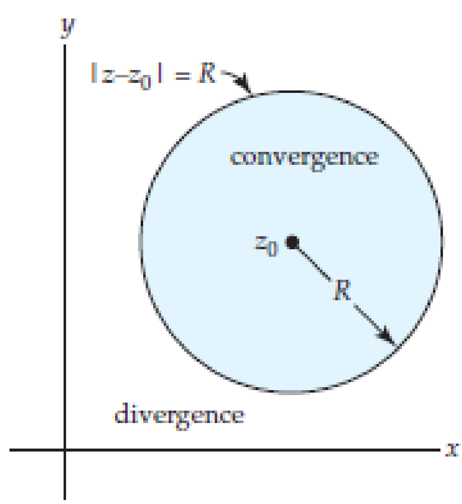

FIG.4.8

Proposition 5.2.13

Consider the geometric series

$$\sum_{k=0}^{\infty} z^k \tag{5.10}$$

(a) The series converges absolutely for $|z| < 1$ to the function

$$f(z) = \frac{1}{1-z}.$$

(b) The series converges uniformly for $|z| \leq r < 1$ to the function

$$f(z) = \frac{1}{1-z}.$$

(c) The series diverges for $|z| \geq 1$.

Proof. 1. We are concerning to showing that

$$\sum_{k=0}^{\infty} |z^k| \tag{5.11}$$

Let the partial sum of series (5.11) is

$$S_n(z) = \sum_{k=0}^{n} |z|^k = \sum_{k=0}^{n} |z^k| = 1 + |z| + |z|^2 + .. + |z|^n$$

. Then

$$S_n(z) = \frac{1-|z|^{n+1}}{1-z} \qquad (|z| < 1)$$

we have

$$\sum_{k=0}^{\infty} |z^k| = \lim_{n \longrightarrow \infty} S_n(z),$$

Since

$$\lim_{n \longrightarrow \infty} |z|^{n+1} = 0 \qquad \text{whenever } |z| < 1,$$

and so

$$\lim_{n \longrightarrow \infty} \frac{1-|z|^{n+1}}{1-z} = \frac{1}{1-z}.$$

Thus, the series (5.10) converges to $\frac{1}{1-z}$ for $(|z| < 1)$.

2. Let $0 < r < 1$ and suppose $|z| \leq r$. Then

$$|z^k| = |z|^k \leq r^k, \quad (r < 1).$$

But the series $\sum_{k=0}^{\infty} r^k$ converge then by theorem 5.2.10 (Weierstrass M-Test) we deduce that the series

$$\sum_{k=0}^{\infty} z^k$$

is uniformly converges for all $|z| \leq r < 1$ to $f(z) = \frac{1}{1-z}$.

3. If $|z| \geq 1$ we have $|z|^k \geq 1$, therefore

$$\lim_{k\longrightarrow\infty} |z|^k \neq 0,$$

thus by theorem 5.2.5(nth Term Test) the series (5.1) diverge for all $|z| \geq 1$.

□

Theorem 5.2.14

For every power series

$$\sum_{k=0}^{\infty} c_k(z-z_0)^k,$$

there exists a positive number R, with $0 \leq R \leq \infty$, which depends only on the coefficients c_k, such that:

(a) the series converges absolutely in $|z-z_0| < R$,

(b) the series converges uniformly in $|z-z_0| \leq R_0 < R$,

(c) the series diverges in $|z-z_0| > R$.

(d) If we denot by

$$\lim_{k\longrightarrow\infty} \left|\frac{c_{k+1}}{c_k}\right| = L, \qquad \text{and} \qquad R = \frac{1}{L} = \lim_{k\longrightarrow\infty} \left|\frac{c_k}{c_{k+1}}\right|.$$

Then, the positive number R is called radius of convergence, and the entire series (1) converges absolutely in the domain $|z-z_0| < R$

(d) The radius of convergence of entire series (5.9) can be also obtained by

$$R = \frac{1}{L} = \lim_{k\longrightarrow\infty} \frac{1}{\sqrt[k]{|c_k|}}.$$

Example 5.2.15

Study the convergence of the series:

$$\text{(a)} \quad \sum_{k=1}^{\infty} \frac{z^k}{k^2}, \qquad \text{(b)} \quad \sum_{k=1}^{\infty} \frac{z^k}{k!}, \qquad \text{(c)} \quad \sum_{k=1}^{\infty} k!\,(z-i)^k.$$

(a) The radius of convergence is given by

$$R = \lim_{k\longrightarrow\infty} \left|\frac{c_k}{c_{k+1}}\right| = \lim_{k\longrightarrow\infty} \frac{(k+1)^2}{k^2} = 1.$$

Thus the series converges absolutely in the domain $|z| < 1$.

On the boundary $|z| = 1$, we have

$$\left|\frac{z^k}{k^2}\right| = \frac{1}{k^2}.$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges, the series

$$\sum_{k=1}^{\infty} \frac{z^k}{k^2}$$

also converges absolutely on the circle $|z| = 1$. Therefore, the series converges absolutely in the closed disk $|z| \leq 1$.

(b) The radius of convergence is given by

$$R = \lim_{k \longrightarrow \infty} \left| \frac{c_k}{c_{k+1}} \right| = \lim_{k \longrightarrow \infty} \frac{(k+1)!}{k!} = \lim_{k \longrightarrow \infty} (k+1) = \infty.$$

Thus the series converges absolutely for all $z \in \mathbb{C}$.

(c) The radius of convergence is given by

$$R = \lim_{k \longrightarrow \infty} \left| \frac{c_k}{c_{k+1}} \right| = \lim_{k \longrightarrow \infty} \frac{k!}{(k+1)!} = \lim_{k \longrightarrow \infty} \frac{1}{k+1} = 0.$$

Hence, the series converges only at the point $z = i$.

Theorem 5.2.16

Let

$$D(z_0, R) = \{z \in \mathbb{C} : |z - z_0| < R\}$$

be the disk of convergence of the power series

$$f(z) := \sum_{k=0}^{\infty} c_k (z - z_0)^k.$$

If C is a contour contained in $D(z_0; R)$, then for every $z \in D(z_0; R)$:

1. $w = f(z)$ is a analytic function.
2. $$f'(z) = \sum_{k=1}^{\infty} k c_k (z - z_0)^{k-1}.$$
3. $$\int_C f(z)\, dz = \int_C \sum_{k=0}^{\infty} c_k (z - z_0)^k\, dz = \sum_{k=0}^{\infty} \int_C c_k (z - z_0)^k\, dz.$$

Definition 5.2.17 Alternative Definition of Analyticity

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$. The function f is said to be ***analytic*** at a point $z_0 \in \mathbb{C}$ if f admits a power series expansion around this point:

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k.$$

5.3 Taylor series expansion.

We have showning that every power series

$$\sum_{k=0}^{\infty} c_k(z-z_0)^k$$

is analytic in its disk of convergence $D(z_0, R)$. In this section, we show that the converse is true: every analytic function in a domain D can be expanded as a power series in a disk $D(z_0; R) \subset D$:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-z_0)^n. \tag{5.12}$$

Theorem 5.3.1 Taylor's Theorem

Let $f\colon D \longrightarrow \mathbb{C}$ be an analytic function in the domain D and $z_0 \in D$. Then f has the series representation as

$$f(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k, \tag{5.13}$$

with

$$c_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{(w-z_0)^{k+1}}\,dw, \quad k = 0, 1, 2, \tag{5.14}$$

valid for $C_R = \{z \in \mathbb{C} : |z-z_0| = R\}$ the largest positively oriented circle centered at z_0 with radius R contained in D. The serie defined in (5.13)-(5.14) is called **Taylor series** for f centered at z_0.

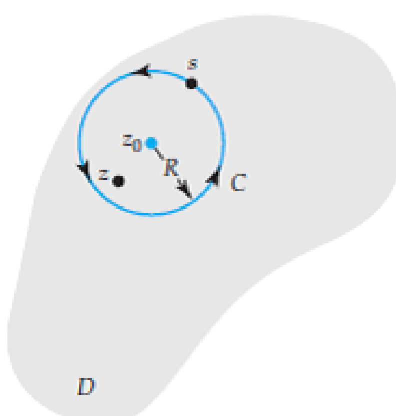

FIG.4.9

Proof. Let z be a fixed point within the circle C and let s denote the variable of integration.The circle C is then described by $|s-z_0| = R$, See Figure 4.9. Let z be such that $|z-z_0| < R$ and let w be such that $|s-z_0| = R$. By Cauchy's Integral Formula, we have

$$f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(s)}{s-z}\,ds.$$

The factor $\frac{1}{s-z}$ can be expressed as a geometric series in terms of $(\frac{z-z_0}{s-z_0})$, where $\left|\frac{z-z_0}{s-z_0}\right| < 1$:

$$\frac{1}{s-z} = \frac{1}{(s-z_0)-(z-z_0)} = \frac{1}{s-z_0} \cdot \frac{1}{1-(\frac{z-z_0}{s-z_0})} = \frac{1}{s-z_0} \sum_{k=0}^{\infty} \left(\frac{z-z_0}{s-z_0}\right)^k.$$

This series converges uniformly to $\frac{1}{s-z}$, and thus

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(s)}{s-z}\, ds = \frac{1}{2\pi i} \int_{C_R} f(s) \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{(s-z_0)^{k+1}}\, ds.$$

By interchanging the sum and the integral, we get

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_R} \frac{f(s)}{(s-z_0)^{k+1}}\, s \right) (z-z_0)^k.$$

By Cauchy's Integral Formula for derivatives, we know that

$$c_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \int_{C_R} \frac{f(s)}{(s-z_0)^{k+1}}\, s, \quad k = 0, 1, 2, \ldots.$$

Thus, we can express $f(z)$ as

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k,$$

which completes the proof of the theorem.

Remark 5.3.2.

- (1)The Taylor series with center $z_0 = 0$ is given by

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k,$$

  and is referred to as the **Maclaurin series**.

- (2)The radius of convergence R of a Taylor series can be computed using the methods from the chapter on power series. However, one can also find the radius R simply as the distance between the center a and the nearest isolated singularity of f to a. If $R = \infty$, then f is necessarily an entire function.

- (3) Follows Some Important Maclaurin Series

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{k=0}^{\infty} z^k, \quad (|z| < 1).$$

$$\ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k, \quad (|z| < 1).$$

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \quad (z \in \mathbb{C}).$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}, \quad (z \in \mathbb{C}).$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}, \quad (z \in \mathbb{K}).$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}, \quad (z \in \mathbb{C}).$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}, \quad (z \in \mathbb{C}).$$

$$(1+z)^\beta = 1 + \beta z + \frac{\beta(\beta-1)}{2!} z^2 + \cdots = 1 + \sum_{k=1}^{\infty} \frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!} z^k, \quad (z \in \mathbb{C}).$$

Example 5.3.3

Example. Suppose the function

$$f(z) = \frac{3-i}{1-i+z}$$

is expanded in a Taylor series with center $z_0 = 5 - 2i$. What is its radius of convergence R? Observe that the function is analytic at every point except at $z = -1+i$, which is an isolated singularity of f. The distance from $z = -1 + i$ to $z_0 = 5 - 2i$ is

$$|z - z_0| = \sqrt{(-1-5)^2 + \left(1-(-2)\right)^2} = \sqrt{43} = R.$$

This number is the radius of convergence R for the Taylor series centered at $4 - 2i$.

Find the Maclaurin expansion of

$$f(z) = \frac{1}{(1-z)^2}.$$

First of all, we have for $|z| < 1$,

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots . \tag{15}$$

If we differentiate both sides of the last result with respect to z, then

$$\frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{d}{dz}(1) + \frac{d}{dz}(z) + \frac{d}{dz}(z^2) + \frac{d}{dz}(z^3) + \cdots$$

or

$$\frac{1}{(1-z)^2} = 0 + 1 + 2z + 3z^2 + \cdots = \sum_{k=1}^{\infty} k z^{k-1}, \quad |z| < 1.$$

Example 5.3.4

Suppose that

$$f(z) = \frac{1}{1-z}$$

is expanded in a Taylor series centered at $z_0 = 4i$. What is the radius of convergence R? Find the Taylor series.

The function f is analytic everywhere except at the point $z = 1$. Thus, the radius of convergence is the distance between $z = 1$ and the center $z_0 = 7i$:

$$R = |1 - 7i| = \sqrt{1^2 + 7^2} = \sqrt{50}.$$

Now,

$$\frac{1}{1-z} = \frac{1}{1-7i-(z-7i)} = \frac{1}{1-7i} \cdot \frac{1}{1-\frac{z-7i}{1-7i}}, \left(\left|\frac{z-7i}{1-7i}\right| < 1\right).$$

Expanding as a geometric series, we obtain

$$\frac{1}{1-z} = \frac{1}{1-7i}\sum_{k=0}^{\infty}\left(\frac{z-7i}{1-7i}\right)^k, \quad |z-7i| < \sqrt{50}.$$

5.4 Laurent series

In this section we will be concerned with a *new kind* of "power series" expansion of involve negative as well as nonnegative integer powers of $z - z_0$.

Definition 5.4.1 singular point

If a complex function f fails to be analytic at a point $z = z_0$, then this point is said to be a **singularity** or **singular point** of the function f.

Definition 5.4.2 Isolated singularity

The point $z = z_0$ is said to be an **isolated singularity** of the function f if there exists some deleted neighborhood, or punctured open disk, $0 < |z - z_0| < R$ of z_0 throughout which f is analytic.

Example 5.4.3

- The complex numbers $z = 5i$ and $z = -5i$ are singularities isolated points of the function f defined by $f(z) = \frac{z}{z^2+25}$ because f is discontinuous at each of these points.
- The complex number $z = 0$ is a non-isolated singularity point of Lnz since every neighborhood of $z = 0$ contains points on the negative real axis.

Now, we introduce new Kind of Series.Indeed if $z = z_0$ is a singularity of a function f, then certainly f cannot be expanded in a power series with z_0 as its center. However, about an isolated singularity $z = z_0$, it is possible to represent f by a series involving both negative and nonnegative integer powers of $z - z_0$; that is,

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots. \tag{5.15}$$

As a very simple example of (5.15) let us consider the function

$$f(z) = \frac{1}{z-1}.$$

As can be seen, the point $z = 1$ is an isolated singularity of f and consequently the function cannot be expanded in a Taylor series centered at that point. Nevertheless, f can be expanded in a series of the form given in (5.15)that is valid for all z near 1:

$$f(z) = \cdots + \frac{0}{(z-1)^2} + \frac{1}{z-1} + 0 + 0 \cdot (z-1) + 0 \cdot (z-1)^2 + \cdots . \tag{5.16}$$

The series representation in (5.16) is valid for $0 < |z-1| < \infty$.

Definition 5.4.4

The series of the form

$$\sum_{k=1}^{\infty} \frac{b_k}{(z-z_0)^k} + \sum_{k=0}^{\infty} a_k (z-z_0)^k. \tag{5.17}$$

is called a **Laurent series** or **Laurent expansion** of f about z_0 on the annulus $r < |zz0| < R$. With,

- The negative powers is called the **principal part** of the Laurent series, and the one containing the positive powers is called the **analytic part** of the Laurent series.

- Let

$$R = \lim_{k \to \infty} \left(\frac{a_k}{a_{k+1}} \right) \quad \text{and} \quad r = \lim_{k \to \infty} \left(\frac{b_{k+1}}{b_k} \right).$$

  The Laurent series (5.17) converges in the annulus $z \in \mathbb{C} : r < |z - z_0| < R$

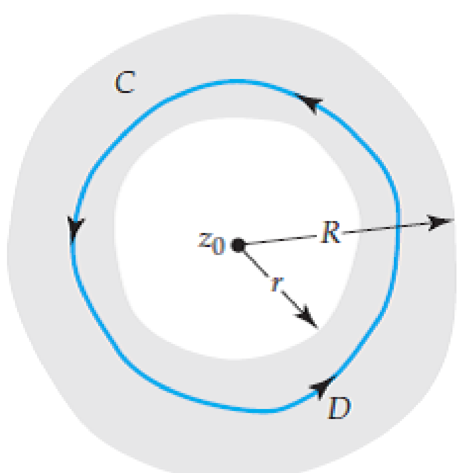

FIG.4.10

Theorem 5.4.5 Laurent's Theorem

Let $f : D \longrightarrow \mathbb{C}$ be an analytic function in the domain

$$D = \{z \in \mathbb{C} : r < |z - z_0| < R\}.$$

Then f can be expanded in a Laurent series

$$\sum_{k=-\infty}^{\infty} c_k (z - z_0)^k = \sum_{k=1}^{\infty} c_{-k}(z - z_0)^{-k} + \sum_{k=0}^{\infty} c_k (z - z_0)^k. \quad \text{for all } z \in D,$$

where the coefficients are given by

$$c_k = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{k+1}} \, ds, \quad k = 0, \pm 1, \pm 2, \ldots$$

Here C is a positively oriented simple closed contour contained in D and enclosing $z = z_0$ in its interior.

Example 5.4.6

The function

$$f(z) = \frac{\sin z}{z^4}$$

is not analytic at the isolated singularity $z = 0$ and hence cannot be expanded in a Maclaurin series. However, $\sin z$ is an entire function, and its Maclaurin series,

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots,$$

converges for $|z| < \infty$.
By dividing this power series by z^4 we obtain a series for f with negative and positive integer powers of z:

$$f(z) = \frac{\sin z}{z^4} = \underbrace{\frac{1}{z^3} - \frac{1}{3!\, z}}_{\text{principal part}} + \underbrace{\frac{z}{5!} - \frac{z^3}{7!} + \frac{z^5}{9!} - \cdots}_{\text{analytic part}}. \tag{5.18}$$

The analytic part of the series in (5.18) converges for $|z| < \infty$. The principal part is valid for $|z| > 0$. Thus, (5.18) converges for all z except at $z = 0$; that is, the series representation is valid for

$$0 < |z| < \infty.$$

Example 5.4.7

Expand

$$f(z) = \frac{1}{z(z-1)}$$

in a Laurent series valid for $1 < |z - 2| < 2$. The specified annular domain is shown in Figure 4.11. The center of this domain, $z = 2$, is the point about which we expand. We decompose f into partial fractions:

$$f(z) = -\frac{1}{z} + \frac{1}{z-1} = f_1(z) + f_2(z).$$

Expansion of $f_1(z)$:

$$f_1(z) = -\frac{1}{z} = -\frac{1}{2+(z-2)} = -\frac{1}{2}\frac{1}{1+\frac{z-2}{2}}.$$

Since $\left|\frac{z-2}{2}\right| < 1$ when $|z-2| < 2$, we use the geometric series:

$$f_1(z) = -\frac{1}{2}\left(1 - \frac{z-2}{2} + \frac{(z-2)^2}{2^2} - \frac{(z-2)^3}{2^3} + \cdots\right).$$

Thus,

$$f_1(z) = -\frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdots.$$

Expansion of $f_2(z)$:

$$f_2(z) = \frac{1}{z-1} = \frac{1}{(z-2)+1} = \frac{1}{z-2}\frac{1}{1+\frac{1}{z-2}}.$$

Since $\left|\frac{1}{z-2}\right| < 1$ when $|z-2| > 1$, we use the geometric series:

$$f_2(z) = \frac{1}{z-2}\left(1 - \frac{1}{z-2} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \cdots\right).$$

Thus,

$$f_2(z) = \frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^4} + \cdots.$$

Combine the two expansions:

$$\begin{aligned} f(z) &= f_1(z) + f_2(z) \\ &= \left(-\tfrac{1}{2} + \tfrac{z-2}{2^2} - \tfrac{(z-2)^2}{2^3} + \tfrac{(z-2)^3}{2^4} - \cdots\right) \\ &\quad + \left(\frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^4} + \cdots\right). \end{aligned}$$

This Laurent expansion is valid in the annulus

$$1 < |z-2| < 2.$$

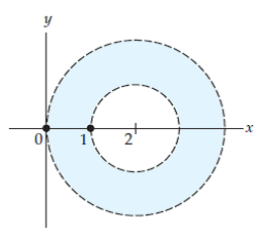

FIG.4.11

Example 5.4.8

Develop $f(z) = e^{1/z}$ in a Laurent series valid in the punctured complex plane $0 < |z| < 1$. We know that the Maclaurin series of the exponential function is

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots.$$

By replacing z with $1/z$, we obtain the Laurent series of f:

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots.$$

This series is valid for every $z \neq 0$, that is, in the punctured complex plane

$$0 < |z| < \infty.$$

5.5 Classification of Isolated Singular Points

An isolated singular point $z = z_0$ of a complex function f is given a classification depending on whether the principal part of its Laurent expansion contains zero, a finite number, or an infinite number of terms.

Definition 5.5.1 Classification of Isolated Singular Points

(i) If the principal part is zero, that is, all the coefficients a_{-k} are zero, then $z = z_0$ is called a **removable singularity**.

(ii) If the principal part contains a finite number of nonzero terms, then $z = z_0$ is called a **pole**. If, in this case, the last nonzero coefficient is a_{-n}, $n \geq 1$, then we say that $z = z_0$ is a pole of order n. If $z = z_0$ is a pole of order 1, then the principal part contains exactly one term with coefficient a_{-1}. A pole of order 1 is commonly called a **simple pole**.

(iii) If the principal part contains infinitely many nonzero terms, then $z = z_0$ is called an **essential singularity**.

Table5.5 summarizes the form of a Laurent series for a function f when $z = z_0$ is one of the above types of isolated singularities. Of course, R in the table could be ∞.

Type of Singularity at $z = z_0$	**Laurent Series for $0 < \|z - z_0\| < R$**
Removable singularity	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Pole of order n	$\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$
Simple pole	$\frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Essential singularity	$\cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$

Tbale Forms of Laurent series near isolated singular points

Example 5.5.2

Analyze the type of singularity at $z = 0$ in the following functions:

$$(a) \quad f_1(z) = \frac{\sin z}{z},$$
$$(b) \quad f_2(z) = \frac{\sin z}{z^2},$$
$$(c) \quad f_3(z) = \sin\left(\frac{1}{z}\right).$$

Solution. The series expansion of $\sin z$ is

$$\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots$$

(a) For

$$f_1(z) = \frac{\sin z}{z} = 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \cdots,$$

all negative powers vanish. Hence $z = 0$ is a **removable singularity**.

(b) For

$$f_2(z) = \frac{\sin z}{z^2} = \frac{1}{z} - \frac{1}{3!}z + \frac{1}{5!}z^3 - \cdots,$$

the principal part contains only one term $\frac{1}{z}$. Thus $z = 0$ is a **simple pole**.

(c) For

$$f_3(z) = \sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots,$$

the principal part has infinitely many terms. Therefore $z = 0$ is an **essential singularity**.

5.6 Exercise Set

Exercise 5.6.1

Determine the radii of convergence of the following power series:

$$\text{(a)} \quad \sum_{n=0}^{\infty} \frac{(-2)^n z^n}{n+1},$$

$$\text{(b)} \quad \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \cdot \frac{z^{2n+1}}{2n+1},$$

$$\text{(c)} \quad \sum_{n=0}^{\infty} \frac{3^n z^n}{5^n+7^n},$$

Solution. (a) According to

$$L = \limsup_{k \longrightarrow \infty} |a_k|^{\frac{1}{k}}.$$

we see that

$$L = \lim_{k \longrightarrow \infty} \left(\frac{2}{k+1}\right)^{\frac{1}{k}} = 2.$$

Therefore, the radius of convergence of the power series is

$$R = \frac{1}{2}.$$

(b) Using the Ratio Test, we see that

$$R = \lim_{k \longrightarrow \infty} \frac{\dfrac{(2k)!}{2^{2k}(k!)^2(2k+1)}}{\dfrac{(2k+2)!}{2^{2k+2}((k+1)!)^2(2k+3)}} = \lim_{k \longrightarrow \infty} \frac{4(k+1)^2(2k+3)}{(2k+1)(2k+2)} = 1,$$

so the radius of convergence of the power series is

$$R = 1.$$

(c) By the Ratio Test, we know that

$$R = \lim_{k \longrightarrow \infty} \frac{\frac{3^k}{5^k+7^k}}{\frac{3^{k+1}}{5^{k+1}+7^{k+1}}} = \frac{1}{3} \lim_{k \longrightarrow \infty} \frac{5^{k+1}+7^{k+1}}{5^k+7^k} = \frac{7}{3}.$$

This means that the radius of convergence of the power series is

$$R = \frac{7}{3}.$$

(a) Since $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$,

$$z\sinh(z^2) = z\frac{e^{z^2} - e^{-z^2}}{2} = \frac{z}{2}\sum_{n=0}^{\infty} \frac{1-(-1)^n}{n!} z^{2n} = \sum_{n=0}^{\infty} \frac{1-(-1)^n}{2} \frac{z^{2n+1}}{n!}.$$

Only odd n contribute; put $n = 2m+1$ to obtain

$$z\sinh(z^2) = \sum_{m=0}^{\infty} \frac{z^{4m+3}}{(2m+1)!}, \qquad R = \infty.$$

Put $w = z - 2$. Then

$$e^z = e^2 e^w = e^2 \sum_{n=0}^{\infty} \frac{w^n}{n!} = e^2 \sum_{n=0}^{\infty} \frac{(z-2)^n}{n!}, \qquad R = \infty.$$

Put $w = z + 1$. Then $z^2 + z = w^2 - w$ and $(1-z)^2 = (2-w)^2$, so

$$\frac{z^2+z}{(1-z)^2} = \frac{w^2 - w}{(2-w)^2} = 1 - \frac{3}{2-w} + \frac{2}{(2-w)^2}.$$

Using $\dfrac{1}{2-w} = \dfrac{1}{2}\sum_{n=0}^{\infty}\left(\dfrac{w}{2}\right)^n$ and $\dfrac{1}{(2-w)^2} = \dfrac{1}{4}\sum_{n=0}^{\infty}(n+1)\left(\dfrac{w}{2}\right)^n$, we get

$$\frac{z^2+z}{(1-z)^2} = 1 - \sum_{n=0}^{\infty} \frac{3\,w^n}{2^{n+1}} + \sum_{n=0}^{\infty} \frac{(n+1)w^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{n-2}{2^{n+1}} w^n.$$

Since the constant term vanishes, this may be written as

$$\frac{z^2+z}{(1-z)^2} = \sum_{n=1}^{\infty} \frac{n-2}{2^{n+1}} (z+1)^n, \qquad R = 2.$$

Exercise 5.6.3

Find the Taylor series of $(\cos z)^2$ at $z = \pi$.

Solution.

Let $w = z - \pi$. Then

$$(\cos z)^2 = (\cos(z+\pi))^2 = (\cos w)^2 = \left(\frac{e^{iw} + e^{-iw}}{2}\right)^2 = \frac{1}{4}\left(e^{2iw} + e^{-2iw} + 2\right)$$

$$= \frac{1}{4}\sum_{n=0}^{\infty} \frac{(2i)^n w^n}{n!} + \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-2i)^n w^n}{n!} + \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(2i)^{2n} w^{2n}}{(2n)!} = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1} w^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} (z-\pi)^{2n}}{(2n)!}.$$

Exercise 5.6.4

Find a power-series expansion of the function $f(z) = \frac{1}{3-z}$ about the point $4i$, and calculate the radius of convergence.

Solution.

$$\frac{1}{3-z} = \frac{1}{(3-4i)-(z-4i)} = \frac{1}{3-4i} \cdot \frac{1}{1-\dfrac{z-4i}{3-4i}} = \frac{1}{3-4i}\sum_{n=0}^{\infty}\left(\frac{z-4i}{3-4i}\right)^n, \quad \text{for } \left|\frac{z-4i}{3-4i}\right| < 1.$$

That is, for $|z-4i| < |3-4i| = 5$. Thus

$$\frac{1}{3-z} = \sum_{n=0}^{\infty}\frac{(z-4i)^n}{(3-4i)^{n+1}},$$

with radius of convergence 5.

Exercise 5.6.5

Find the Laurent series of the function

$$f(z) = \frac{z+4}{z^2\big(z^2+3z+2\big)}$$

Solution.

$$f(z) = \frac{z+4}{z^2(z^2+3z+2)} = -\frac{5}{2z} + \frac{2}{z^2} + \frac{3}{z+1} - \frac{1}{2(z+2)}.$$

For $0 < |z| < 1$:

$$\frac{3}{z+1} = 3\sum_{n=0}^{\infty}(-1)^n z^n, \qquad -\frac{1}{2(z+2)} = -\frac{1}{4}\frac{1}{1+(z/2)} = -\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n}.$$

Hence

$$f(z) = -\frac{5}{2z} + \frac{2}{z^2} + 3\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n} = \frac{2}{z^2} - \frac{5}{2z} + \sum_{n=0}^{\infty}(-1)^n\left(3 - \tfrac{1}{4\cdot 2^n}\right)z^n.$$

For $1 < |z| < 2$:

$$\frac{3}{z+1} = \frac{3}{z}\cdot\frac{1}{1+(1/z)} = \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n z^{-n}, \qquad -\frac{1}{2(z+2)} = -\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n}$$

(valid for $|z| < 2$ for the last series). Thus

$$f(z) = -\frac{5}{2z} + \frac{2}{z^2} + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n z^{-n} - \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n}.$$

For $|z| > 2$:

$$\frac{3}{z+1} = \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n z^{-n}, \qquad -\frac{1}{2(z+2)} = -\frac{1}{2z}\frac{1}{1+(2/z)} = -\frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n 2^n z^{-n},$$

so

$$f(z) = -\frac{5}{2z} + \frac{2}{z^2} + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n z^{-n} - \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n 2^n z^{-n}.$$

For $0 < |z+1| < 1$ (put $w = z+1$, so $z = w-1$):

$$\frac{5}{2(1-w)} = \frac{5}{2}\sum_{n=0}^{\infty} w^n, \qquad \frac{2}{(1-w)^2} = \sum_{n=0}^{\infty} 2(n+1)w^n, \qquad -\frac{1}{2(w+1)} = -\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n w^n.$$

Hence

$$f(z) = \frac{3}{w} + \sum_{n=0}^{\infty}\Big(2(n+1) + \frac{9}{2} - \frac{(-1)^n}{2}\Big)w^n = \frac{3}{z+1} + \sum_{n=0}^{\infty}\Big(2n + \frac{9}{2} - \frac{(-1)^n}{2}\Big)(z+1)^n, \qquad 0 < |z+1| < 1.$$

Exercise 5.6.6

$$\text{Let } f(z) = \frac{z^2}{z^2 - z - 2}.$$

Find the Laurent series of $f(z)$ in each of the following domains:

(a) $1 < |z| < 2$

(b) $0 < |z-2| < 1$

Solution. we have

$$f(z) = \frac{z^2}{z^2 - z - 2} = 1 + \frac{4}{3(z-2)} - \frac{1}{3(z+1)}.$$

For $1 < |z| < 2$ (Laurent series about 0):

$$\frac{4}{3(z-2)} = -\frac{2}{3}\cdot\frac{1}{1-\frac{z}{2}} = -\frac{2}{3}\sum_{n=0}^{\infty}\Big(\frac{z}{2}\Big)^n, \qquad |z| < 2,$$

$$-\frac{1}{3(z+1)} = -\frac{1}{3z}\cdot\frac{1}{1+\frac{1}{z}} = -\frac{1}{3z}\sum_{n=0}^{\infty}(-1)^n z^{-n}, \qquad |z| > 1.$$

Hence

$$\boxed{f(z) = 1 - \frac{2}{3}\sum_{n=0}^{\infty} 2^{-n} z^n - \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n z^{-n-1}, \quad 1 < |z| < 2.}$$

For $0 < |z-2| < 1$ (Laurent series about 2; set $w = z-2$):

$$\frac{4}{3(z-2)} = \frac{4}{3w}, \qquad -\frac{1}{3(z+1)} = -\frac{1}{3(3+w)} = -\frac{1}{9}\cdot\frac{1}{1+\frac{w}{3}} = -\frac{1}{9}\sum_{n=0}^{\infty}(-1)^n 3^{-n} w^n,$$

valid for $|w| < 3$ (in particular for $|w| < 1$). Therefore

$$\boxed{f(z) = 1 + \frac{4}{3(z-2)} - \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n 3^{-n}(z-2)^n, \quad 0 < |z-2| < 1.}$$

Exercise 5.6.7

verify that $|e^z - 1| \le e^{|z|} - 1 \le |z|\, e^{|z|}$.

Solution.

We want to verify that $|e^z - 1| \le e^{|z|} - 1 \le |z|\, e^{|z|}$.

We want to verify that $|e^z - 1| \le e^{|z|} - 1 \le |z|\, e^{|z|}$.

By the triangle inequality, on the one hand, we have

$$|e^z - 1| = \left|\sum_{n=1}^{\infty}\frac{z^n}{n!}\right| = \left|z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right| \le |z| + \frac{|z|^2}{2!} + \frac{|z|^3}{3!} + \cdots = \sum_{n=1}^{\infty}\frac{|z|^n}{n!} = e^{|z|} - 1.$$

On the other hand, we see that

$$e^{|z|} - 1 = |z| + \frac{|z|^2}{2!} + \frac{|z|^3}{3!} + \cdots = |z|\sum_{n=1}^{\infty}\frac{|z|^{n-1}}{n!} = |z|\sum_{n=0}^{\infty}\frac{|z|^n}{(n+1)!} \le |z|\sum_{n=0}^{\infty}\frac{|z|^n}{n!} = |z|\, e^{|z|}.$$

Hence we complete the proof of the problem.

Exercise 5.6.8

Let $f(z) = \dfrac{z}{(z-1)(z+2)}$.

(1) Find constants a and b such that

$$f(z) = \frac{a}{z-1} + \frac{b}{z+2}.$$

(2) Develop the Laurent series of $f(z)$ about 0. Write the expansions valid in the following annuli:

(a) $0 < |z| < 1$,

(b) $1 < |z| < 2$,

(c) $|z| > 2$.

Solution. 1) We consider the function

$$f(z) = \frac{z}{(z-1)(z+2)} = \frac{a}{z-1} + \frac{b}{z+2}.$$

Thus,

$$f(z) = \frac{az + 2a + bz - b}{(z-1)(z+2)} = \frac{(a+b)z + (2a-b)}{(z-1)(z+2)}.$$

By comparing coefficients we obtain the system

$$\begin{cases} a + b = 1, \\ 2a - b = 0. \end{cases}$$

Solving gives

$$a = \tfrac{1}{3}, \qquad b = \tfrac{2}{3}.$$

Chapter 6

Residue Theorem and its Applications

Chapter contents

In the last section, we saw that if a complex function f has an isolated singularity at a point z_0, then f has a Laurent series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots, \tag{6.1}$$

which converges for all z near z_0. More precisely, the representation is valid in some deleted neighborhood of z_0 or punctured open disk $0 < |z-z_0| < R$.

Now, our entire focus will be on the coefficient a_{-1} and its importance in the evaluation of contour integrals.

Definition 6.0.1 Residue

The coefficient a_{-1} of $\left(\frac{1}{z-z_0}\right)$ in (6.1) the Laurent series given in is called the **residue** of the function f at the isolated singularity z_0 and will be denoted by

$$\mathbf{a_{-1}} = \mathrm{Res}(\mathbf{f}(\mathbf{z}), \mathbf{z_0})$$

to denote the residue of f at z_0.

Proposition 6.0.2

Let f be a analytic function in $0 < |z - z_0| < R$. Then for every closed contour γ contained in $0 < |z - z_0| < R$, we have

$$\int_\gamma f(z)\, dz = 2\pi i \ \mathrm{Res}(f; z_0).,$$

Proof. Since f is analytic in $0 < |z - z_0| < R$, it admits a Laurent series expansion there

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k.$$

Thus,

$$\int_\gamma f(z)\, dz = \int_\gamma \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k\, dz = \sum_{k=-\infty}^{\infty} c_k \int_\gamma (z - z_0)^k\, dz.$$

But only the term with $k = -1$ contributes:

$$= c_{-1} \int_\gamma (z - z_0)^{-1}\, dz.$$

Therefore,

$$\int_\gamma f(z)\, dz = 2\pi i \ \mathrm{Res}(f; z_0).$$

□

Example 6.0.3

We can see that $z = 0$ is an essential singularity of

$$f(z) = e^{\frac{3}{z}}.$$

because, the inspection of the Laurent series is given by,

$$e^{\frac{3}{z}} = 1 + \frac{3}{z} + \frac{3^2}{2!\, z^2} + \frac{3^3}{3!\, z^3} + \cdots, \qquad 0 < |z| < \infty,$$

shows that the coefficient of $\frac{1}{z}$ is

$$a_{-1} = \mathrm{Res}(f(z), 0) = 3.$$

6.1 Residue Theorem.

We come now to the reason why the residue concept is important. The next theorem states that, under some circumstances, we can evaluate complex integrals $\oint_C f(z)\,dz$ by summing the residues at the isolated singularities of f within the closed contour C.

Theorem 6.1.1 Cauchy's Residue Theorem

Let D be a simply connected domain and C a simple closed contour lying entirely within D. If a function f is analytic on and within C, except at a finite number of isolated singular points $z_1, z_2, \ldots, z_n$ within C, then

$$\int_C f(z)\,dz \;=\; 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k). \tag{6.2}$$

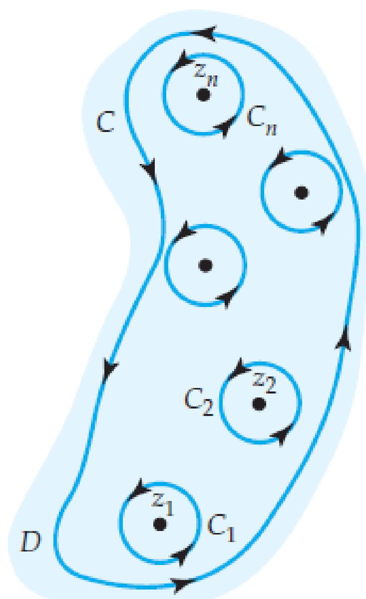

FIG.4.12

6.2 Residue Calculus.

Proposition 6.2.1 Residue Calculation

Let $z = z_0$ be an isolated singularity of f. Then:

1. If $z = z_0$ is a simple pole of f,

$$\mathrm{Res}(f, z_0) = \lim_{z \longrightarrow z_0} (z - z_0) f(z).$$

2. If
$$f(z) = \frac{g(z)}{h(z)},$$
where $g(z_0) \neq 0$, $h(z_0) = 0$ and $h'(z_0) \neq 0$, then
$$\text{Res}(f, z_0 = \frac{g(z_0)}{h'(z_0)}.$$

3. If $z = z_0$ is a pole of order n of f,
$$\text{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \longrightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} \big[(z - z_0)^n f(z)\big].$$

Example 6.2.2

Consider the function
$$f(z) = \frac{1}{(z-1)^2(z-3)}.$$
It has a simple pole at $z = 3$ and a pole of order 2 at $z = 1$. We use above proposition to compute the residues.

a)**Residue at $z = 3$** Since $z = 3$ is a simple pole, we use the formula
$$\text{Res}(f(z), 3) = \lim_{z \longrightarrow 3} (z - 3) f(z).$$
Thus,
$$\text{Res}(f(z), 3) = \lim_{z \longrightarrow 3} \frac{1}{(z-1)^2} = \frac{1}{(3-1)^2} = \frac{1}{4}.$$
b)**Residue at $z = 1$:** Since $z = 1$ is a pole of order 2, we apply
$$\text{Res}(f(z), 1) = \frac{1}{1!} \lim_{z \longrightarrow 1} \frac{d}{dz}\Big[(z-1)^2 f(z)\Big].$$
Now,
$$(z-1)^2 f(z) = \frac{1}{z-3},$$
so
$$\frac{d}{dz}\left(\frac{1}{z-3}\right) = -\frac{1}{(z-3)^2}.$$
Therefore,
$$\text{Res}(f(z), 1) = \lim_{z \longrightarrow 1} -\frac{1}{(z-3)^2} = -\frac{1}{4}.$$

Example 6.2.3 Evaluation by the Residue Theorem

Let
$$f(z) = \frac{1}{(z-1)^2(z-3)}.$$

Evaluate $\oint_C f(z)\,dz$ in the following cases:

(a) C is the rectangle with sides $x = 0,\ x = 4,\ y = -1,\ y = 1.$

(b) C is the circle $|z| = 2.$

(a) C is the rectangle with sides $x = 0$, $x = 4$, $y = -1$, $y = 1$.
Both singularities $z = 1$ and $z = 3$ lie inside this rectangle. By the Residue Theorem,

$$\oint_C f(z)\,dz = 2\pi i\big(\operatorname{Res}(f,1) + \operatorname{Res}(f,3)\big).$$

Compute the residues.
Residue at $z = 3$ (simple pole):

$$\operatorname{Res}(f,3) = \lim_{z\longrightarrow 3}(z-3)f(z) = \lim_{z\longrightarrow 3}\frac{1}{(z-1)^2} = \frac{1}{(3-1)^2} = \frac{1}{4}.$$

Residue at $z = 1$ (double pole): For a pole of order 2 at $z = 1$,

$$\operatorname{Res}(f,1) = \frac{1}{1!}\lim_{z\longrightarrow 1}\frac{d}{dz}\big[(z-1)^2 f(z)\big] = \lim_{z\longrightarrow 1}\frac{d}{dz}\left(\frac{1}{z-3}\right) = \lim_{z\longrightarrow 1}\left(-\frac{1}{(z-3)^2}\right) = -\frac{1}{(1-3)^2} = -\frac{1}{4}.$$

Thus

$$\oint_C f(z)\,dz = 2\pi i\left(-\frac{1}{4} + \frac{1}{4}\right) = 0.$$

(b) C is the circle $|z| = 2$. The circle $|z| = 2$ contains $z = 1$ but not $z = 3$. Hence by the Residue Theorem,

$$\oint_C f(z)\,dz = 2\pi i\ \operatorname{Res}(f,1) = 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi}{2}\,i.$$

Example 6.2.4 Evaluation by the Residue Theorem

Evaluate

$$\oint_C \frac{2z+6}{z^2+4}\,dz,$$

where C is the circle $|z - i| = 2$. By factoring the denominator as

$$z^2 + 4 = (z-2i)(z+2i),$$

we see that the integrand has simple poles at $2i$ and $2i$. Because only $2i$ lies within the contour C, it follows

$$\operatorname{Res}\left(\frac{2z+6}{z^2+4}, 2i\right) = \lim_{z\longrightarrow 2i}(z-2i)\frac{2z+6}{(z-2i)(z+2i)} = \frac{2(2i)+6}{2i+2i} = \frac{4i+6}{4i} = \frac{3+2i}{2i}.$$

Simplify:

$$\frac{3+2i}{2i} = 1 - \frac{3}{2}i.$$

By the Residue Theorem,

$$\oint_C \frac{2z+6}{z^2+4}\,dz = 2\pi i\ \operatorname{Res}\left(\frac{2z+6}{z^2+4}, 2i\right) = 2\pi i\left(1 - \frac{3}{2}i\right) = \pi(3+2i).$$

Hence

$$\boxed{\oint_C \frac{2z+6}{z^2+4}\,dz = \pi(3+2i).}$$

Example 6.2.5 Evaluation by the Residue Theorem

Evaluate

$$\oint_C \frac{2z+6}{z^2+4}\,dz,$$

where C is the circle $|z-i|=2$. Writing $z^4+5z^3 = z^3(z+5)$, we see that the integrand

$$f(z) = \frac{e^z}{z^3(z+5)}$$

has a pole of order 3 at $z=0$ and a simple pole at $z=-5$. Only $z=0$ lies within the contour $|z|=2$. From the Residue Theorem,

$$\oint_C \frac{e^z}{z^4+5z^3}\,dz = 2\pi i\ \mathrm{Res}(f,0).$$

For a pole of order 3 at $z=0$,

$$\mathrm{Res}(f,0) = \frac{1}{2!}\lim_{z\longrightarrow 0}\frac{d^2}{dz^2}\left(z^3\cdot\frac{e^z}{z^3(z+5)}\right) = \frac{1}{2}\lim_{z\longrightarrow 0}\frac{d^2}{dz^2}\left(\frac{e^z}{z+5}\right).$$

Differentiating:

$$\frac{d}{dz}\left(\frac{e^z}{z+5}\right) = \frac{e^z(z+4)}{(z+5)^2},$$

$$\frac{d^2}{dz^2}\left(\frac{e^z}{z+5}\right) = \frac{(z^2+8z+17)e^z}{(z+5)^3}.$$

Thus,

$$\mathrm{Res}(f,0) = \tfrac{1}{2}\cdot\frac{(0^2+8\cdot 0+17)e^0}{5^3} = \frac{17}{250}.$$

Therefore,

$$\oint_C \frac{e^z}{z^4+5z^3}\,dz = 2\pi i\cdot\frac{17}{250} = \frac{17\pi i}{125}.$$

6.3 Applications to integral calculus and series summation.

In this section, we will explore how residue theory can be applied to evaluate real integrals of the following types:

$$\int_0^{2\pi} F(\cos\theta,\sin\theta)\,d\theta, \tag{6.3}$$

$$\int_{-\infty}^{\infty} f(x)\,dx, \tag{6.4}$$

and

$$\int_{-\infty}^{\infty} f(x)\cos(\alpha x)\,dx, \qquad \int_{-\infty}^{\infty} f(x)\sin(\alpha x)\,dx, \tag{6.5}$$

where F in the first integral and f in the second and third integrals are rational functions.

For the rational function $f(x) = \frac{p(x)}{q(x)}$ appearing in the latter two cases, we assume that the polynomials $p(x)$ and $q(x)$ have no common factors. Moreover, establish the relationship between the residue theory and the zeros of an analytic function and a consideration of how residues can, in certain cases, be used to find the sum of an infinite series.

6.3.1 Integrals of the form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$ (F trigonometric rational functions)

Let the integral

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta \tag{6.6}$$

The main idea is to transform a real trigonometric integral of the form (6.6) into a complex contour integral. To do this, we shall parametrize this contour, let

$$z = e^{i\theta}, \quad 0 \le \theta \le 2\pi.$$

We can then write

$$dz = ie^{i\theta}\, d\theta, \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

It follows that

$$d\theta = \frac{dz}{iz}, \qquad \cos\theta = \tfrac{1}{2}\left(z + \frac{1}{z}\right), \qquad \sin\theta = \tfrac{1}{2i}\left(z - \frac{1}{z}\right).$$

The given integral then becomes

$$I = \int_C F\left(\frac{1}{2}\left(z + z^{-1}\right), \frac{1}{2i}\left(z - z^{-1}\right)\right) \frac{dz}{iz},$$

where C is the unit circle $|z| = 1$.

By applying the residue theorem, we obtain

$$I = 2\pi i \sum_{|z_k|<1} \operatorname{Res}\left(\frac{1}{iz} F\left(\tfrac{1}{2}\left(z + \tfrac{1}{z}\right),\ \tfrac{1}{2i}\left(z - \tfrac{1}{z}\right)\right), z_k\right), \tag{1}$$

where z_k are the singularities of the function

$$\frac{1}{iz} F\left(\tfrac{1}{2}\left(z + \tfrac{1}{z}\right),\ \tfrac{1}{2i}\left(z - \tfrac{1}{z}\right)\right)$$

that lie inside the unit circle $|z| < 1$.

Example 6.3.1

We evaluate

$$I = \int_0^{2\pi} \frac{d\theta}{(2 + \cos\theta)^2}.$$

Using the substitution $z = e^{i\theta}$, so that $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}(z + z^{-1})$, we obtain

$$I = \oint_C \frac{1}{\left(2 + \frac{1}{2}(z + z^{-1})\right)^2} \cdot \frac{dz}{iz},$$

where C is the unit circle traversed counterclockwise.

Simplifying the integrand gives

$$I = \oint_C \frac{4z}{i(z^2+4z+1)^2}\,dz = \frac{4}{i}\oint_C \frac{z}{(z^2+4z+1)^2}\,dz.$$

Factoring the denominator,

$$z^2+4z+1 = (z-z_1)(z-z_2), \quad z_1 = -2-\sqrt{3},\ z_2 = -2+\sqrt{3}.$$

Since $|z_2| < 1$ and $|z_1| > 1$, only z_2 lies inside C. Thus,

$$I = \frac{4}{i}\cdot 2\pi i\ \text{Res}\left(\frac{z}{(z-z_1)^2(z-z_2)^2}, z_2\right).$$

Because $z = z_2$ is a pole of order 2, we use

$$\text{Res}(f(z), z_2) = \lim_{z\longrightarrow z_2}\frac{d}{dz}\left[(z-z_2)^2 f(z)\right] = \lim_{z\longrightarrow z_2}\frac{d}{dz}\left(\frac{z}{(z-z_1)^2}\right).$$

Differentiating,

$$\frac{d}{dz}\left(\frac{z}{(z-z_1)^2}\right) = \frac{(z-z_1)^2 - 2z(z-z_1)}{(z-z_1)^3} = \frac{-z-z_1}{(z-z_1)^3}.$$

At $z = z_2$,

$$\text{Res}(f(z), z_2) = \frac{-z_2-z_1}{(z_2-z_1)^3} = \frac{1}{6\sqrt{3}}.$$

Therefore,

$$I = \frac{4}{i}\cdot 2\pi i\cdot\frac{1}{6\sqrt{3}} = \frac{4\pi}{3\sqrt{3}}.$$

6.3.2 Integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$

We begin this section by developing methods to evaluate real integrals of the form

$$I = \int_{-\infty}^{\infty} f(x)\,dx \tag{6.7}$$

where f is a real valued function and will be specified later. Integrals with infinite endpoints converge depending on the existence of a limit; namely, we

We say that I converges if the two limits in

$$I = \lim_{L\longrightarrow\infty}\int_{-L}^{\alpha} f(x)\,dx + \lim_{R\longrightarrow\infty}\int_{\alpha}^{R} f(x)\,dx, \quad \alpha \text{ finite}, \tag{6.8}$$

exist.

When evaluating integrals in complex analysis, it is useful (as we will see) to consider a more restrictive limit by taking $L = R$, and this is sometimes referred to as the *Cauchy Principal Value at Infinity*, I_p:

$$I_p = \text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\longrightarrow\infty}\int_{-R}^{R} f(x)\,dx. \tag{6.9}$$

Note that, If integral (6.8) is convergent, then $I = Ip$ by simply taking as a special case $L = R$. It is possible for Ip to exist but not the more general limit (6.8) . For example, if f is odd and nonzero at infinity ($e.g. f(x) = x$), then$Ip = 0$ but I will not exist. For this reason we shall only consider integrals with infinite

limits whose convergence can be established in the sense of (6.8).
To evaluate an integral

$$\int_{-\infty}^{\infty} f(x)\,dx,$$

where the rational function $f(x) = \frac{p(x)}{q(x)}$ is continuous on $(-\infty, \infty)$, by residue theory we replace x by the complex variable z and integrate the complex function

$$f(z) = \frac{p(z)}{q(z)}$$

over a closed contour C that consists of the interval $[-R, R]$ on the real axis and a semicircle C_R of radius large enough to enclose all the poles of $f(z)$ in the upper half-plane $\mathrm{Im}(z) > 0$. See Figure FIG.13, so in this situation, we have

$$\int_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k).$$

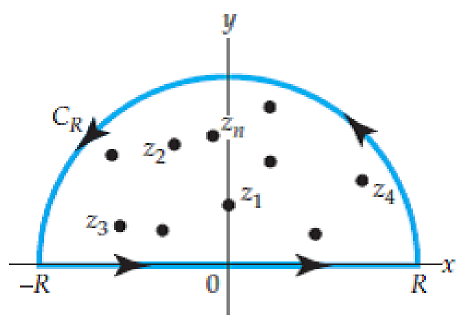

FIG.13

Remark 6.3.2.

Remark 6.3.2. Remark that :

1. If $f(x)$ is an even function, then

$$\int_{0}^{+\infty} f(x)\,dx = \tfrac{1}{2}\int_{-\infty}^{+\infty} f(x)\,dx.$$

2. If we choose the semicircle contained in the lower half-plane, we would obtain an analogous formula with the poles in the lower half-plane:

$$I^* = \int_{-\infty}^{+\infty} f(x)\,dx = -2\pi i \sum_{\mathrm{Im}\, z_k < 0} \mathrm{Res}\big(f(z), z_k\big).$$

Example 6.3.3

Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+9)}.$$
Let $f(z) = \frac{1}{(z^2+1)(z^2+9)}$. Since $(z^2+1)(z^2+9) = (z-i)(z+i)(z-3i)(z+3i)$, we take C to be the closed contour consisting of the interval $[-R, R]$ on the real axis and the semicircle C_R of radius $R > 3$. As seen from Figure F.14,
$$\int_C \frac{dz}{(z^2+1)(z^2+9)} = \underbrace{\int_{-R}^{R} \frac{dx}{(x^2+1)(x^2+9)}}_{I_1} + \underbrace{\int_{C_R} \frac{dz}{(z^2+1)(z^2+9)}}_{I_2} = I_1 + I_2$$
and by the residue theorem,
$$I_1 + I_2 = 2\pi i\Big[\operatorname{Res}(f(z), i) + \operatorname{Res}(f(z), 3i)\Big].$$
At the simple poles $z = i$ and $z = 3i$, we find
$$\operatorname{Res}(f(z), i) = \frac{1}{16i}, \quad \operatorname{Res}(f(z), 3i) = -\frac{1}{48i}.$$
Thus,
$$I_1 + I_2 = 2\pi i\left(\frac{1}{16i} - \frac{1}{48i}\right) = \frac{\pi}{12}.$$
To let $R \longrightarrow \infty$, we use the ML-inequality on C_R:
$$\left|\int_{C_R} \frac{dz}{(z^2+1)(z^2+9)}\right| \leq \frac{\pi R}{(R^2-1)(R^2-9)} \longrightarrow 0 \quad \text{as } R \longrightarrow \infty.$$
Hence,
$$\lim_{R \longrightarrow \infty} I_1 = \frac{\pi}{12},$$
or equivalently,
$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+9)} = \frac{\pi}{12}.$$

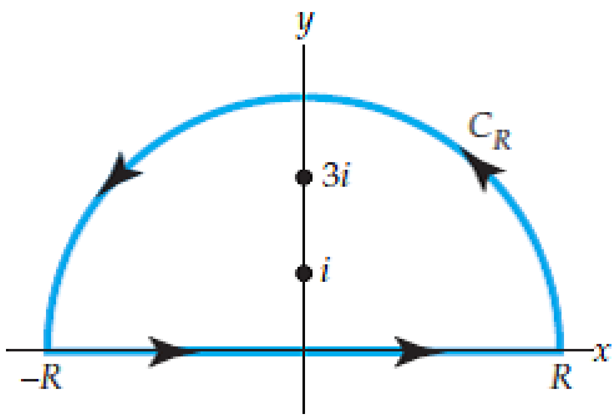

FIG.14

It is often not easy to prove that the circumferential integral along C_R approaches zero when $R \longrightarrow \infty$. The sufficient conditions under which this behavior is always true are summarized in the following theorem

Theorem 6.3.4 Behavior of the Integral as $R \longrightarrow \infty$

Suppose $f(z) = \frac{p(z)}{q(z)}$ is a rational function, where the degree of $p(z)$ is n and the degree of $q(z)$ is $m \geq n+2$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$. Then

$$\int_{C_R} f(z)\, dz \longrightarrow 0 \quad \text{as } R \longrightarrow \infty.$$

Find the Taylor series of the following functions and their radii of convergence:

$$\text{(a)} \qquad z \sinh(z^2) = \sum_{m=0}^{\infty} \frac{z^{4m+3}}{(2m+1)!}.$$

$$\text{(b)} \qquad e^z = e^2 e^{z-2} = e^2 \sum_{n=0}^{\infty} \frac{(z-2)^n}{n!},$$

$$\text{(c)} \qquad \frac{z^2+z}{(1-z)^2}, \quad w := z+1,$$

$$\frac{z^2+z}{(1-z)^2} = \frac{w^2-w}{(2-w)^2} = \frac{1}{4}(w^2-w)\sum_{n=0}^{\infty}(n+1)\Big(\frac{w}{2}\Big)^n$$

$$= -\frac{1}{4}(z+1) + \sum_{k=2}^{\infty} \frac{k-2}{2^{k+1}}(z+1)^k.$$

Exercise 5.6.2

6.3.3 Integrals of the form $\int_{-\infty}^{\infty} f(x)\cos\alpha x$and $\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$

By Euler's formula, $e^{i\alpha x} = \cos(\alpha x) + i\sin(\alpha x)$, where α is a positive real number, we can write

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx = \int_{-\infty}^{\infty} f(x)\cos(\alpha x)\,dx + i\int_{-\infty}^{\infty} f(x)\sin(\alpha x)\,dx. \tag{6.10}$$

whenever both integrals on the right-hand side converge.

Suppose $f(x) = \frac{p(x)}{q(x)}$ is a rational function that is continuous on $(-\infty,\infty)$. Then both Fourier integrals in (6.10)can be evaluated at the same time by considering the complex integral

$$\int_C f(z)e^{i\alpha z}\,dz,$$

where $\alpha > 0$, and the contour C again consists of the interval $[-R,R]$ on the real axis and a semicircular contour C_R with radius large enough to enclose the poles of $f(z)$ in the upper-half plane.

Before proceeding, we give, without proof, sufficient conditions under which the contour integral along C_R approaches zero as $R \longrightarrow \infty$.

Theorem 6.3.6 Behavior of Integral as $R \longrightarrow \infty$

Suppose $f(z) = \frac{p(z)}{q(z)}$ is a rational function, where the degree of $p(z)$ is n and the degree of $q(z)$ is $m \geq n+2$. Let C_R be a semicircular contour

$$z = Re^{i\theta}, \quad 0 \leq \theta \leq \pi,$$

and let $\alpha > 0$. Then

$$\int_{C_R} f(z)e^{i\alpha z}\,dz \longrightarrow 0 \quad \text{as } R \longrightarrow \infty.$$

Using the closed curves $C = C_R \cup [-R,R]$ and applying the above theorem

$$I = \int_{-\infty}^{\infty} f(x)e^{i\beta x}\,dx = 2\pi i \sum_{\text{Im } z_k>0} \text{Res}\left(f(z)e^{i\alpha z}; z_k\right),$$

Example 6.3.7 P.V. and even function

We want to evaluate

$$I = \int_0^{\infty} \frac{x\sin x}{x^2+9}\,dx.$$

Note that $x/(x^2+9)$ is an odd function and $\sin x$ is odd, hence the product is even. Therefore

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+9}\,dx = 2\int_0^{\infty} \frac{x\sin x}{x^2+9}\,dx = 2I.$$

Consider the complex integral

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2+9}\,dz,$$

where P.V. denotes the Cauchy principal value. By the residue theorem, closing the contour in the upper half-plane (since e^{iz} decays there for $\Im z > 0$) we pick up the pole at $z = 3i$. The integrand has a simple pole at $z = 3i$, and its residue is

$$\text{Res}\left(\frac{ze^{iz}}{z^2+9}, z = 3i\right) = \lim_{z\longrightarrow 3i} \frac{(z-3i)ze^{iz}}{(z-3i)(z+3i)} = \frac{3i\,e^{i(3i)}}{6i} = \frac{1}{2}e^{-3}.$$

Hence

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2+9}\,dz = 2\pi i \cdot \frac{1}{2}e^{-3} = \pi i e^{-3}.$$

Split the integral into real and imaginary parts:

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2+9}\,dz = \int_{-\infty}^{\infty} \frac{x\cos x}{x^2+9}\,dx + i\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+9}\,dx.$$

The first integral vanishes because $x\cos x/(x^2+9)$ is odd. Therefore

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+9}\,dx = \text{Im}\left(\pi i e^{-3}\right) = \pi e^{-3}.$$

Recalling, $2I = \int_0^{\infty} \frac{x\sin x}{x^2+9}\,dx$, we obtain

$$I = \int_0^{\infty} \frac{x\sin x}{x^2+9}\,dx = \frac{\pi}{2}e^{-3}.$$

Finally, in view of the fact that the integrand is an even function, we obtain the value of the prescribed integral:

$$\boxed{\int_0^{\infty} \frac{x\sin x}{x^2+9}\,dx = \frac{\pi}{2}e^{-3}}$$

Note 6.3.8

(1) The improper integrals of forms (6.4) and (6.5) that we have considered up to this point were continuous on the interval$(-\infty,\infty)$. In other words, the complex function $f(z) = \frac{p(z)}{q(z)}$ did not have poles on the real axis.

(2) to evaluate by residues when $f(z)$ has a pole at$z = c$, where c is a real number, we use an **indented contour** as illustrated in Figure FIG.15. The symbol C_r denotes a semicircular contour centered at$z = c$ and oriented in the positive direction.The next theorem is important to this discussion.

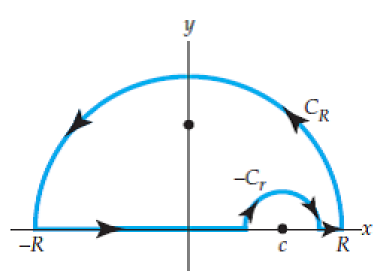

FIG.15

6.3.4 Jordan's lemma

In the evaluation of integrals of the type (6.5), it is sometimes necessary to use **Jordan's lemma**, which is stated just below as a theorem.

Theorem 6.3.9

Suppose that

(a) A function $f(z)$ is analytic at all points in the upper half-plane $y \geq 0$ that are exterior to a circle $|z| = R_0$;

(b) C_R denotes a semicircle $z = Re^{i\theta}$ $(0 \leq \theta \leq \pi)$, where $R > R_0$ (see FIG.17);

(c) for all points z on C_R, there is a positive constant M_R such that $|f(z)| \leq M_R$ and

$$\lim_{R \longrightarrow \infty} M_R = 0.$$

Then, for every positive constant a,

$$\lim_{R \longrightarrow \infty} \int_{C_R} f(z) e^{iaz} \, dz = 0.$$

Example 6.3.10 Summing an Infinite Series

We want to evaluate

$$S = \sum_{k=0}^{\infty} \frac{1}{k^2+4}.$$

6.4 Argument Principle

Theorem 6.4.1 Argument Principle

Let C be a simple closed contour lying entirely within a domain D. Suppose f is analytic in D except at a finite number of poles inside C, and that $f(z) \neq 0$ on C. Then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = N_0 - N_p,$$

where N_0 is the total number of zeros of f inside C and N_p is the total number of poles of f inside C. In determining N_0 and N_p, zeros and poles are counted according to their order or multiplicities.

6.5 Rouche's Theorem.

The main result in this section is known as **Rouche's Theorem** and is a consequence of the argument principle. It can be useful in locating regions of the complex plane in which a given analytic function has zeros. So, this theorem is helpful in determining the number of zeros of an analytic function.

Theorem 6.5.1 Rouch´e's Theorem

Let C denote a simple closed contour, and suppose that

(a) two functions f and g are analytic inside and on C ;

(b) $|f(z)| > |g(z)|$ at each point on C.

Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, inside C .

Example 6.5.2 Locate the zeros of the polynomial function

In order to determine the number of roots, counting multiplicities, of the equation

$$z^4 + 3z^3 + 6 = 0 \tag{6.11}$$

inside the circle $|z| = 2$, write $f(z) = 3z^3$ and $g(z) = z^4 + 6$.
Then observe that when $|z| = 2$,

$$|f(z)| = 3|z|^3 = 24 \quad \text{and} \quad |g(z)| \leq |z|^4 + 6 = 16 + 6 = 22.$$

The conditions in Rouché's Theorem are thus satisfied. Consequently, since $f(z)$ has three zeros, counting multiplicities, inside the circle $|z| = 2$, so does $f(z) + g(z)$.
That is, equation (6.11) has three roots there, counting multiplicities.

6.6 Exercises Set

Exercise 6.6.1

Evaluate

$$I = \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}.$$

Solution. Use the substitution $z = e^{i\theta}$. Then

$$\cos\theta = \frac{1}{2}\Big(z + \frac{1}{z}\Big), \qquad d\theta = \frac{dz}{iz},$$

and the integral over $\theta \in [0, 2\pi]$ becomes a contour integral over the unit circle $|z| = 1$:

$$I = \oint_{|z|=1} \frac{1}{5 + 4 \cdot \frac{1}{2}\left(z + \frac{1}{z}\right)} \cdot \frac{dz}{iz} = \oint_{|z|=1} \frac{dz}{i(2z^2 + 5z + 2)},$$

since multiplying numerator and denominator by z gives

$$\frac{1}{5 + 2\left(z + \frac{1}{z}\right)} \cdot \frac{1}{iz} = \frac{z}{i(2z^2 + 5z + 2)} \cdot \frac{1}{z} = \frac{1}{i(2z^2 + 5z + 2)}.$$

The denominator polynomial $2z^2 + 5z + 2$ factors (or solve the quadratic):

$$2z^2 + 5z + 2 = 0 \quad \Rightarrow \quad z = \frac{-5 \pm \sqrt{25 - 16}}{4} = \frac{-5 \pm 3}{4},$$

so the roots are $z = -\frac{1}{2}$ and $z = -2$. Only $z = -\frac{1}{2}$ lies inside the unit circle.

The integrand has a simple pole at $z = -\frac{1}{2}$. Writing $p(z) = 2z^2 + 5z + 2$, the residue at $z = -\frac{1}{2}$ is

$$\operatorname{Res}\left(\frac{1}{i\,p(z)}, -\tfrac{1}{2}\right) = \frac{1}{i\,p'(-\frac{1}{2})} = \frac{1}{i(4z+5)\big|_{z=-1/2}} = \frac{1}{i \cdot 3} = \frac{1}{3i}.$$

By the Residue Theorem,

$$I = 2\pi i \cdot \operatorname{Res}\left(\frac{1}{i\,p(z)}, -\tfrac{1}{2}\right) = 2\pi i \cdot \frac{1}{3i} = \frac{2\pi}{3}.$$

Exercise 6.6.2

Evaluate

$$I = \int_0^{2\pi} \frac{d\theta}{(2 + \cos\theta)^2}.$$

Solution. Use the substitution $z = e^{i\theta}$. Then

$$\cos\theta = \frac{1}{2}\Big(z + \frac{1}{z}\Big), \qquad d\theta = \frac{dz}{iz},$$

and the integral becomes a contour integral over the unit circle $|z| = 1$:

$$I = \oint_{|z|=1} \frac{1}{\left(2 + \frac{1}{2}(z + z^{-1})\right)^2} \frac{dz}{iz}.$$

Simplify the integrand. Note

$$2+\tfrac{1}{2}\Big(z+\frac{1}{z}\Big)=\frac{z^2+4z+1}{2z},$$

so

$$\frac{1}{\big(2+\frac{1}{2}(z+z^{-1})\big)^2}=\frac{4z^2}{(z^2+4z+1)^2}.$$

Therefore

$$I=\oint_{|z|=1}\frac{4z^2}{(z^2+4z+1)^2}\cdot\frac{dz}{iz}=\frac{4}{i}\oint_{|z|=1}\frac{z}{(z^2+4z+1)^2}\,dz.$$

Factor the quadratic:

$$z^2+4z+1=(z-z_1)(z-z_2),\qquad z_1=-2-\sqrt{3},\quad z_2=-2+\sqrt{3}.$$

Numerically $z_1\approx-3.732$ (outside $|z|=1$) and $z_2\approx-0.2679$ (inside $|z|=1$). Hence only z_2 contributes.

Write the integrand as

$$\frac{z}{(z^2+4z+1)^2}=\frac{z}{(z-z_1)^2(z-z_2)^2}.$$

Since z_2 is a pole of order 2, the residue at z_2 is

$$\operatorname{Res}\Big(\frac{z}{(z^2+4z+1)^2},z_2\Big)=\lim_{z\longrightarrow z_2}\frac{d}{dz}\Big[(z-z_2)^2\cdot\frac{z}{(z-z_1)^2(z-z_2)^2}\Big]=\lim_{z\longrightarrow z_2}\frac{d}{dz}\frac{z}{(z-z_1)^2}.$$

Differentiate:

$$\frac{d}{dz}\frac{z}{(z-z_1)^2}=\frac{(z-z_1)^2-2z(z-z_1)}{(z-z_1)^4}=\frac{-z-z_1}{(z-z_1)^3}.$$

Evaluating at $z=z_2$ gives

$$\operatorname{Res}=\frac{-z_2-z_1}{(z_2-z_1)^3}.$$

Use the symmetric relations for the quadratic z^2+4z+1:

$$z_1+z_2=-4\quad\Rightarrow\quad -z_2-z_1=4,$$

and

$$z_2-z_1=2\sqrt{3}\quad\Rightarrow\quad (z_2-z_1)^3=(2\sqrt{3})^3=24\sqrt{3}.$$

Thus

$$\operatorname{Res}=\frac{4}{24\sqrt{3}}=\frac{1}{6\sqrt{3}}.$$

Now apply the Residue Theorem. Since

$$I=\frac{4}{i}\oint_{|z|=1}\frac{z}{(z^2+4z+1)^2}\,dz=\frac{4}{i}\cdot 2\pi i\cdot\operatorname{Res}=8\pi\cdot\operatorname{Res},$$

we obtain

$$I=8\pi\cdot\frac{1}{6\sqrt{3}}=\frac{4\pi}{3\sqrt{3}}.$$

Rationalizing the denominator gives the equivalent form

$$I=\frac{4\pi\sqrt{3}}{9}.$$

Solution (Another proof of the fundamental theorem of algebra)**.** We consider a polynomial

$$P(z)=a_0+a_1z+a_2z^2+\cdots+a_nz^n\quad(a_n\neq0)\tag{6.12}$$

of degree n $(n\geq1)$. Show that P has n zeros, counting multiplicities. article amsmath, amssymb

We write

$$f(z) = a_n z^n, \quad g(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1},$$

and let z be any point on a circle $|z| = R$, where $R > 1$. When such a point is taken, we see that

$$|f(z)| = |a_n| R^n.$$

Also,

$$|g(z)| \le |a_0| + |a_1| R + |a_2| R^2 + \cdots + |a_{n-1}| R^{n-1}.$$

Consequently, since $R > 1$,

$$|g(z)| \le (|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|)\, R^{n-1},$$

and it follows that

$$\frac{|g(z)|}{|f(z)|} \le \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n| R} < 1,$$

if, in addition to being greater than unity,

$$R > \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n|}. \tag{4}$$

That is, $|f(z)| > |g(z)|$ when $R > 1$ and inequality (4) is satisfied. Rouché's Theorem then tells us that $f(z)$ and $f(z) + g(z)$ have the same number of zeros, namely n, inside C. Hence we may conclude that $P(z)$ has precisely n zeros, counting multiplicities, in the complex plane.

Exercise 6.6.3

Evaluate

$$I = \int_0^{\infty} \frac{1}{\sqrt{x(x+1)}}\, dx.$$

Solution. First observe that the real integral is improper for two reasons: there is an infinite discontinuity at $x = 0$ and an infinite limit of integration. However, since the integrand behaves like $x^{-1/2}$ near the origin and like $x^{-3/2}$ as $x \longrightarrow \infty$, the integral converges.

We form the contour integral

$$\oint_C \frac{1}{z^{1/2}(z+1)}\, dz,$$

where C is the closed contour shown in FIG.16 consisting of four components: C_r and C_R are small and large circular arcs, and AB and ED are line segments along opposite sides of the branch cut on the positive real axis. The function

$$f(z) = \frac{1}{z^{1/2}(z+1)}$$

is analytic inside and on C, except for the simple pole at $z = -1 = e^{\pi i}$. Hence

$$\oint_C \frac{dz}{z^{1/2}(z+1)} = 2\pi i \,\mathrm{Res}(f(z), -1).$$

On the lower and upper sides of the branch cut we have

$$\int_{ED} = \int_r^R \frac{(xe^{2\pi i})^{-1/2}}{xe^{2\pi i} + 1} (e^{2\pi i} dx) = -\int_r^R \frac{x^{-1/2}}{x+1}\, dx,$$

and

$$\int_{AB} = \int_R^r \frac{(xe^{0i})^{-1/2}}{xe^{0i} + 1} (e^{0i} dx) = \int_r^R \frac{x^{-1/2}}{x+1}\, dx.$$

As $r \longrightarrow 0$ and $R \longrightarrow \infty$, the integrals over C_r and C_R vanish, and so we obtain

$$2\int_0^\infty \frac{dx}{\sqrt{x(x+1)}} = 2\pi i \operatorname{Res}(f(z), -1).$$

Finally, the residue at $z = -1$ is

$$\operatorname{Res}(f(z), -1) = \lim_{z \longrightarrow -1} (z+1)\frac{1}{z^{1/2}(z+1)} = \left.\frac{1}{z^{1/2}}\right|_{z=e^{\pi i}} = e^{-\pi i/2} = -i.$$

Therefore,

$$2I = 2\pi i(-i) = 2\pi \quad \Longrightarrow \quad I = \int_0^\infty \frac{dx}{\sqrt{x(x+1)}} = \pi.$$

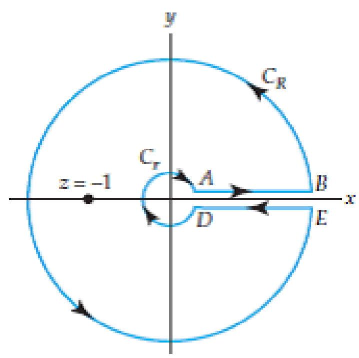

FIG.16

Exercise 6.6.4

Prove the Jordan lemma.

we begin by the Jordan's inequality:

$$\int_0^\pi e^{-R\sin\theta}\, d\theta < \frac{\pi}{R} \qquad (R > 0). \tag{6.13}$$

To verify inequality (6.13), we first note from the graphs of the functions $y = \sin\theta$ and $y = \frac{2}{\pi}\theta$ that

$$\sin\theta \geq \frac{2}{\pi}\theta \qquad (0 \leq \theta \leq \tfrac{\pi}{2}).$$

Consequently, since $R > 0$,

$$e^{-R\sin\theta} \leq e^{-2R\theta/\pi} \qquad (0 \leq \theta \leq \tfrac{\pi}{2}).$$

Therefore,

$$\int_0^{\pi/2} e^{-R\sin\theta}\, d\theta \;\leq\; \int_0^{\pi/2} e^{-2R\theta/\pi}\, d\theta = \frac{\pi}{2R}\left(1 - e^{-R}\right), \qquad (R > 0). \tag{2}$$

Hence,

$$\int_0^{\pi/2} e^{-R\sin\theta}\, d\theta \leq \frac{\pi}{2R}, \qquad (R > 0).$$

But this is just another form of inequality (6.13), since the graph of $y = \sin\theta$ is symmetric with respect to the vertical line $\theta = \pi/2$ on the interval $0 \le \theta \le \pi$. keeping in mind statements (a)–(b) of theorem (6.3.9) of its hypothesis, we write

$$\int_{C_R} f(z)e^{iaz}\,dz = \int_0^{\pi} f(Re^{i\theta})\exp\!\big(iaRe^{i\theta}\big)\,Re^{i\theta} i\,d\theta.$$

Since

$$\left|f(Re^{i\theta})\right| \le M_R \quad \text{and} \quad \left|\exp(iaRe^{i\theta})\right| \le e^{-aR\sin\theta},$$

it follows that

$$\left|\int_{C_R} f(z)e^{iaz}\,dz\right| \le M_R R\int_0^{\pi} e^{-aR\sin\theta}\,d\theta.$$

In view of Jordan's inequality (1), we obtain

$$\left|\int_{C_R} f(z)e^{iaz}\,dz\right| < \frac{M_R\pi}{a}.$$

The final limit in the theorem is now evident, since
$M_R \longrightarrow 0$ as $R \longrightarrow \infty$.

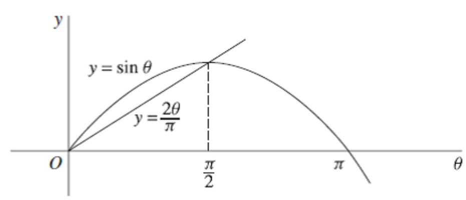

FIG.18

Exercise 6.6.5 Dirichlet's Integral

show that

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Solution. We obtain Dirichlet's integral by integrating $\frac{e^{iz}}{z}$ around the closed contour illustrated in FIG.19

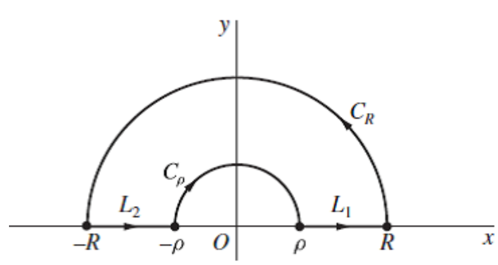

FIG.19

In this figure, ρ and R are positive real numbers with $\rho < R$. The segments L_1 and L_2 correspond to the intervals $\rho \leq x \leq R$ and $-R \leq x \leq -\rho$ on the real axis, respectively. The arcs C_ρ and C_R are the semicircles indicated in the diagram. The small semicircle C_ρ is included to bypass the singularity of the function e^{iz}/z at the origin. The Cauchy–Goursat theorem tells us that

$$\int_{L_1} \frac{e^{iz}}{z}\,dz + \int_{C_R} \frac{e^{iz}}{z}\,dz + \int_{L_2} \frac{e^{iz}}{z}\,dz + \int_{C_\rho} \frac{e^{iz}}{z}\,dz = 0,$$

or

$$\int_{L_1} \frac{e^{iz}}{z}\,dz + \int_{L_2} \frac{e^{iz}}{z}\,dz = -\int_{C_\rho} \frac{e^{iz}}{z}\,dz - \int_{C_R} \frac{e^{iz}}{z}\,dz. \tag{5}$$

Moreover, since the legs L_1 and $-L_2$ have parametric representations

$$z = re^{i0} = r \quad (\rho \leq r \leq R), \qquad z = re^{i\pi} = -r \quad (\rho \leq r \leq R),$$

respectively, the left-hand side of equation (5) can be written

$$\int_{L_1} \frac{e^{iz}}{z}\,dz - \int_{-L_2} \frac{e^{iz}}{z}\,dz = \int_\rho^R \frac{e^{ir}}{r}\,dr - \int_\rho^R \frac{e^{-ir}}{r}\,dr = \int_\rho^R \frac{e^{ir} - e^{-ir}}{r}\,dr.$$

Hence,

$$\int_\rho^R \frac{e^{ir} - e^{-ir}}{r}\,dr = 2i\int_\rho^R \frac{e^{ir} - e^{-ir}}{2ir}\,dr = 2i\int_\rho^R \frac{\sin r}{r}\,dr.$$

Consequently, equation (5) reduces to

$$2i\int_\rho^R \frac{\sin r}{r}\,dr = -\int_{C_\rho} \frac{e^{iz}}{z}\,dz - \int_{C_R} \frac{e^{iz}}{z}\,dz. \tag{7}$$

Now, from the Laurent series representation

$$\frac{e^{iz}}{z} = \frac{1}{z}\left(1 + \frac{iz}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots\right) = \frac{1}{z} + i + \frac{i^2}{2!}z + \frac{i^3}{3!}z^2 + \cdots, \quad (0 < |z| < \infty),$$

it is clear that e^{iz}/z has a simple pole at the origin with residue 1. So, according to the theorem at the beginning of this section,

$$\lim_{\rho \longrightarrow 0} \int_{C_\rho} \frac{e^{iz}}{z}\,dz = -\pi i.$$

Also, since

$$\left|\frac{1}{z}\right| = \frac{1}{|z|} = \frac{1}{R}, \quad \text{when } z \text{ is a point on } C_R,$$

we know from Jordan's lemma, we get

$$\lim_{R \longrightarrow \infty} \int_{C_R} \frac{e^{iz}}{z}\,dz = 0.$$

Thus, by letting $\rho \longrightarrow 0$ in equation (7) and then letting $R \longrightarrow \infty$, we arrive at the result

$$2i\int_0^\infty \frac{\sin r}{r}\,dr = \pi i \Longleftrightarrow \int_0^\infty \frac{\sin r}{r}\,dr = \frac{\pi}{2}.$$

which is, in fact, the same of desired results.

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