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**Qualitative Study of a Class Fractional Differential Equations
and Inclusions in a Banach Space**

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Abstract

The main goal of this thesis is to present a set of results on the existence and uniqueness of certain classes of the initial value problems and boundary value problems for differential problems involving Caputo, Riemann-Liouville and Hilfer derivatives, under certain conditions. The results have been proven analytically, where the existence results are based on some classical fixed point theorems (Banach, Schaefer, Krasnoselskii) as well as Mönch's fixed point theorem combined with the technique of Kuratowski's measure of noncompactness. To support our results, we provide different illustrative examples in each chapter.

Key words and phrases: Fractional calculus, Fractional Volterra integral equation, χ -fractional integral, The measure of Kuratowski, Fixed point theorem, Existence and uniqueness.

Résumé

L'objectif principal de cette thèse est de présenter un ensemble de résultats concernant l'existence et l'unicité de certaines classes de problèmes de Cauchy et de problèmes aux limites pour des équations différentielles impliquant les dérivées de Caputo, de Riemann–Liouville et de Hilfer, sous certaines conditions.

Les résultats ont été établis analytiquement ; ceux relatifs à l'existence reposent sur des théorèmes classiques de point fixe (Banach, Schaefer, Krasnoselskii), ainsi que sur le théorème du point fixe de Mönch combiné à la technique de la mesure de non-compacité de Kuratowski.

Afin d'étayer ces résultats, nous présentons différents exemples illustratifs dans chaque chapitre.

Mots et expressions clés: Calcul fractionnaire, Équation intégrale de Volterra fractionnaire, Intégrale fractionnaire χ , La mesure de Kuratowski, Théorème du point fixe, Existence et unicité.

الملخص

تهدف هذه الأطروحة بشكل أساسي إلى عرض مجموعة من النتائج المتعلقة بوجود ووحداية حلول فئات معينة من مسائل القيمة الابتدائية ومسائل القيمة الحدية للمسائل التفاضلية التي تتضمن مشتقات كابوتو، وريمان-ليوفيل، وهيلفر، وذلك في ظل شروط محددة. وقد تم إثبات هذه النتائج تحليليًا، حيث تستند نتائج الوجود إلى بعض نظريات النقطة الثابتة الكلاسيكية (باناخ، وشيفر، وكراسنوسيلسكي)، بالإضافة إلى نظرية مونش للنقطة الثابتة مقترنة بتقنية مقياس كوراتوفسكي لعدم التراص. ولدعم نتائجنا، نقدم أمثلة توضيحية متنوعة في كل فصل.

الكلمات المفتاحية: حساب التفاضل والتكامل الكسري، معادلة فولتيرا التكاملية الكسرية، التكامل الكسري χ ، مقياس كوراتوفسكي، نظرية النقطة الثابتة، الوجود والوحداية.

The list of works

1. Belqassim Azzouz and Moustafa Beddani, *Existence and Uniqueness of Solutions for a Class From Fractional Differential Equations Involving a Left χ -Hilfer Derivative*, *Mathematical Methods in the Applied Sciences*, **48** (16) (2025), 15132–15139.
2. Moustafa Beddani, Hamid Beddani and Belqassim Azzouz, *On the existence of χ -differentiable solutions for sequential differential system involving a mixed derivative in Banach space*, *Filomat*, **38** (26) (2024), 9223–9236.
3. Belqassim Azzouz, Moustafa Beddani and Hamid Beddani, *Nonlocal Integral Boundary Value Problem to Sequential Differential Equation Involving a Fractional Mixed Derivatives*, *Miskolc Mathematical Notes*. Accepted, (2025).
4. Belqassim Azzouz and Moustafa Beddani, *New results of the right (k, χ) -Hilfer*. Under Review, (2025).

Communication national

- 1-On the solution of a system associated with fractional differential equations. Univ. Mostaganem, (Algeria) 23rd june 2024.
- 2-Fractional order differential systems involving right Caputo and left Riemann-Liouville fractional derivatives with nonlocal coupled conditions. Univ. Relizane, (Algeria)19th – 20th December 2023.
- 3-The existence and uniqueness of solutions for a coupled system involving the right caputo and left Riemann-Liouville fractional derivative equations can be established under certain conditions. Univ. Oum El Bouaghi, (Algeria)10th – 11th February 2024.
- 4-Study for a coupled problem of k-Hilfer. Univ. Mostaganem, (Algeria)11th – 12th December 2024.

5-Study of a coupled system of fractional differential equations. L'ESE D'oran, (Algeria)01st – 02nd June 2024.

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Communication international

1- Solution of a coupled system of fractional differential equations with nonlocal conditions. Univ. Bordj Bou Arréridj, (Algeria)14th – 15th May 2024.

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Introduction

The fractional calculus is a generalization of ordinary calculus to an arbitrary order. Fractional calculus is an important and developing field in both pure and applied mathematics. Applications of the theory of fractional calculus in both basic sciences and engineering are very diverse. They appear increasingly in various fields of research (see [29, 30, 33]). We are witnessing a significant development in fractional calculus and methods in the theory of differential equations. It should be noted that most of the papers and books on fractional calculus are devoted to the solvability of fractional differential equations. Very recently, there have been some papers concerning fractional differential equations (for example, Refs. [5], [13], [17], [24] and [34]). Fractional differential equations have gained a considerable importance due to their applications in science and engineering fields such as control theory, electrochemistry, viscoelasticity and optics, see [29, 30, 33, 35, 36, 38]. Although many different definitions of fractional derivatives exist, the choice of which definition to use depends mainly on the problem at hand. There is no single best definition of the fractional derivative, one should select the most suitable one when modeling a process or considering a mathematical problem, we mention for example Riemann-Liouville, Caputo, Hadamard, Caputo-Fabrizio and Hilfer fractional derivatives, for details, we refer the reader to [5, 13, 17, 24, 31, 34].

There are two measures which are the most important ones. the Kuratowski measure of noncompactness $\vartheta(B)$ of a bounded set B in a metric space is defined as infimum of numbers $r > 0$ such that B can be covered with a finite number of sets of diameter smaller than r , the Hausdorff measure of noncompactness $\chi(B)$ defined as infimum of numbers r such that B can be covered with a finite number of balls of radius smaller than r . Several authors have studied the measures of noncompactness in Banach spaces. See, for example, the books such as [4, 9] and the articles [15, 14, 2]. In recent

decades, fractional differential equations with initial boundary conditions have been studied by many researchers. Recent studies have investigated the existence and uniqueness of solutions of differential equations using fractional Hilfer derivatives (see, for example [11], [21], [42] and references therein), marking a notable advancement in fractional calculus and opening new avenues for mathematical exploration and application (for details, see [12], [22], [25], [28], [32], [37] and references therein).

In this thesis, we investigate the existence of solutions to several problems involving fractional differential equations by using various fixed point theorems. The thesis consists of an introduction, five chapters, and a conclusion with some perspectives, organized as follows:

The first chapter provides the necessary mathematical tools. It begins with elementary notions of fractional calculus, followed by the definition of the space of solutions. Next, it recalls some definitions and properties of the measure of noncompactness, and concludes with a presentation of several fixed point theorems.

The second chapter addresses the existence and uniqueness of solutions for a class of fractional differential equations involving a left χ -Hilfer derivative. It introduces the system, presents the representation formula for the solution, discusses existence and uniqueness results, and ends with an illustrative example.

The third chapter presents new results concerning the solvability of the right (k, χ) -Hilfer fractional differential equation. It formulates the problem, outlines the necessary preliminaries, develops the main results, and concludes with an example illustrating the theory.

The fourth chapter focuses on nonlocal integral boundary value problems for sequential differential equations involving fractional mixed derivatives. It begins with general results and preliminaries, then establishes two key results on the existence of solutions and compactness of (4.1.1)-(4.1.2) using fixed point theorems, and concludes with an example supporting the results.

The fifth chapter applies the Mönch fixed point theorem combined with the technique of measure of noncompactness to study the existence of χ -differentiable solutions for a sequential differential system involving mixed derivatives in Banach spaces. It presents the system, derives the corresponding integral equation, proves the existence of solutions, and provides an illustrative example. This approach has recently proven to be an effective tool for solving fractional differential equations in Banach spaces.

Finally, the thesis summarizes the results and outlines future research directions.

Chapter 1

Some mathematical tools

This chapter will focus on the initial definitions and basic concepts related to fractional calculus, such as fractional integration and fractional differentiation, and some definitions, principles, and theorems that will be used in subsequent chapters. These definitions and properties are taken from the references [1, 40, 41].

1.1 Functional spaces

Let $\chi: [a, b] \rightarrow \mathbb{R}$ be an increasing differentiable function such that $\chi'(\gamma) > 0$, for all $\gamma \in [a, b]$. We define the space $C_{1-\xi, \chi}[a, b]$ as follows

$$C_{1-\xi, \chi}[a, b] = \left\{ g: [a, b] \rightarrow \mathbb{R}, (\chi(\cdot) - \chi(a))^{1-\xi} g(\cdot) \in C[a, b] \right\}, \quad 0 < \xi < 1.$$

Where $C[a, b]$ is the space of the continuous function g on the interval $[a, b]$.

We note that space $C_{1-\xi, \chi}[a, b]$ is a Banach space with the following norm

$$\|g\|_{C_{1-\xi, \chi}[a, b]} = \max_{\gamma \in [a, b]} \left| (\chi(\gamma) - \chi(a))^{1-\xi} g(\gamma) \right|.$$

Moreover, the following space

$$C_{1-\xi, \chi}^{\rho, \theta}[a, b] = \left\{ g: \left(\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right)^{\rho-1} g(\gamma) \in C[a, b] \text{ and } \left(\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right)^{\rho} g(\gamma) \in C_{1-\xi, \chi}[a, b] \right\},$$

is also a Banach space. We define the following auxiliary spaces which will be used later

$$C_{1-\xi, \chi}^{\rho, \theta}[a, b] = \left\{ g: [a, b] \rightarrow \mathbb{R}, g \in C_{1-\xi, \chi}[a, b] \text{ and } {}^{RL}\mathfrak{D}_{a+}^{\rho, \theta} g \in C_{1-\xi, \chi}[a, b] \right\},$$

and

$$C_{1-\xi, \chi}^{\xi}[a, b] = \left\{ g :]a, b[\rightarrow \mathbb{R}, g \in C_{1-\xi, \chi}[a, b] \text{ and } {}^{RL}\mathfrak{D}_{a+}^{\xi, \chi} g \in C_{1-\xi, \chi}[a, b] \right\}.$$

1.2 Elementary Notions of Fractional Calculus

Many mathematical definitions of fractional integration and differentiation exist. Although they do not always lead to identical results, they are considered equivalent for a wide class of functions. In this section, we introduce the fractional integration operator and the two most commonly used definitions of fractional derivatives: the Riemann–Liouville and Hilfer derivatives. We also highlight some basic properties of these two concepts.

First, we introduce the two functions that will be used in fractional calculus theory.

The gamma function

Euler’s function is a special function, which extends the range of the operator to include real and complex values.

Definition 1.2.1. [35] *Let $\gamma \in \mathbb{C}$ such that $\text{Re}(\gamma) > 0$. Then the Gamma function Γ is defined by the integral*

$$\Gamma(\gamma) = \int_0^{\infty} t^{\gamma-1} e^{-t} dt.$$

Proposition 1.2.1. *The function Γ (The gamma function) restricted to $(0, \infty)$ is characterized by the following properties*

(a) $\Gamma(1) = 1$, $\Gamma(0^+) = +\infty$, and also $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(b) $\Gamma(\gamma + 1) = \gamma\Gamma(\gamma)$.

(c) In particular, if $\gamma = n \in \mathbb{N}^*$, so $\Gamma(n) = (n - 1)!$.

The beta function

In some cases, using the beta function is convenient.

Definition 1.2.2. [35] Let $\gamma, w \in \mathbb{C}$, such that $\operatorname{Re}(\gamma) > 0$, $\operatorname{Re}(w) > 0$,

The beta function β is given by the integral

$$\beta(\gamma, w) = \int_0^1 t^{\gamma-1} (1-t)^{w-1} dt.$$

Proposition 1.2.2. The beta function β satisfies the following identities

(i) $\beta(\gamma, w) = \beta(w, \gamma)$.

(ii) $\beta(\gamma, w) = \frac{\Gamma(\gamma) \Gamma(w)}{\Gamma(\gamma+w)}$, $\operatorname{Re}(\gamma) > 0$, $\operatorname{Re}(w) > 0$.

1.2.1 The left χ -Riemann-Liouville approach

In this part, we will review the essential definitions and properties of the χ -Riemann-Liouville fractional integral.

Definition 1.2.3. [41] Let $g \in L^1[a, b]$. Then, the left χ -Riemann-Liouville fractional integral of order $\rho > 0$ of the function g is defined by

$${}_{a+}^{\rho, \chi} g(\gamma) = \frac{1}{\Gamma(\rho)} \int_a^\gamma \chi'(t) (\chi(\gamma) - \chi(t))^{\rho-1} g(t) dt.$$

Definition 1.2.4. [41] Let $q-1 < \rho \leq q$, $\chi \in C^q[a, b]$, $\chi'(\gamma) = 0$, $\gamma \in [a, b]$ and $g \in C[a, b]$.

Then, the left χ -Riemann-Liouville fractional derivative of a function g of order ρ is defined by

$${}^{RL} \mathfrak{D}_{a+}^{\rho, \chi} g(\gamma) = \left(\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right)^q {}_{a+}^{q-\rho, \chi} g(\gamma).$$

Lemma 1.2.1. [41] Let $l > 0$, $\xi \in \mathbb{R}$ such that $\xi > -1$, the χ -fractional integral and derivative of a power function are given by

$${}_{a+}^{l, \chi} (\chi(\gamma) - \chi(a))^\xi = \frac{\Gamma(\xi+1)}{\Gamma(\xi+1+l)} (\chi(\gamma) - \chi(a))^{\xi+l},$$

and

$${}^{RL} \mathfrak{D}_{a+}^{l, \chi} (\chi(\gamma) - \chi(a))^\xi = \frac{\Gamma(\xi+1)}{\Gamma(\xi+1-l)} (\chi(\gamma) - \chi(a))^{\xi-l}.$$

Lemma 1.2.2. [41] Let $l_1, l_2 > 0$, and $\rho > 0$. We have the following semigroup property

$${}_{a+}^{l_1, \chi} {}_{a+}^{l_2, \chi} = {}_{a+}^{l_1+l_2, \chi} \quad \text{and} \quad {}^{RL} \mathfrak{D}_{a+}^{\rho, \chi} {}_{a+}^{\rho, \chi} g(\gamma) = g(\gamma).$$

Theorem 1.2.1. [41] Let $l, \sigma > 0$ with $\sigma < 1$, assume that $g \in C_{\sigma, \chi}[a, b]$ and ${}_{a+}^{q-l, \chi} g \in C_{\sigma, \chi}^q[a, b]$.

$${}_{a+}^{l, \chi} \left({}^{RL} \mathfrak{D}_{a+}^{l, \chi} g(\gamma) \right) = g(\gamma) - \sum_{i=1}^q \frac{(\chi(\gamma) - \chi(a))^{l-i}}{\Gamma(l-i+1)} \left[\left(\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right)^{q-i} {}_{a+}^{q-l, \chi} g(\gamma) \right]_{\gamma=a}.$$

Theorem 1.2.2. [41] Let $0 < l, 0 < \xi \leq 1$ and $g \in C_{1-\xi, \chi}[a, b]$. If $l > \xi$. Then, ${}_{a+}^{l, \chi} \in C[a, b]$ and

$${}_{a+}^{l, \chi} g(a) = \lim_{\gamma \rightarrow a+} {}_{a+}^{l, \chi} g(\gamma) = 0.$$

Theorem 1.2.3. [40] Let $0 < \rho < 1$, $0 \leq \theta \leq 1$ and $\xi = \rho + \theta - \theta\rho$, the left χ -Riemann-Liouville fractional integral ${}_{a+}^{\rho, \chi}(\cdot)$ is bounded from $C_{1-\xi, \chi}[a, b]$ into $C_{1-\xi, \chi}[a, b]$.

Lemma 1.2.3 ([24, 26]). Let $\rho > 0$ and $0 < \gamma < 1$. If $\gamma \leq \rho$. Then ${}_{a+}^{\rho, \chi}$ is bounded from $C_{\gamma, \chi}([a, b])$ into $C([a, b])$.

Lemma 1.2.4. Let $q \in \mathbb{N}$, and $g \in C([a, b])$. Then,

$$\left(\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right)^q {}_{a+}^{q, \chi} g(\gamma) = g(\gamma).$$

Lemma 1.2.5. [41] Let $\rho > 0$. Then, for $g \in C_{q-\xi, \chi}[a, b]$, we get

$${}^{RL} \mathfrak{D}_{a+}^{\rho, \chi} {}_{a+}^{\rho, \chi} g(\gamma) = g(\gamma).$$

1.2.2 The left χ -Hilfer approach

This subsection focuses on the basic concepts and properties of the χ -Hilfer fractional derivative and integral. We notice that the fractional integration is a generalization of the concept of integer integration.

Definition 1.2.5. [41] Let $\rho > 0$, $\theta \in [0, 1]$, $\chi \in C^q[a, b]$, $\chi'(\gamma) \neq 0$, $\gamma \in [a, b]$ and $g \in C^q[a, b]$.

Then, the left χ -Hilfer fractional derivative of a function g of order ρ and type θ is defined by

$${}^H \mathfrak{D}_{a+}^{\rho, \theta, \chi} g(\gamma) = {}_{a+}^{\theta(q-\theta), \chi} \left(\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right)^q {}_{a+}^{(1-\theta)(q-\rho), \chi} g(\gamma). \quad (1.2.1)$$

Remark 1.2.1. If we take $\xi = \rho + \theta(q - \rho)$ in equation 1.2.1, we get

$$\begin{aligned} {}^H\mathfrak{D}_{a^+}^{\rho, \theta, \chi} g(\gamma) &= \xi^{-p, \chi} \left(\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right)^q {}^{(q-\xi), \chi}_{a^+} g(\gamma) \\ &= \xi^{-\rho, \chi} {}^{RL}\mathfrak{D}_{a^+}^{\xi, \chi} g(\gamma). \end{aligned}$$

Theorem 1.2.4. [7] Let $0 < \rho < 1$, $\theta \in [0, 1]$ and $\xi = \rho + \theta - \theta\rho$. If $g \in C_{1-\xi, \chi}^\xi[a, b]$. Then,

$${}^{\xi, \chi}_{a^+} {}^{RL}\mathfrak{D}_{a^+}^{\xi, \chi} g(\gamma) = {}^{\rho, \chi}_{a^+} {}^H\mathfrak{D}_{a^+}^{\rho, \theta, \chi} g(\gamma), \quad {}^{RL}\mathfrak{D}_{a^+}^{\xi, \chi} {}^{\rho, \chi}_{a^+} g(\gamma) = {}^{RL}\mathfrak{D}_{a^+}^{\theta - \theta\rho, \chi} g(\gamma).$$

Theorem 1.2.5. [41] Let $\rho > 0$, $0 \leq \theta \leq 1$, $g \in L^1[a, b]$. Assume that ${}^{RL}\mathfrak{D}_{a^+}^{\theta(1-\rho), \chi} g(\gamma)$ exists it lies in $L^1[a, b]$. Then,

$${}^H\mathfrak{D}_{a^+}^{\rho, \theta, \chi} {}^{\rho, \chi}_{a^+} g(\gamma) = {}^{\theta(1-\rho), \chi}_{a^+} {}^{RL}\mathfrak{D}_{a^+}^{\theta(1-\rho), \chi} g(\gamma), \quad \gamma \in (a, b].$$

Moreover, if $g \in C_{1-\xi, \chi}[a, b]$, ${}^{\theta(1-\rho), \chi}_{a^+} g \in C_{1-\xi, \chi}^1[a, b]$, then ${}^H\mathfrak{D}_{a^+}^{\rho, \theta, \chi} {}^{\rho, \chi}_{a^+} g(\cdot)$ exists on $(a, b]$ and

$${}^H\mathfrak{D}_{a^+}^{\rho, \theta, \chi} {}^{\rho, \chi}_{a^+} g(\gamma) = g(\gamma), \quad \gamma \in (a, b].$$

1.3 Measure of noncompactness

We recall here some definitions and properties of measure of noncompactness. For more details, we refer the reader to Deimling [18] and Kamenskii [23].

Definition 1.3.1. Let Y^+ be the positive cone of an ordered Banach space (Y, \cdot) . A function m defined on the set of all bounded subsets of the Banach space X with values in Y^+ is called a measure of noncompactness (MNC) on X , if $m(\overline{\text{co}}\Omega) = m(\Omega)$, for all bounded subsets $\Omega \subset X$.

Definition 1.3.2. Let X be a metric space and let $P(X) = \{Y \subset X : Y \text{ bounded}\}$.

A measure of noncompactness m is called:

(i) monotone if $A, B \in P_b(X)$, $A \subset B$ implies $m(A) \leq m(B)$,

where $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$,

(ii) nonsingular if $m(\{a\} \cup A) = m(A)$ for every $a \in X$, $A \in P_b(X)$,

(iii) regular if $m(A) = 0$ is equivalent to the relative compactness of Ω .

One of the most important examples of MNC is Hausdorff MNC χ defined on each bounded subset Ω of X by

$$\chi(\Omega) = \inf \{ \varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net} \}.$$

Without confusion, Kuratowski MNC ϑ is defined on each bounded subset Ω of X by

$$\vartheta(\Omega) = \inf \{ \varepsilon > 0 : \Omega \text{ admits a finite cover by sets of diameter } \leq \varepsilon \}.$$

It is well known that Hausdorff MNC χ and Kuratowski MNC ϑ enjoy the above properties (i) – (iii) and other properties.

$$(iv) \quad m(A + B) = m(A) + m(B),$$

$$(v) \quad m(c.B) = |c|m(B), \quad c \in \mathbb{R},$$

$$(vi) \quad m(coB) = m(B).$$

Remark 1.3.1. For every $A \in P_b(X)$, we have $\chi(A) \leq \vartheta(A) \leq 2\chi(A)$.

In the following, an example of useful measure of noncompactness in spaces of continuous functions is presented.

Example 1.3.1. We consider the general example of MNC in the space of continuous functions $C([a, b], X)$ defined for all $\Omega \in C([a, b], X)$ by

$$m(\Omega) = \sup_{t \in [a, b]} \chi(\Omega(t)),$$

where χ is Hausdorff MNC in X and $\Omega(t) = \{y(t) : y \in \Omega\}$.

Lemma 1.3.1. [39] Let $J = [0, \infty)$. If $H \in C(I, X)$ is bounded and equicontinuous, then $\vartheta(H(\cdot))$ is continuous on I , and

$$\vartheta_C(H) = \max_{t \in I} \vartheta(H(t)), \quad \vartheta \left(\int_I x(t) dt, x \in H \right) = \int_I \vartheta(H(t)) dt,$$

where $H(t) = \{x(t), x \in H\}$, $t \in I$, I is a compact interval of J and ϑ is the Kuratowski non-compactness measure on the space X .

1.4 On fixed point theorems

Fixed point theory offers basic tools for examining the existence and uniqueness of solutions to various non-linear problems. It often depends on specific properties (e.g, contraction, complete continuity, etc). Recently, fixed point theory has proven to be a highly useful and significant instrument in studying a variety of phenomena across a wide range of scientific and engineering domains. Fixed point theory plays an essential role in solving fractional differential equations and their applications. In this subsection, we will present some fixed point theorems that we will need for this thesis, such as Banach's theorem, Schaefer's theorem, and Krasnoselskii's theorem, in addition to the fixed point theorem of Mönch combined with Kuratowski's measure of non-compactness.

Definition 1.4.1. [20](Fixed point) Let $(X, \|\cdot\|_x)$ be a Banach space and let a map $T : X \rightarrow X$. We say that $x \in X$ is a fixed point of T , if x satisfies the equation $T(x) = x$.

Theorem 1.4.1. [24] (Banach fixed point theorem) Let $(X; d)$ be a nonempty complete metric space with $T : X \rightarrow X$ is a contraction mapping. Then map T has a fixed point.

Theorem 1.4.2. [16] (Krasnoselskii's fixed point theorem) Let Ω be a closed convex and nonempty subset of a Banach space X , let T_1, T_2 be the operators such that

i) $T_1x + T_2y \in \Omega$ for every pair $x, y \in \Omega$.

ii) T_1 is compact and continuous.

iii) T_2 is a contraction mapping.

Then there exists $z \in \Omega$ such that $z = T_1z + T_2z$.

Theorem 1.4.3. [43](Schaefer's fixed point theorem) Let $T : X \rightarrow X$ be a completely continuous operator. If the set $E(T) = \{x \in X : x = \lambda^*Tx \text{ for some } \lambda^* \in [0, 1]\}$ is bounded, then T has fixed points.

Theorem 1.4.4. [3] Let E be a Banach space and D a closed and convex subset of E such that D is bounded and contains 0 , and let $N : D \rightarrow D$ be a continuous mapping. If the following implication:

$$V = N(V) \setminus \{0\} \text{ or } V = \overline{\text{conv}} N(V) \implies \rho_i(V) = 0,$$

is satisfied for every subset V of D , then N has at least one fixed point.

Chapter 2

Existence and uniqueness of solutions for a class of fractional differential equations involving a left χ -Hilfer derivative

2.1 Introduction

We study the existence and uniqueness of solutions to the following boundary value problem presented in [6]:

$$(PLS) \begin{cases} {}^H\mathfrak{D}_{a+}^{\rho,\theta,\chi}\mathbf{y}(\gamma) = \mathfrak{h}(\gamma, \mathbf{y}(\gamma)), & \gamma \in]a, b], \\ {}^{1-\xi,\chi}_{a+}\mathbf{y}(a) = \sum_{i=1}^m \lambda_i \mathbf{y}(\mu_i), & \mu_i \in]a, b], \end{cases}$$

where $\chi : [a, b] \rightarrow \mathbb{R}$ be an increasing differentiable function such that $\chi'(\gamma) > 0$, for all $\gamma \in]a, b]$, $\xi = \rho + \theta - \theta\rho$, ${}^H\mathfrak{D}_{a+}^{\rho,\theta,\chi}$ represents the left χ -Hilfer fractional derivative of order ρ with $0 < \rho < 1$ and type θ with $0 \leq \theta \leq 1$, ${}^{1-\xi,\chi}_{a+}$ is the left χ -Riemann-Liouville fractional integral of order $1 - \xi$, $\mathfrak{h} :]a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function. The points μ_i , $i = 1, \dots, m$ are prefixed points satisfying $a < \mu_1 < \dots < \mu_m < b$, λ_i , $i = 1, \dots, m$ are real numbers.

2.2 Representation formula for the solution

This section contains the following lemmas, which play an important role in building an equivalent fractional integral equation of the boundary value problem (PLS), which is essential for the rest of the chapter.

Lemma 2.2.1. (see [21]) Let $\hbar :]a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $\hbar(\cdot, \mathbf{y}(\cdot)) \in C_{1-\xi, \chi}[a, b]$, for any $\mathbf{y} \in C_{1-\xi, \chi}[a, b]$. A function $\mathbf{y} \in C_{1-\xi, \chi}^\xi[a, b]$ is a solution of fractional initial value problem

$$\begin{cases} {}^H\mathfrak{D}_{a+}^{\rho, \theta; \chi} \mathbf{y}(\gamma) = \hbar(\gamma, \mathbf{y}(\gamma)), & \gamma \in (a, b], 0 < \rho < 1, 0 \leq \theta \leq 1, \\ \mathbf{y}(a) = \mathbf{y}_a \in \mathbb{R}, & \xi = \rho + \theta(1 - \rho), \end{cases}$$

if and only if \mathbf{y} satisfies the following volterra integral equation

$$\mathbf{y}(\gamma) = \frac{(\chi(\gamma) - \chi(a))^{\xi-1}}{\Gamma(\xi)} \mathbf{y}_a + \frac{1}{\Gamma(\rho)} \int_a^\gamma \chi'(t) (\chi(\gamma) - \chi(t))^{\rho-1} \hbar(t, \mathbf{y}(t)) dt, \quad \gamma \in (a, b]. \quad (2.2.1)$$

According to Lemma 2.2.1, a new and important equivalent mixed-type integral equation for our problem can be established. We adopt some ideas from (see [42]) to establish an equivalent mixed-type integral equation

$$\begin{aligned} \mathbf{y}(\gamma) = & \frac{(\chi(\gamma) - \chi(a))^{\xi-1}}{\Gamma(\xi) - \sum_{i=1}^m \lambda_i [\chi(\mu_i) - \chi(a)]^{\xi-1}} \left[\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) (\chi(\mu_i) - \chi(t))^{\rho-1} \right. \\ & \left. \times \hbar(t, \mathbf{y}(t)) dt \right] + \frac{1}{\Gamma(\rho)} \int_a^\gamma \chi'(t) (\chi(\gamma) - \chi(t))^{\rho-1} \hbar(t, \mathbf{y}(t)) dt, \quad \gamma \in]a, b]. \end{aligned}$$

To simplify notation, we use the abbreviated symbol :

$$\Psi^{\xi-1}(\cdot, a) = (\chi(\cdot) - \chi(a))^{\xi-1}, \quad \chi'(t) \Psi^{\rho-1}(\cdot, t) = \chi'(t) (\chi(\cdot) - \chi(t))^{\rho-1},$$

and

$$\Psi^{1-\xi}(\cdot, a) = (\chi(\cdot) - \chi(a))^{-\xi+1}, \quad \Lambda = \Gamma(\xi) - \sum_{i=1}^m \lambda_i \Psi^{\xi-1}(\mu_i, a) = 0.$$

Now, we can write

$$\mathbf{y}(\gamma) = \frac{\Psi^{\xi-1}(\gamma, a)}{\Lambda} \left[\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) g(t) dt \right] + \frac{1}{\Gamma(\rho)} \int_a^\gamma \chi'(t) \Psi^{\rho-1}(\gamma, t) g(t) dt. \quad (2.2.2)$$

For brevity, we shall take

$$\hbar(t, \mathbf{y}(t)) = g(t).$$

Lemma 2.2.2. Let $\hbar :]a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $\hbar(\cdot, \mathbf{y}(\cdot)) \in C_{1-\xi, \chi}[a, b]$, for any $\mathbf{y} \in C_{1-\xi, \chi}[a, b]$. A function $\mathbf{y} \in C_{1-\xi, \chi}^\xi[a, b]$ is a solution of the problem (PLS) if and only if \mathbf{y} satisfies the mixed-type integral 2.2.2.

Proof. According to Lemma 2.2.1, a solution of problem (PLS) can be expressed by

$$\mathbf{y}(\gamma) = \frac{\Psi^{\xi-1}(\gamma, a)}{\Gamma(\xi)} \left[{}_{a+}^{1-\xi, \chi} \mathbf{y}(\gamma) \Big|_{\gamma=a} + {}_{a+}^{\rho, \chi} g(\gamma), \right] \quad (2.2.3)$$

now, we substitute $\gamma = \mu_i$ in 2.2.3 into the above equation

$$\mathbf{y}(\mu_i) = \frac{\Psi^{\xi-1}(\mu_i, a)}{\Gamma(\xi)} \left[{}_{a+}^{1-\xi, \chi} \mathbf{y}(\gamma) \Big|_{\gamma=a} + {}_{a+}^{\rho, \chi} g(\mu_i) \right] \quad (2.2.4)$$

by multiply by λ_i to both sides of 2.2.4, we can write

$$\lambda_i \mathbf{y}(\mu_i) = \lambda_i \frac{\Psi^{\xi-1}(\mu_i, a)}{\Gamma(\xi)} \left[{}_{a+}^{1-\xi, \chi} \mathbf{y}(\gamma) \Big|_{\gamma=a} + \lambda_i {}_{a+}^{\rho, \chi} g(\mu_i) \right].$$

Thus, we deduce that

$${}_{a+}^{1-\xi, \chi} \mathbf{y}(a) = \sum_{i=1}^m \lambda_i \mathbf{y}(\mu_i) = \sum_{i=1}^m \lambda_i \frac{\Psi^{\xi-1}(\mu_i, a)}{\Gamma(\xi)} \left[{}_{a+}^{1-\xi, \chi} \mathbf{y}(\gamma) \Big|_{\gamma=a} + \sum_{i=1}^m \lambda_i {}_{a+}^{\rho, \chi} g(\mu_i) \right].$$

We find

$${}_{a+}^{1-\xi, \chi} \mathbf{y}(a) = \frac{\Gamma(\xi) \sum_{i=1}^m \lambda_i {}_{a+}^{\rho, \chi} g(\mu_i)}{\Lambda}. \quad (2.2.5)$$

Substituting 2.2.5 into 2.2.3, we conclude that $\mathbf{y}(\gamma)$ satisfies 2.2.2, when \mathbf{y} is a solution of (PLS).

The necessity has been already proved. Next, we are ready to prove its sufficiency. Applying ${}_{a+}^{1-\xi, \chi}$ to both sides 2.2.2, we obtain

$$\begin{aligned} {}_{a+}^{1-\xi, \chi} \mathbf{y}(\gamma) &= \frac{\Gamma(\xi)}{\Lambda} \left[\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) g(t) dt \right] \\ &\quad + \frac{1}{\Gamma(\rho - \xi + 1)} \int_a^\gamma \chi'(t) \Psi^{\rho-\xi}(\gamma, t) g(t) dt. \end{aligned}$$

Since $1 - \xi < \rho - \xi + 1$, Theorem 1.2.2 can be used when taking the limit $\gamma \rightarrow a$,

$${}_{a+}^{1-\xi, \chi} \mathbf{y}(a) = \frac{\Gamma(\xi)}{\Lambda} \left[\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) g(t) dt \right]. \quad (2.2.6)$$

Substituting $\gamma = \mu_i$ into 2.2.2, we have

$$\begin{aligned} \mathbf{y}(\mu_i) &= \frac{\Psi^{\xi-1}(\mu_i, a)}{\Lambda} \left[\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) g(t) dt \right] \\ &\quad + \frac{1}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) g(t) dt. \end{aligned}$$

Then, we drive

$$\begin{aligned} \sum_{i=1}^m \lambda_i \mathbf{y}(\mu_i) &= \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) g(t) dt \left[1 + \sum_{i=1}^m \frac{\lambda_i}{\Lambda} \Psi^{\xi-1}(\mu_i, a) \right] \\ &= \frac{\Gamma(\xi)}{\Lambda} \left[\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) g(t) dt \right]. \end{aligned} \quad (2.2.7)$$

From Equations 2.2.6 and 2.2.7, we obtain

$${}_{a+}^{1-\xi, \chi} \mathbf{y}(a) = \sum_{i=1}^m \lambda_i \mathbf{y}(\mu_i).$$

This proves the initial condition 2.1 is verified.

Now by applying ${}^{RL} \mathfrak{D}_{a+}^{\xi, \chi}$ to both sides of 2.2.2, it follows from the Theorem 1.2.4 and Lemma 1.2.1 that

$$\begin{aligned} {}^{RL} \mathfrak{D}_{a+}^{\xi, \chi} \mathbf{y}(\gamma) &= {}^{RL} \mathfrak{D}_{a+}^{\xi, \chi} \left(\frac{\Psi^{\xi-1}(\gamma, a)}{\Lambda} \left[\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) g(t) dt \right] \right. \\ &\quad \left. + \frac{1}{\Gamma(\rho)} \int_a^{\gamma} \chi'(t) \Psi^{\rho-1}(\gamma, t) g(t) dt \right) \\ &= {}^{RL} \mathfrak{D}_{a+}^{\xi, \chi} \rho, \chi g(\gamma) = {}^{RL} \mathfrak{D}_{a+}^{\theta(1-\rho), \chi} g(\gamma). \end{aligned} \quad (2.2.8)$$

Since $\mathbf{y} \in C_{1-\xi, \chi}^{\xi}[a, b]$, and by Definition of $C_{1-\xi, \chi}^{\xi}[a, b]$, we have ${}^{RL} \mathfrak{D}_{a+}^{\xi, \chi} \mathbf{y} \in C_{1-\xi, \chi}[a, b]$. Therefore, from 2.2.8 it follows that

$${}^{RL} \mathfrak{D}_{a+}^{\theta(1-\rho), \chi} \bar{h}(\gamma, \mathbf{y}(\gamma)) \in C_{1-\xi, \chi}[a, b]. \quad (2.2.9)$$

Since $\rho < 1$, $\theta \in [0, 1]$ and $0 < 1 - \rho < 1$, we obtain $\theta(1 - \rho) < 1$. Therefore

$$[\theta(1 - \rho)] + 1 = 1.$$

In this case the definition of χ -Riemann-Liouville derivative reduce to

$${}^{RL} \mathfrak{D}_{a+}^{\theta(1-\rho), \chi} \bar{h}(\gamma, \mathbf{y}(\gamma)) = \left(\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right) {}_{a+}^{1-\theta(1-\rho), \chi} \bar{h}(\gamma, \mathbf{y}(\gamma)). \quad (2.2.10)$$

Clearly, by 2.2.9 and 2.2.10, we obtain

$$\left(\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right) {}_{a+}^{1-\theta(1-\rho), \chi} \bar{h}(\gamma, \mathbf{y}(\gamma)) \in C_{1-\xi, \chi}[a, b]. \quad (2.2.11)$$

Since $\xi = \rho + \theta - \theta\rho > \theta(1 - \rho)$, we have $1 - \xi < 1 - \theta + \theta\rho$.

Since $g(\cdot) \in C_{1-\xi, \chi}[a, b]$, by applying Theorem 1.2.2, we get

$${}_{a+}^{1-\theta(1-\rho), \chi} g(\cdot) \in C[a, b]. \quad (2.2.12)$$

Using the remark 1.2.1, from Equation 2.2.11 and 2.2.12, it follows that

$${}_{a+}^{1-\theta+\theta\rho, \chi} g(\cdot) \in C_{1-\xi, \chi}^1[a, b].$$

By applying ${}_{a+}^{\theta(1-\rho), \chi}$ on both sides of Equation 2.2.8 and using Theorem 1.2.1 with

$l = \theta - \theta\rho$ and $q = 1$, we have

$$\begin{aligned} {}_{a+}^{\theta(1-\rho), \chi} RL\mathfrak{D}_{a+}^{\xi, \chi} \mathbf{y}(\gamma) &= {}_{a+}^{\theta(1-\rho), \chi} RL\mathfrak{D}_{a+}^{\theta(1-\rho), \chi} g(\gamma) \\ &= g(\gamma) - \frac{(\chi(\gamma) - \chi(a))^{\theta(1-\rho)-1}}{\Gamma(\theta(1-\rho))} \left[{}_{a+}^{1-\theta(1-\rho), \chi} g(a) \right]. \end{aligned}$$

Using the theorem 1.2.2 with $l = 1 - \theta(1 - \rho)$, we obtain

$$\left[{}_{a+}^{1-\theta(1-\rho), \chi} g(a) \right] = 0.$$

Comparing the last Equality with 2.1, we get

$${}^H\mathfrak{D}_{a+}^{\rho, \theta, \chi} \mathbf{y}(\gamma) = g(\gamma), \quad \gamma \in (a, b].$$

The proof is completed. □

2.3 Existence and uniqueness

We are concerned with the existence and uniqueness of solutions for the problem (PLS).

Theorem 2.3.1. Assume that two following the hypotheses $[H_1]$ and $[H_2]$ are fulfilled

$[H_1]$ Let $\tilde{h} :]a, b[\times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $\tilde{h}(\cdot, \mathbf{y}(\cdot)) \in C_{1-\xi, \chi}^{\theta(1-\rho)}[a, b]$, for any $\mathbf{y} \in C_{1-\xi, \chi}[a, b]$, and there exists $L > 0$ such that

$$|\tilde{h}(\gamma, \mathbf{y}_1) - \tilde{h}(\gamma, \mathbf{y}_2)| \leq L|\mathbf{y}_1 - \mathbf{y}_2|, \text{ for all } \gamma \in]a, b[\text{ and } \mathbf{y}_i \in \mathbb{R} (i = 1, 2). \quad (2.3.1)$$

$[H_2]$ The constant

$$\sigma := \frac{LB(\xi, \rho)}{\Lambda \Gamma(\rho)} \left(\sum_{i=1}^m \lambda_i \Psi^{\rho+\xi-1}(\mu_i, a) + \Psi^\rho(b, a) \right) < 1. \quad (2.3.2)$$

Then, the problem (PLS) has at least one solution in the space $C_{1-\xi, \chi}^\xi[a, b] \cap C_{1-\xi, \chi}^{\rho, \theta}[a, b]$.

Proof. Consider the operator $T : C_{1-\xi, \chi}[a, b] \rightarrow C_{1-\xi, \chi}[a, b]$, it is well defined and given by

$$\begin{aligned} (T\mathbf{y})(\gamma) &= \frac{\Psi^{\xi-1}(\gamma, a)}{\Lambda} \left[\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) F_{\mathbf{y}}(t) dt \right] \\ &\quad + \frac{1}{\Gamma(\rho)} \int_a^{\gamma} \chi'(t) \Psi^{\rho-1}(\gamma, t) F_{\mathbf{y}}(t) dt. \end{aligned} \quad (2.3.3)$$

Where $F_{\mathbf{y}}(t) := \tilde{h}(t, \mathbf{y}(t))$.

Consider the ball $B_r = \left\{ \mathbf{y} \in C_{1-\xi, \chi}[a, b] : \|\mathbf{y} - F_0\|_{C_{1-\xi, \chi}} \leq r \right\}$, $\tilde{h}(t) = F_0(t) := \tilde{h}(t, 0)$, and $r = \frac{w}{1-\sigma}$, where $\sigma < 1$ and

$$w := \frac{B(\xi, \rho)}{\Lambda \Gamma(\rho)} \left[\sum_{i=1}^m \lambda_i \Psi^{\rho+\xi-1}(\mu_i, a) + \Psi^{\rho}(b, a) \right] \|\tilde{h}\|_{C_{1-\xi, \chi}}.$$

Now, we need to analyze the operator T into sum two operators $T_1 + T_2$ as follows

$$T_1\mathbf{y}(\gamma) = \frac{\Psi^{\xi-1}(\gamma, a)}{\Lambda} \left[\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) F_{\mathbf{y}}(t) dt \right]$$

and

$$T_2\mathbf{y}(\gamma) = \frac{1}{\Gamma(\rho)} \int_a^{\gamma} \chi'(t) \Psi^{\rho-1}(\gamma, t) F_{\mathbf{y}}(t) dt.$$

Step 1 : we prove that $T_1\mathbf{y} + T_2u \in B_r$ for every $\mathbf{y}, u \in B_r$.

For operator T_1 , by our hypotheses, we have

$$\begin{aligned} \|\Psi^{1-\xi}(\gamma, a) T_1\mathbf{y}(\gamma)\| &\leq \frac{1}{\Lambda} \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) \left[\|F_{\mathbf{y}}(t) - F_0(t)\| + \|F_0(t)\| \right] dt \\ &\leq \frac{1}{\Lambda} \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) \left(L \|\mathbf{y}(t)\| + \|\tilde{h}(t)\| \right) dt \\ &\leq \frac{B(\xi, \rho)}{\Lambda} \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\rho)} \chi'(t) \Psi^{\rho+\xi-1}(\mu_i, a) \left(L \|\mathbf{y}\|_{C_{1-\xi, \chi}} + \|\tilde{h}\|_{C_{1-\xi, \chi}} \right). \end{aligned}$$

Hence, for every $\mathbf{y} \in B_r$, we find that

$$\|T_1\mathbf{y}\|_{C_{1-\xi, \chi}} \leq \frac{B(\xi, \rho)}{\Lambda} \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\rho)} \chi'(t) \Psi^{\rho+\xi-1}(\mu_i, a) \left(L \|\mathbf{y}\|_{C_{1-\xi, \chi}} + \|\tilde{h}\|_{C_{1-\xi, \chi}} \right). \quad (2.3.4)$$

As for operator T_2 , by using the previous hypotheses, we have

$$\begin{aligned} \|\Psi^{1-\xi}(\gamma, a) T_2u(\gamma)\| &\leq \frac{\Psi^{1-\xi}(\gamma, a)}{\Gamma(\rho)} \int_a^{\gamma} \chi'(t) \Psi^{\rho-1}(\gamma, t) \left[\|F_u(t) - F_0(t)\| + \|F_0(t)\| \right] dt \\ &\leq \frac{\Psi^{1-\xi}(\gamma, a)}{\Gamma(\rho)} \Psi^{\rho+\xi-1}(\gamma, a) B(\xi, \rho) \left(L \|\mathbf{y}\|_{C_{1-\xi, \chi}} + \|\tilde{h}\|_{C_{1-\xi, \chi}} \right). \end{aligned}$$

In view of Lemma 1.2.1 and Inequality 2.3.4, then for every $u \in B_r$, we get :

$$T_2 u \in C_{1-\xi, x} \quad \frac{B(\xi, \rho)}{\Gamma(\rho)} \left(L \|\mathbf{y} - u\|_{C_{1-\xi, x}} + \tilde{h} \right) [\Psi^\rho(\gamma, a)]. \quad (2.3.5)$$

Linking 2.3.5 and 2.3.4, for every $\mathbf{y}, u \in B_r$

$$T_1 \mathbf{y} + T_2 u \in C_{1-\xi, x} \quad T_1 \mathbf{y} \in C_{1-\xi, x} + T_2 u \in C_{1-\xi, x} \quad \sigma r + w \in r.$$

This proves that $T_1 \mathbf{y} + T_2 u \in B_r$ for every $\mathbf{y}, u \in B_r$.

Step 2 : T_1 is a contraction mapping.

For any $\mathbf{y}, u \in B_r$, we have

$$\begin{aligned} & \left| \Psi^{1-\xi}(\gamma, a) T_1 \mathbf{y}(\gamma) - \Psi^{1-\xi}(\gamma, a) T_1 u(\gamma) \right| \\ & \quad \sum_{i=1}^m \frac{\lambda_i}{\Lambda} \frac{1}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) |F_{\mathbf{y}}(t) - F_u(t)| dt \\ & \quad \sum_{i=1}^m \frac{\lambda_i}{\Lambda} \frac{1}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) L \|\mathbf{y}(t) - u(t)\| dt \\ & \quad \frac{B(\xi, \rho)}{\Gamma(\rho)} L \sum_{i=1}^m \frac{\lambda_i}{\Lambda} \Psi^{\rho+\xi-1}(\mu_i, a) \|\mathbf{y} - u\|_{C_{1-\xi, x}}. \end{aligned}$$

This gives

$$\|T_1 \mathbf{y} - T_1 u\|_{C_{1-\xi, x}} \leq \frac{B(\xi, \rho)}{\Gamma(\rho)} L \sum_{i=1}^m \frac{\lambda_i}{\Lambda} \Psi^{\rho+\xi-1}(\mu_i, a) \|\mathbf{y} - u\|_{C_{1-\xi, x}}.$$

The operator T_1 is contraction mapping due to hypothesis $[H_2]$.

Step 3 : The operator T_2 is completely continuous on B_r .

According to Step 1, for $\mathbf{y} \in B_r$, we know that

$$T_2 \mathbf{y} \in C_{1-\xi, x} \quad \left[\frac{B(\xi, \rho)}{\Gamma(\rho)} \left(L \|\mathbf{y} - u\|_{C_{1-\xi, x}} + \tilde{h} \right) [\Psi^\rho(b, a)] \right] := l.$$

So operator T_2 is uniformly bounded on B_r . Now, we show that $(T_2 B_r)$ is equicontinuous.

Let $\mathbf{y} \in B_r$ and for $a < \gamma_1 < \gamma_2 < b$, we obtain

$$\begin{aligned} & \left| \Psi^{1-\xi}(\gamma_2, a) T_2 \mathbf{y}(\gamma_2) - \Psi^{1-\xi}(\gamma_1, a) T_2 \mathbf{y}(\gamma_1) \right| \\ & \quad \left| \frac{\Psi^{1-\xi}(\gamma_2, a)}{\Gamma(\rho)} \int_a^{\gamma_2} \chi'(t) \Psi^{\rho-1}(\gamma_2, t) [\Psi^{\xi-1}(t, a)] \max_{t \in [a, b]} \Psi^{1-\xi}(t, a) F_{\mathbf{y}}(t) dt \right. \\ & \quad \left. - \frac{\Psi^{1-\xi}(\gamma_1, a)}{\Gamma(\rho)} \int_a^{\gamma_1} \chi'(t) \Psi^{\rho-1}(\gamma_1, t) [\Psi^{\xi-1}(t, a)] \max_{t \in [a, b]} \Psi^{1-\xi}(t, a) F_{\mathbf{y}}(t) dt \right| \\ & \quad F_{\mathbf{y}} \in C_{1-\xi, x} [a, b] \frac{B(\xi, \rho)}{\Gamma(\rho)} \left| \left[\Psi^{\rho+\xi-1}(\gamma_2, a) \right] - \left[\Psi^{\rho+\xi-1}(\gamma_1, a) \right] \right|. \end{aligned}$$

Tending to zero as $\gamma_2 - \gamma_1$. Thus $(T_2 B_r)$ is equicontinuous. Hence, the operator $(T_2 B_r)$ is compact. By the Arzela-Ascoli Theorem. It follows Theorem 1.4.2 that the problem (PLS) has at least one solution.

Finally, we show that such a solution is indeed in $C_{1-\xi, \chi}^\xi[a, b]$ By applying ${}^{RL}\mathfrak{D}_{a+}^{\xi, \chi}$ on both sides of 2.2.2, we get

$$\begin{aligned} {}^{RL}\mathfrak{D}_{a+}^{\xi, \chi} \mathbf{y}(\gamma) &= {}^{RL}\mathfrak{D}_{a+}^{\xi, \chi} \rho, \chi g(\gamma) = {}^{RL}\mathfrak{D}_{a+}^{\xi - \rho, \chi} \bar{h}(\gamma, \mathbf{y}(\gamma)) \\ &= {}^{RL}\mathfrak{D}_{a+}^{\theta(1-\rho), \chi} \bar{h}(\gamma, \mathbf{y}(\gamma)). \end{aligned}$$

Since $\bar{h}(\cdot, \mathbf{y}(\cdot)) \in C_{1-\xi, \chi}^{\theta(1-\rho)}[a, b]$, it follows by definition of the space $C_{1-\xi, \chi}^{\theta(1-\rho)}[a, b]$ that ${}^{RL}\mathfrak{D}_{a+}^{\xi, \chi} \mathbf{y}(\gamma) \in C_{1-\xi, \chi}^\xi[a, b]$ which implies that $\mathbf{y}(\gamma) \in C_{1-\xi, \chi}^\xi[a, b]$. \square

Theorem 2.3.2. Assume that hypotheses $[H_1]$ and $[H_2]$ are fulfilled,

if $\sigma < 1$. Then, the problem (PLS) has a unique solution where σ is defined as in Theorem 2.3.1.

Proof. Let the operator $T : C_{1-\xi, \chi}[a, b] \rightarrow C_{1-\xi, \chi}[a, b]$ defined as in Equation 2.3.3.

By Lemma 2.2.2, it is clear that the fixed point of T are solution of problem (PLS) .

Let $\mathbf{y}, u \in C_{1-\xi, \chi}[a, b]$ and $\gamma \in (a, b]$, we have

$$\begin{aligned} & \left| \Psi^{1-\xi}(\gamma, a) T \mathbf{y}(\gamma) - \Psi^{1-\xi}(\gamma, a) T u(\gamma) \right| \\ & \quad \sum_{i=1}^m \frac{\lambda_i}{\Lambda} \frac{1}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) |F_{\mathbf{y}}(t) - F_u(t)| dt \\ & \quad + \frac{\Psi^{1-\xi}(\gamma, a)}{\Gamma(\rho)} \int_a^\gamma \chi'(t) \Psi^{\rho-1}(\gamma, t) |F_{\mathbf{y}}(t) - F_u(t)| dt \\ & \quad \sum_{i=1}^m \frac{\lambda_i}{\Lambda} \frac{L}{\Gamma(\rho)} \int_a^{\mu_i} \chi'(t) \Psi^{\rho-1}(\mu_i, t) \Psi^{\xi-1}(t, a) \left[\mathbf{y} - u \right]_{C_{1-\xi, \chi}} dt \\ & \quad + \frac{L \Psi^{1-\xi}(\gamma, a)}{\Gamma(\rho)} \int_a^\gamma \chi'(t) \Psi^{\rho-1}(\gamma, t) \Psi^{\xi-1}(t, a) \left[\mathbf{y} - u \right]_{C_{1-\xi, \chi}} dt \\ & \quad \left[\sum_{i=1}^m \frac{\lambda_i}{\Lambda} \Psi^{\rho+\xi-1}(\mu_i, a) + \Psi^\rho(b, a) \right] \frac{B(\xi, \rho)}{\Gamma(\rho)} L \left[\mathbf{y} - u \right]_{C_{1-\xi, \chi}}. \end{aligned}$$

This gives, $\|T \mathbf{y} - T u\|_{C_{1-\xi, \chi}} \leq \sigma \|\mathbf{y} - u\|_{C_{1-\xi, \chi}}$. Since $\sigma < 1$, the operator $T : C_{1-\xi, \chi}[a, b] \rightarrow C_{1-\xi, \chi}[a, b]$ is a contraction mapping. Hence by Banach fixed point theorem, it follows that T has a unique fixed point. which is a solution of problem (PLS) . \square

2.4 Illustrative Example

We provide an example to illustrate our main results.

$$(\Theta) \begin{cases} {}^H\mathfrak{D}_{1+}^{\rho, \theta, \chi} \mathbf{y}(\gamma) = \frac{1}{2025e^{\gamma+2024}(1+|\mathbf{y}(\gamma)|)}, & \gamma \in [1, 2], \\ {}_{1+}^{1-\xi, \chi} \mathbf{y}(1) = \frac{2}{5} \mathbf{y}\left(\frac{2}{3}\right), & \xi = \rho + \theta - \theta\rho, \end{cases}$$

Now

$$\bar{h}(\gamma, \mathbf{y}(\gamma)) = \frac{1}{2025e^{\gamma+2024}(1+|\mathbf{y}(\gamma)|)}, \quad \gamma \in (1, 2],$$

and see that $a = 1$, $b = 2$, $\rho = \frac{2}{4}$, $\theta = \frac{2}{6}$, $\chi = \gamma$, $\lambda_1 = \frac{2}{5}$, $\mu_1 = \frac{2}{3}$ and $\xi = \frac{2}{3}$.

Clearly, the function \bar{h} is continuous, and for $|\bar{h}(\gamma, \mathbf{y}_1) - \bar{h}(\gamma, \mathbf{y}_2)| \leq \frac{1}{2025e^{2025}} |\mathbf{y}_1 - \mathbf{y}_2|$

Hence, the condition $[H_2]$ is satisfied with $L = \frac{1}{2025e^{2025}}$. Thus:

$$\begin{aligned} \sigma &:= \frac{LB(\xi, \rho)}{\Gamma(\rho)} \left(\lambda_1 \Psi^{\rho+\xi-1}(\mu_1, a) + \Psi^\rho(b, a) \right) \\ &= \frac{LB\left(\frac{2}{3}, \frac{2}{4}\right)}{\Gamma\left(\frac{2}{4}\right)} \left(\frac{2}{5} \Psi^{\frac{2}{4}+\frac{2}{3}-1}\left(\frac{2}{3}, 1\right) + \Psi^{\frac{2}{4}}(2, 1) \right) < 1. \end{aligned}$$

It follows from Theorem 2.3.2 that the problem (Θ) has a unique solution on $(1, 2]$.

Chapter 3

New results of the right (k, χ) -Hilfer

3.1 Introduction

We aim to establish conditions ensuring the existence and uniqueness of solutions to problems involving fractional differential equations of the right (k, χ) -Hilfer type, as presented in [8]:

$$(\mathbb{P}\mathbb{R}\tau) \begin{cases} {}_{k,H}\mathfrak{D}_{b^-}^{p,\theta,\chi}\mathbf{y}(\gamma) = \mathfrak{h}\left(\gamma, \mathbf{y}(\gamma), \mathbf{y}(\tau\gamma), {}_{k,H}\mathfrak{D}_{b^-}^{p,\theta,\chi}\mathbf{y}(\tau\gamma)\right), & \gamma \in I := [a, b], \\ {}_k^{k-\xi_k,\chi}\mathbf{y}(b) = \sum_{i=1}^m \lambda_i {}_k^{\theta,\chi}\mathbf{y}(\mu_i), & \mu_i \in]a, b[, \end{cases}$$

where ${}_{k,H}\mathfrak{D}_{b^-}^{p,\theta,\chi}(\cdot)$ represents the right (k, χ) -Hilfer fractional derivative of order p with $0 < p < k$ and type θ with $0 < \theta < 1$, ${}_k^{k-\xi_k,\chi}(\cdot)$ and ${}_k^{\theta,\chi}(\cdot)$ are the right (k, χ) -Riemann-Liouville fractional integral of order $k - \xi_k$ and $\theta > 0$ respectively with $\xi_k = p + \theta k - \theta p$, $\mathfrak{h} : [a, b] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is a continuous function, $0 < \tau < 1$, $\mu_i (i = 1, \dots, m)$ are prefixed points satisfying $a < \mu_1 < \dots < \mu_m < b$, λ_i is real numbers.

3.2 Prerequisites

We introduce some definitions and preliminary results that will be used throughout this paper. Let $0 < \xi_k < k$, $q \in \mathbb{N}$, $I_1 = [a, b]$ and $\chi : [a, b] \rightarrow \mathbb{R}$ be an increasing function such that $\chi'(a) = 0$, by $C_{1-\frac{\xi_k}{k},\chi}[a, b]$, $C_{1-\frac{\xi_k}{k},\chi}^q[a, b]$ and $C_{1-\frac{\xi_k}{k},\chi}^{\xi_k}[a, b]$, we denote the weighted spaces of continuous functions

defined by

$$C_{1-\frac{\xi_k}{k},\chi}[a,b] = \left\{ g : [a,b] \rightarrow \mathbb{R}, \mathfrak{I}_{\chi,-}^{-\frac{\xi_k}{k}}(b,\gamma) g(\gamma) \in C[a,b] \right\}, \quad 0 < \frac{\xi_k}{k} < 1,$$

where

$$\mathfrak{I}_{\chi,-}^{-\frac{\xi_k}{k}}(b,\gamma) = (\chi(b) - \chi(\gamma))^{-\frac{\xi_k}{k}+1},$$

with the norm

$$\|g\|_{C_{1-\frac{\xi_k}{k},\chi}[a,b]} = \max_{\gamma \in [a,b]} \left| \mathfrak{I}_{\chi,-}^{-\frac{\xi_k}{k}}(b,\gamma) g(\gamma) \right|.$$

$$C_{1-\frac{\xi_k}{k},\chi}^q[a,b] = \left\{ g : \left[\frac{-k}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^{q-1} g(\gamma) \in C[a,b] \text{ and } \left[\frac{-k}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q g(\gamma) \in C_{1-\frac{\xi_k}{k},\chi}[a,b] \right\},$$

$$\text{and } C_{1-\frac{\xi_k}{k},\chi}^{\xi_k}[a,b] = \left\{ g : [a,b] \rightarrow \mathbb{R}, g \in C_{1-\frac{\xi_k}{k},\chi}[a,b] \text{ and } {}_{k,RL}\mathfrak{D}_{b-}^{\xi_k,\chi} g \in C_{1-\frac{\xi_k}{k},\chi}[a,b] \right\}.$$

Definition 3.2.1. [27] The right (k,χ) -Riemann-Liouville fractional integral of order $p > 0$ of the function $g \in L^1[a,b]$ is defined by

$${}_{k,b-}^{p,\chi} g(\gamma) = \frac{1}{k\Gamma_k(p)} \int_{\gamma}^b \mathfrak{I}_{\chi',-}^{\frac{p}{k}}(t,\gamma) g(t) dt,$$

with

$$\mathfrak{I}_{\chi',-}^{\frac{p}{k}}(t,\gamma) = \chi'(t)(\chi(t) - \chi(\gamma))^{\frac{p}{k}-1},$$

where $\Gamma_k(\cdot)$ is the k -Gamma function defined by

$$\Gamma_k(\gamma) = \int_0^{\infty} t^{\gamma-1} e^{-\frac{t^k}{k}} dt, \quad k > 0,$$

as is characterized by the following properties

$$\Gamma_k(\gamma + k) = \gamma \Gamma_k(\gamma), \quad \Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1} \Gamma\left(\frac{\gamma}{k}\right).$$

Lemma 3.2.1. Let $p_1, p_2 > 0$ and each $g \in C_{q-\frac{\xi_k}{k},\chi}[a,b]$, we have ${}_{k,b-}^{p_1,\chi} g \in C[a,b]$, and

$$({}_{k,b-}^{p_1,\chi} {}_{k,b-}^{p_2,\chi} g)(\gamma) = ({}_{k,b-}^{p_1+p_2,\chi} g)(\gamma). \text{ For all } \gamma \in [a,b].$$

Proof. The proof of Lemma 3.2.1 uses the definition of the right (k,χ) -Riemann-Liouville fractional integral, Dirichlet's formula, the substitution $\chi(t) = \chi(\gamma) + z(\chi(b) - \chi(\gamma))$ and the properties outlined in Definition 3.2.1. □

Definition 3.2.2. The right (k, χ) -Riemann-Liouville fractional derivative of a function $g \in C[a, b]$ of order $q - 1 < p \leq q$ is defined by

$${}_{k,RL}\mathfrak{D}_{b^-}^{p,\chi}g(\gamma) = \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q {}_k \mathfrak{I}_{b^-}^{q-p,\chi}g(\gamma), \quad q \in \mathbb{N}.$$

Lemma 3.2.2. Let $q \in \mathbb{N}$, $k > 0$ and $g \in C([a, b])$. Then,

$$(i_1) \quad \left[-\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q {}_k \mathfrak{I}_{b^-}^{q,\chi}g(\gamma) = g(\gamma).$$

$$(i_2) \quad \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q {}_k \mathfrak{I}_{b^-}^{k,\chi}g(\gamma) = g(\gamma).$$

Proof. (i₁) By the lemma(2.4) [24], For $q - 1 < l \leq q \in \mathbb{N}$ and $g \in C[a, b]$, we get

$$g(\gamma) = {}_{RL}\mathfrak{D}_{b^-}^{l,\chi} {}_k \mathfrak{I}_{b^-}^{l,\chi}g(\gamma)$$

If we take $l = q$, we get

$$\begin{aligned} g(\gamma) &= {}_{RL}\mathfrak{D}_{b^-}^{q,\chi} {}_k \mathfrak{I}_{b^-}^{q,\chi}g(\gamma) \\ &= \left[-\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q {}_k \mathfrak{I}_{b^-}^{q-q,\chi} {}_k \mathfrak{I}_{b^-}^{q,\chi}g(\gamma) \\ &= \left[-\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q {}_k \mathfrak{I}_{b^-}^{q,\chi}g(\gamma). \end{aligned}$$

(i₂) Using the definition 3.2.1, furthermore (i₁), we get

$$\begin{aligned} \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q {}_k \mathfrak{I}_{b^-}^{k,\chi}g(\gamma) &= \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q \frac{1}{k\Gamma_k(qk)} \int_{\gamma}^b \mathfrak{A}_{\chi',-}^q(t, \gamma) g(t) dt \\ &= \left[-\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q \frac{k^q}{kk^{q-1}\Gamma(q)} \int_{\gamma}^b \mathfrak{A}_{\chi',-}^q(t, \gamma) g(t) dt \\ &= \left[-\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q {}_k \mathfrak{I}_{b^-}^{q,\chi}g(\gamma) \\ &= g(\gamma). \end{aligned}$$

□

Lemma 3.2.3. Let $l \in]0, k[$, $\xi_k \in]0, k[$ and $g \in C_{q-\frac{\xi_k}{k}, \chi}[a, b]$. Then the following expression leads to the right inverse operator as follows

$${}_{k,RL}\mathfrak{D}_{b^-}^{l,\chi} {}_k \mathfrak{I}_{b^-}^{l,\chi}g(\gamma) = g(\gamma).$$

Proof. By applying Definition 3.2.2 and Lemma 3.2.2-(i₂), we get

$$\begin{aligned} {}_{RL}\mathfrak{D}_{b^-}^{l,\chi} {}_k^{l,\chi}g(\gamma) &= \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q {}_k^{qk-l,\chi} {}_k^{l,\chi}g(\gamma) \\ &= \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q {}_k^{qk,\chi}g(\gamma) \\ &= g(\gamma). \end{aligned}$$

□

Definition 3.2.3. The right (k, χ) -Hilfer fractional derivative of a function $g \in C^q[a, b]$ of order $p > 0$ and type $\theta \in [0, 1]$ is defined by

$${}_{k,H}\mathfrak{D}_{b^-}^{p,\theta,\chi} g(\gamma) = {}_k^{\theta(qk-p),\chi} \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q {}_k^{(1-\theta)(qk-p),\chi} g(\gamma),$$

on the other hand, we get

$${}_{k,H}\mathfrak{D}_{b^-}^{p,\theta,\chi} g(\gamma) = {}_k^{\xi_k-p,\chi} \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q {}_k^{(qk-\xi_k),\chi} g(\gamma).$$

Theorem 3.2.1. Let $0 < l, k > 0$ and $\sigma = k - \xi_k > 0$ with $k - \xi_k = l < k$. Then, the right (k, χ) -Riemann-Liouville fractional integral ${}_k^{l,\chi}(\cdot)$ is bounded from $C_{1-\frac{\xi_k}{k},\chi}[a, b]$ into $C[a, b]$.

$$\left\| {}_k^{l,\chi} g \right\|_{\infty} \leq \|g\|_{C_{\frac{\sigma}{k},\chi}[a,b]} \frac{\Gamma_k(k-\sigma)}{\Gamma_k(k-\sigma+l)} \mathfrak{A}_{\chi,-}^{\frac{l-\sigma}{k}+1}(a,b).$$

Proof. Let $g \in C_{\frac{\sigma}{k},\chi}[a, b]$, and $\gamma_1, \gamma_2 \in I = [a, b]$ with $\gamma_2 > \gamma_1$. Then, we have

$$\begin{aligned} & \left| {}_k^{l,\chi}g(\gamma_2) - {}_k^{l,\chi}g(\gamma_1) \right| \\ &= \left| \frac{1}{k\Gamma_k(l)} \int_{\gamma_2}^b \mathfrak{A}_{\chi',-}^{\frac{l}{k}}(t, \gamma_2)g(t)dt - \frac{1}{k\Gamma_k(l)} \int_{\gamma_1}^b \mathfrak{A}_{\chi',-}^{\frac{l}{k}}(t, \gamma_1)g(t)dt \right| \\ &= \left| \frac{1}{k\Gamma_k(l)} \int_{\gamma_2}^b \mathfrak{A}_{\chi',-}^{\frac{l}{k}}(t, \gamma_2) \mathfrak{A}_{\chi,-}^{-\frac{\sigma}{k}+1}(b,t) \left| \mathfrak{A}_{\chi,-}^{\frac{\sigma}{k}+1}(b,t)g(t) \right| dt \right. \\ &\quad \left. - \frac{1}{k\Gamma_k(l)} \int_{\gamma_1}^b \mathfrak{A}_{\chi',-}^{\frac{l}{k}}(t, \gamma_1) \mathfrak{A}_{\chi,-}^{-\frac{\sigma}{k}+1}(b,t) \left| \mathfrak{A}_{\chi,-}^{\frac{\sigma}{k}+1}(b,t)g(t) \right| dt \right| \\ &\leq \|g\|_{C_{\frac{\sigma}{k},\chi}[a,b]} \left| {}_k^{l,\chi} \mathfrak{A}_{\chi,-}^{-\frac{\sigma}{k}+1}(b, \gamma_2) - {}_k^{l,\chi} \mathfrak{A}_{\chi,-}^{-\frac{\sigma}{k}+1}(b, \gamma_1) \right| \\ &\leq \|g\|_{C_{\frac{\sigma}{k},\chi}[a,b]} \frac{\Gamma_k(k-\sigma)}{\Gamma_k(k-\sigma+l)} \left| \mathfrak{A}_{\chi,-}^{\frac{l-\sigma}{k}+1}(b, \gamma_2) - \mathfrak{A}_{\chi,-}^{\frac{l-\sigma}{k}+1}(b, \gamma_1) \right|. \end{aligned}$$

Since $\sigma = l < k$, using the continuity of χ , we get

$$\left| {}_k^{l,\chi} g(\gamma_2) - {}_k^{l,\chi} g(\gamma_1) \right| = 0 \text{ as } |\gamma_2 - \gamma_1| \rightarrow 0.$$

Consequently, ${}_k \mathcal{I}_{b^-}^{l,\chi} g \in C[a, b]$, following similar type of steps as above, we can see that

$$\left\| {}_k \mathcal{I}_{b^-}^{l,\chi} g \right\|_{C[a,b]} \leq \|g\|_{C_{\frac{\sigma}{k}, \chi}[a,b]} \frac{\Gamma_k(k-\sigma)}{\Gamma_k(k-\sigma+l)} \mathfrak{A}_{\chi, -}^{\frac{l-\sigma}{k}+1}(a, b).$$

Thus the proof is complete. \square

Theorem 3.2.2. Let $l, \sigma > 0, k > 0$ with $\sigma < k$ and $q = \left\lfloor \frac{l}{k} \right\rfloor$, assume that $g \in C_{\frac{\sigma}{k}, \chi}[a, b]$ and ${}_k \mathcal{I}_{b^-}^{qk-l, \chi} g$

$\in C_{\frac{\sigma}{k}, \chi}^q[a, b]$. Then,

$${}_k \mathcal{I}_{b^-}^{l,\chi} \left({}_k \mathcal{H} \mathfrak{D}_{b^-}^{l,\chi} g(\gamma) \right)$$

$$= (-1)^{q+1} \left[g(\gamma) - \sum_{j=1}^{j=q} \frac{\mathfrak{A}_{\chi, -}^{\frac{l}{k}-j+1}(b, \gamma)}{\Gamma_k(l-jk+k)} \left(\left[\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^{q-j} {}_k \mathcal{I}_{b^-}^{qk-l, \chi} g(\gamma) \right) \right]_{\gamma=b}.$$

Proof. First, by the lemma 3.2.2-(i₂), Taking $q=1$, we have

$$\begin{aligned} (-1)^1 \left[\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right] {}_k \mathcal{I}_{b^-}^{k, \chi} g(\gamma) &= g(\gamma) \\ \left[\frac{-k}{\chi'(\gamma)} \frac{d}{d\gamma} \right] {}_k \mathcal{I}_{b^-}^{k, \chi} g(\gamma) &= g(\gamma). \end{aligned} \quad (3.2.1)$$

Using the equation 3.2.1 with $g(\gamma)$ replaced by ${}_k \mathcal{I}_{b^-}^{l,\chi} \left({}_k \mathcal{H} \mathfrak{D}_{b^-}^{l,\chi} g(\gamma) \right)$, we have

$${}_k \mathcal{I}_{b^-}^{l,\chi} \left({}_k \mathcal{H} \mathfrak{D}_{b^-}^{l,\chi} g(\gamma) \right) = \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right] {}_k \mathcal{I}_{b^-}^{k, \chi} \left({}_k \mathcal{I}_{b^-}^{l,\chi} \left({}_k \mathcal{H} \mathfrak{D}_{b^-}^{l,\chi} g(\gamma) \right) \right). \quad (3.2.2)$$

From the relation 3.2.2 and the definition 3.2.1 and the definition of the right (k, χ) -Riemann-liouville fractional integrals, and derivative, we have

$$\begin{aligned} & {}_k \mathcal{I}_{b^-}^{l,\chi} \left[{}_k \mathcal{H} \mathfrak{D}_{b^-}^{l,\chi} g(\gamma) \right] \\ &= \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right] \left({}_k \mathcal{I}_{b^-}^{k+l, \chi} \left[{}_k \mathcal{H} \mathfrak{D}_{b^-}^{l,\chi} g(\gamma) \right] \right) \\ &= \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right] \left(\frac{1}{k\Gamma_k(k+l)} \int_{\gamma}^b \mathfrak{A}_{\chi, -}^{\frac{l}{k}+1}(t, \gamma) {}_k \mathcal{H} \mathfrak{D}_{b^-}^{l,\chi} g(t) dt \right) \\ &= \left[-\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right] \left(\frac{1}{k^{\frac{l}{k}} \Gamma(\frac{l}{k}+1)} \int_{\gamma}^b \mathfrak{A}_{\chi, -}^{\frac{l}{k}+1}(t, \gamma) {}_k \mathcal{H} \mathfrak{D}_{b^-}^{l,\chi} g(t) dt \right) \\ &= \left[-\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right] \left(\frac{(-1)^q}{k^{\frac{l}{k}} \Gamma(\frac{l}{k}+1)} \int_{\gamma}^b \mathfrak{A}_{\chi, -}^{\frac{l}{k}+1}(t, \gamma) \left[\frac{k}{\chi'(t)} \frac{d}{dt} \right]^q {}_k \mathcal{I}_{b^-}^{qk-l, \chi} g(t) dt \right). \end{aligned} \quad (3.2.3)$$

Integration by parts the relation

$$\frac{1}{k^{\frac{l}{k}} \Gamma(\frac{l}{k}+1)} \int_{\gamma}^b \mathfrak{A}_{\chi, -}^{\frac{l}{k}+1}(t, \gamma) \left[\frac{k}{\chi'(t)} \frac{d}{dt} \right]^q {}_k \mathcal{I}_{b^-}^{qk-l, \chi} g(t) dt, \quad ()$$

we obtain

$$\begin{aligned}
() &= \frac{k}{k^{\frac{l}{k}} \Gamma(\frac{l}{k} + 1)} \int_{\gamma}^b \mathfrak{A}_{\chi, -}^{\frac{l}{k} + 1}(t, \gamma) \frac{d}{dt} \left(\left[\frac{k}{\chi'(t)} \frac{d}{dt} \right]^{q-1} {}_k q_{b-}^{k-l, \chi} g(t) \right) dt \\
&= \frac{k^{1 - \frac{l}{k}}}{\Gamma(\frac{l}{k} + 1)} \mathfrak{A}_{\chi, -}^{\frac{l}{k} + 1}(b, \gamma) \left(\left[\frac{k}{\chi'(t)} \frac{d}{dt} \right]^{q-1} {}_k q_{b-}^{k-l, \chi} g(t) \right)_{t=b} \\
&\quad - \frac{k^{2 - \frac{l}{k}}}{\Gamma(\frac{l}{k})} \int_{\gamma}^b \mathfrak{A}_{\chi, -}^{\frac{l}{k}}(t, \gamma) \frac{d}{dt} \left(\left[\frac{k}{\chi'(t)} \frac{d}{dt} \right]^{q-2} {}_k q_{b-}^{k-l, \chi} g(t) \right) dt.
\end{aligned}$$

Repeating the process of integration by parts n^{th} step, we have

$$\begin{aligned}
() &= \sum_{j=1}^{j=q} \frac{k^{j - \frac{l}{k}}}{\Gamma(\frac{l}{k} - j + 2)} \mathfrak{A}_{\chi, -}^{\frac{l}{k} - j + 2}(b, t) \left(\left[\frac{k}{\chi'(t)} \frac{d}{dt} \right]^{q-j} {}_k q_{b-}^{k-l, \chi} g(t) \right)_{t=b} \\
&\quad - \frac{k^{q - \frac{l}{k}}}{\Gamma(\frac{l}{k} - (q-1))} \int_{\gamma}^b \mathfrak{A}_{\chi, -}^{\frac{l}{k} - q + 1}(t, \gamma) \left[\frac{k}{\chi'(t)} \frac{d}{dt} \right]^{q-q} {}_k q_{b-}^{k-l, \chi} g(t) dt.
\end{aligned}$$

Now, let us consider the definition of the right fractional integral (k, χ) -Riemann-Liouville, the quasi-group property, and the properties given in Definition 3.2.1, we have

$$\begin{aligned}
() &= \sum_{j=1}^{j=q} \frac{k^{j - \frac{l}{k}}}{\Gamma(\frac{l}{k} - j + 2)} \mathfrak{A}_{\chi, -}^{\frac{l}{k} - j + 2}(b, \gamma) \left(\left[\frac{k}{\chi'(t)} \frac{d}{dt} \right]^{q-j} {}_k q_{b-}^{k-l, \chi} g(t) \right)_{t=b} \\
&\quad - \frac{k}{k \Gamma_k(l + k - qk)} \int_{\gamma}^b \mathfrak{A}_{\chi', -}^{(l+k-qk)}(t, \gamma) {}_k q_{b-}^{k-l, \chi} g(t) dt \\
&= \sum_{j=1}^{j=q} \frac{k^{j - \frac{l}{k}}}{\Gamma(\frac{l}{k} - j + 2)} \mathfrak{A}_{\chi, -}^{\frac{l}{k} - j + 2}(b, \gamma) \left(\left[\frac{k}{\chi'(t)} \frac{d}{dt} \right]^{q-j} {}_k q_{b-}^{k-l, \chi} g(t) \right)_{t=b} \\
&\quad - k {}_k {}_{b-}^{l+k-qk, \chi} {}_k q_{b-}^{k-l, \chi} g(\gamma) \\
&= \sum_{j=1}^{j=q} \frac{k^{j - \frac{l}{k}}}{\Gamma(\frac{l}{k} - j + 2)} \mathfrak{A}_{\chi, -}^{\frac{l}{k} - j + 2}(b, \gamma) \left(\left[\frac{k}{\chi'(t)} \frac{d}{dt} \right]^{q-j} {}_k q_{b-}^{k-l, \chi} g(t) \right)_{t=b} \\
&\quad - \int_{\gamma}^b \chi'(t) g(t) dt. \tag{3.2.4}
\end{aligned}$$

Using the expression 3.2.4 in the expression () and the properties given in Definition 3.2.1, we get

$$\begin{aligned}
& {}_k \mathcal{D}_{b^-}^{l, \chi} g(\gamma) \\
&= \frac{(-1)^{q+1}}{\chi'(\gamma)} \frac{d}{d\gamma} \left\{ - \int_{\gamma}^b \chi'(t) g(t) dt \right. \\
&\quad \left. + \sum_{j=1}^{j=q} \frac{k^{j-\frac{l}{k}}}{\Gamma(\frac{l}{k} - j + 2)} \mathfrak{A}_{\chi, -}^{\frac{l}{k}-j+2}(b, \gamma) \left(\left[\frac{k}{\chi'(t)} \frac{d}{dt} \right]^{q-j} {}_k \mathcal{D}_{b^-}^{qk-l, \chi} g(t) \right)_{t=b} \right\} \\
&= (-1)^{q+1} \left[g(\gamma) - \sum_{j=1}^{j=q} \frac{k^{j-\frac{l}{k}}}{\Gamma(\frac{l}{k} - j + 1)} \mathfrak{A}_{\chi, -}^{\frac{l}{k}-j+1}(b, \gamma) \left(\left[\frac{k}{\chi'(t)} \frac{d}{dt} \right]^{q-j} {}_k \mathcal{D}_{b^-}^{qk-l, \chi} g(t) \right)_{t=b} \right] \\
&= (-1)^{q+1} \left[g(\gamma) \right. \\
&\quad \left. - \sum_{j=1}^{j=q} \frac{k^{j-\frac{l}{k}}}{k^{1-(\frac{l}{k}-j+1)} \Gamma_k(l-jk+k)} \mathfrak{A}_{\chi, -}^{\frac{l}{k}-j+1}(b, \gamma) \left(\left[\frac{k}{\chi'(t)} \frac{d}{dt} \right]^{q-j} {}_k \mathcal{D}_{b^-}^{qk-l, \chi} g(t) \right)_{t=b} \right] \\
&= (-1)^{q+1} \left[g(\gamma) - \sum_{j=1}^{j=q} \frac{\mathfrak{A}_{\chi, -}^{\frac{l}{k}-j+1}(b, \gamma)}{\Gamma_k(l-jk+k)} \left(\left[\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^{q-j} {}_k \mathcal{D}_{b^-}^{qk-l, \chi} g(\gamma) \right)_{\gamma=b} \right].
\end{aligned}$$

□

Theorem 3.2.3. Let $p, k > 0$ with $p < k$, $\theta \in [0, 1]$ and $\xi_k = p + \theta k - \theta p$. Then,

$${}_k \mathcal{D}_{b^-}^{\xi_k, \chi} g(\gamma) = {}_k \mathcal{D}_{b^-}^{p, \chi} g(\gamma), \quad g \in C_{1-\frac{\xi_k}{k}, \chi}^{\xi_k}[a, b].$$

Proof. By the definition 3.2.3 and Lemma 3.2.1, we have

$$\begin{aligned}
{}_k \mathcal{D}_{b^-}^{\xi_k, \chi} g(\gamma) &= {}_k \mathcal{D}_{b^-}^{p+\theta(k-p), \chi} g(\gamma) \\
&= {}_k \mathcal{D}_{b^-}^{p, \chi} {}_k \mathcal{D}_{b^-}^{\theta(k-p), \chi} g(\gamma) \\
&= {}_k \mathcal{D}_{b^-}^{p, \chi} {}_k \mathcal{D}_{b^-}^{\xi_k-p, \chi} g(\gamma) \\
&= {}_k \mathcal{D}_{b^-}^{p, \chi} g(\gamma).
\end{aligned}$$

□

Theorem 3.2.4. Let $0 < l, 0 < \xi_k < k$ and $g \in C_{1-\frac{\xi_k}{k}, \chi}^{\xi_k}[a, b]$. If $l > \xi_k$, then ${}_k \mathcal{D}_{b^-}^{l, \chi}(\cdot) \in C[a, b]$ and

$${}_k \mathcal{D}_{b^-}^{l, \chi} g(b) = \lim_{\gamma \rightarrow b^-} {}_k \mathcal{D}_{b^-}^{l, \chi} g(\gamma) = 0.$$

Proof. Let $g \in C_{\frac{\sigma}{k}, \chi}^{\sigma}[a, b]$. Then, by the theorem 3.2.1, we have ${}_k \mathcal{D}_{b^-}^{l, \chi} g \in C[a, b]$.

Also,

$$\begin{aligned}
|{}_k I_{b^-}^{l,\chi} g(\gamma)| &= \left| \frac{1}{k\Gamma_k(l)} \int_{\gamma}^b \mathfrak{A}_{\chi',-}^{\frac{l}{k}}(t, \gamma) g(t) dt \right| \\
&= \frac{1}{k\Gamma_k(l)} \int_{\gamma}^b \mathfrak{A}_{\chi',-}^{\frac{l}{k}}(t, \gamma) \mathfrak{A}_{\chi,-}^{-\frac{\sigma}{k}+1}(b, t) \mathfrak{A}_{\chi,-}^{\frac{\sigma}{k}+1}(b, t) g(t) dt \\
&= g C_{\frac{\sigma}{k}, \chi}^{[a, b]} k I_{b^-}^{l,\chi} \mathfrak{A}_{\chi,-}^{-\frac{\sigma}{k}+1}(b, t) \\
&= g C_{\frac{\sigma}{k}, \chi}^{[a, b]} \frac{\Gamma_k(k - \sigma)}{\Gamma_k(k - \sigma + l)} \mathfrak{A}_{\chi,-}^{\frac{l-\sigma}{k}+1}(b, \gamma).
\end{aligned} \tag{3.2.5}$$

Let $\sigma < l$, by using the continuity of χ and inequality 3.2.5, we get

$$\lim_{\gamma \rightarrow b^-} |{}_k I_{b^-}^{l,\chi} g(\gamma)| = 0.$$

This provides the required equation as given in Theorem 3.2.4. \square

Definition 3.2.4. [19] Let $\gamma, w \in \mathbb{C}$, such that $\text{Re}(\gamma) > 0$, $\text{Re}(w) > 0$.

(a) The k -beta function β_k is given by the integral

$$\beta_k(\gamma, w) = \frac{1}{k} \int_0^1 t^{\frac{\gamma}{k}-1} (1-t)^{\frac{w}{k}-1} dt, \quad k > 0.$$

(b) The k -beta function β_k satisfies the following identities

$$\beta_k(\gamma, w) = \frac{1}{k} \beta\left(\frac{\gamma}{k}, \frac{w}{k}\right), \quad \beta_k(\gamma, w) = \frac{\Gamma_k(\gamma)\Gamma_k(w)}{\Gamma_k(s+w)}.$$

Lemma 3.2.4. Let $l > 0$, $k > 0$ and $\xi \in \mathbb{R}$ such that $\frac{\xi}{k} > -1$, if $g \in C[a, b]$. Then,

$$(a) \quad {}_k I_{b^-}^{l,\chi} g(\gamma) = k^{-\frac{l}{k}} I_{b^-}^{\frac{l}{k}, \chi} g(\gamma).$$

$$(b) \quad {}_{k, RL} \mathfrak{D}_{b^-}^{l,\chi} g(\gamma) = k^{\frac{l}{k}} {}_{k, RL} \mathfrak{D}_{b^-}^{\frac{l}{k}, \chi} g(\gamma).$$

Proof. Now, we will demonstrate axiom (a). By using the properties given in Definition 3.2.1, we obtain:

$$\begin{aligned}
{}_k I_{b^-}^{l,\chi} g(\gamma) &= \frac{1}{k\Gamma_k(l)} \int_{\gamma}^b \mathfrak{A}_{\chi',-}^{\frac{l}{k}}(t, \gamma) g(t) dt \\
&= \frac{1}{k k^{\frac{l}{k}-1} \Gamma_k(\frac{l}{k})} \int_{\gamma}^b \mathfrak{A}_{\chi',-}^{\frac{l}{k}}(t, \gamma) g(t) dt \\
&= \frac{1}{k^{\frac{l}{k}} \Gamma_k(\frac{l}{k})} \int_{\gamma}^b \mathfrak{A}_{\chi',-}^{\frac{l}{k}}(t, \gamma) g(t) dt \\
&= k^{-\frac{l}{k}} I_{b^-}^{\frac{l}{k}, \chi} g(\gamma).
\end{aligned}$$

(b) Using the lemma 3.2.4(a), we get

$$\begin{aligned}
{}_{k,RL}\mathfrak{D}_{b^-}^{l,\chi} g(\gamma) &= \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q {}_k q^{k-l,\chi} g(\gamma) \\
&= \left[-\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q k^q k^{-(\frac{qk-l}{k})} \frac{q^{k-l}}{b^-} g(\gamma) \\
&= \left[-\frac{1}{\chi'(\gamma)} \frac{d}{d\gamma} \right]^q k^{\frac{l}{k}} {}_{b^-} q^{-\frac{l}{k},\chi} g(\gamma) \\
&= k^{\frac{l}{k}} {}_{k,RL}\mathfrak{D}_{b^-}^{\frac{l}{k},\chi} g(\gamma).
\end{aligned}$$

□

Theorem 3.2.5. Let $l > 0$, $k > 0$ and let $\xi \in \mathbb{R}$ such that $\frac{\xi}{k} > -1$. Then (k, χ) -fractional integral and derivative of a power function are given by

$$\begin{aligned}
(a) {}_k I_{b^-}^{l,\chi} \mathfrak{A}_{\chi,-}^{\frac{\xi}{k}+1}(b, \gamma) &= \frac{\Gamma_k(\xi + k)}{\Gamma_k(\xi + k + l)} \mathfrak{A}_{\chi,-}^{\frac{\xi+l}{k}+1}(b, \gamma). \\
(b) {}_{k,RL}\mathfrak{D}_{b^-}^{l,\chi} \mathfrak{A}_{\chi,-}^{\frac{\xi}{k}+1}(b, \gamma) &= \frac{\Gamma_k(\xi + k)}{\Gamma_k(\xi + k - l)} \mathfrak{A}_{\chi,-}^{\frac{\xi-l}{k}+1}(b, \gamma).
\end{aligned}$$

Proof. Now, we show the axiom (a), we easily find the result by applying the substitution $\chi(t) = \chi(\gamma) + z(\chi(b) - \chi(\gamma))$, the definition 3.2.4.

(b) Using the definition 3.2.2 and Lemma 3.2.4.

□

Theorem 3.2.6. Let $p > 0$, $k > 0$, with $p < k$, $\theta \in [0, 1]$ and $\xi_k = p + \theta k - \theta p$.

Then, for $g \in C_{1-\frac{\xi_k}{k}, \chi}^{\xi_k}[a, b]$

$${}_{k,RL}\mathfrak{D}_{b^-}^{\xi_k, \chi} {}_k I_{b^-}^{p, \chi} g(\gamma) = {}_{k,RL}\mathfrak{D}_{b^-}^{\theta k - \theta p, \chi} g(\gamma).$$

Proof. We take $p = \xi_k$, $q = 1$ by the definition 3.2.2 and using Lemma 3.2.1, we get

$$\begin{aligned}
{}_{k,RL}\mathfrak{D}_{b^-}^{\xi_k, \chi} {}_k I_{b^-}^{p, \chi} g(\gamma) &= \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right] {}_k I_{b^-}^{k-\xi_k, \chi} {}_k I_{b^-}^{p, \chi} g(\gamma) \\
&= \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right] {}_k I_{b^-}^{k-\xi_k+p, \chi} g(\gamma) \\
&= \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right] {}_k I_{b^-}^{k-\theta(k-p), \chi} g(\gamma) \\
&= {}_{k,RL}\mathfrak{D}_{b^-}^{\theta(k-p), \chi} g(\gamma).
\end{aligned}$$

□

3.3 Main Results

3.3.1 Representation formula for the solution

We use the fundamental notations, definitions, lemmas, and properties introduced in the first chapter to derive the integral solution to problem (\mathbb{P}') , and subsequently to problem $(\mathbb{P}\mathbb{R}\tau)$.

Lemma 3.3.1. *Let $R : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $R \in C_{1-\frac{\xi_k}{k}, \chi}^q[a, b]$, for any $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^1[a, b]$. A function $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^1[a, b]$ is a solution of fractional initial value problem*

$$(\mathbb{P}') \begin{cases} {}_{k,H} \mathfrak{D}_{b-}^{p, \theta, \chi} \mathbf{y}(\gamma) = R(\gamma, \mathbf{y}(\gamma)), & \gamma \in [a, b], \quad 0 < p < k, \quad 0 < \theta < 1, \\ {}_k I_{b-}^{k-\xi_k, \chi} \mathbf{y}(b) = \mathbf{y}_b \in \mathbb{R}, & \xi_k = p + \theta(k - p). \end{cases}$$

If and only if \mathbf{y} satisfies the following integral equation

$$\mathbf{y}(\gamma) = \frac{\mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \gamma)}{\Gamma_k(\xi_k)} \mathbf{y}_b + \frac{1}{k\Gamma_k(p)} \int_{\gamma}^b \mathfrak{A}_{\chi', -}^{\frac{p}{k}}(t, \gamma) R(t, \mathbf{y}(t)) dt. \quad (3.3.1)$$

We also note that in order to make it easy to concentrate, we note

$$\mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \gamma) = (\chi(b) - \chi(\gamma))^{\frac{\xi_k}{k}-1}, \quad \mathfrak{A}_{\chi', -}^{\frac{p}{k}}(t, \gamma) = \chi'(t)(\chi(t) - \chi(\gamma))^{\frac{p}{k}-1}.$$

Proof. Suppose $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^{\xi_k}[a, b]$ is a solution to the problem (\mathbb{P}') , we prove that \mathbf{y} satisfy the fractional integral Equation 3.3.1.

Since $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^{\xi_k}[a, b]$, we have $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^1[a, b]$ and

$$\left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right] {}_k I_{b-}^{k-\xi_k, \chi} \mathbf{y} = {}_{k,H} \mathfrak{D}_{b-}^{\xi_k, \chi} \mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^1[a, b]. \quad (3.3.2)$$

Further, by applying Theorem 3.2.1 with $\sigma = l = k - \xi_k$, we get

$${}_k I_{b-}^{k-\xi_k, \chi} \mathbf{y} \in C[a, b]. \quad (3.3.3)$$

According to formulas 3.3.2 and 3.3.3, and the definition of weighted space $C_{1-\frac{\xi_k}{k}, \chi}^q[a, b]$, we obtain

$${}_k I_{b-}^{k-\xi_k, \chi} \mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^1[a, b].$$

Since $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^1[a, b]$ and ${}_k \mathcal{I}_{b-}^{k-\xi_k, \chi} \mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^1[a, b]$, by applying the theorem 3.2.2 with $\sigma = k - \xi_k$, $l = \xi_k$ and $q = \left\lceil \frac{\xi_k}{k} \right\rceil = 1$, we get

$$\begin{aligned} {}_k \mathcal{I}_{b-}^{\xi_k, \chi} {}_{k, H} \mathcal{D}_{b-}^{\xi_k, \chi} \mathbf{y}(\gamma) &= \mathbf{y}(\gamma) - \frac{\mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \gamma)}{\Gamma_k(\xi_k)} [{}_k \mathcal{I}_{b-}^{k-\xi_k, \chi}]_{\gamma=b} \mathbf{y}(\gamma) \\ &= \mathbf{y}(\gamma) - \frac{\mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \gamma)}{\Gamma_k(\xi_k)} \mathbf{y}_b. \end{aligned} \quad (3.3.4)$$

Since $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^{\xi_k}[a, b]$, by Theorem 3.2.3, we get

$$\begin{aligned} {}_k \mathcal{I}_{b-}^{\xi_k, \chi} {}_{k, H} \mathcal{D}_{b-}^{\xi_k, \chi} \mathbf{y}(\gamma) &= {}_k \mathcal{I}_{b-}^{p, \chi} {}_{k, H} \mathcal{D}_{b-}^{p, \theta; \chi} \mathbf{y}(\gamma) \\ &= {}_k \mathcal{I}_{b-}^{p, \chi} R(\gamma, \mathbf{y}(\gamma)). \end{aligned} \quad (3.3.5)$$

Now, we can write

$$\begin{aligned} \mathbf{y}(\gamma) &= \frac{\mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \gamma)}{\Gamma_k(\xi_k)} \mathbf{y}_b + {}_k \mathcal{I}_{b-}^{p, \chi} R(\gamma, \mathbf{y}(\gamma)) \\ &= \frac{\mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \gamma)}{\Gamma_k(\xi_k)} \mathbf{y}_b + \frac{1}{k\Gamma_k(p)} \int_{\gamma}^b \mathfrak{A}_{\chi', -}^{\frac{p}{k}}(t, \gamma) R(t, \mathbf{y}(t)) dt. \end{aligned}$$

Which is desired fractional integral Equation 3.3.1.

Conversely, suppose that $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^{\xi_k}[a, b]$ satisfy Equation 3.3.1. Then,

$$\mathbf{y}(\gamma) = \frac{\mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \gamma)}{\Gamma_k(\xi_k)} \mathbf{y}_b + {}_k \mathcal{I}_{b-}^{p, \chi} R(\gamma, \mathbf{y}(\gamma)), \quad \gamma \in [a, b].$$

Inserting ${}_k \mathcal{I}_{b-}^{\xi_k, \chi}$ on both sides of above equation, we get

$${}_k \mathcal{I}_{b-}^{\xi_k, \chi} \mathbf{y}(\gamma) = {}_k \mathcal{I}_{b-}^{\xi_k, \chi} \frac{\mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \gamma)}{\Gamma_k(\xi_k)} \mathbf{y}_b + {}_k \mathcal{I}_{b-}^{\xi_k, \chi} {}_k \mathcal{I}_{b-}^{p, \chi} R(\gamma, \mathbf{y}(\gamma)).$$

By applying the lemma 3.2.4 and Theorem 3.2.6, we obtain

$$\begin{aligned} {}_k \mathcal{I}_{b-}^{\xi_k, \chi} \mathbf{y}(\gamma) &= \frac{\mathbf{y}_b}{\Gamma_k(\xi_k)} {}_k \mathcal{I}_{b-}^{\frac{\xi_k}{k}, \chi} {}_{RL} \mathcal{D}_{b-}^{\frac{\xi_k}{k}, \chi} \mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \gamma) + {}_k \mathcal{I}_{b-}^{\theta(k-p), \chi} R(\gamma, \mathbf{y}(\gamma)) \\ &= {}_k \mathcal{I}_{b-}^{\theta(k-p), \chi} R(\gamma, \mathbf{y}(\gamma)). \end{aligned} \quad (3.3.6)$$

Since $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^{\xi_k}[a, b]$, we have ${}_k \mathcal{I}_{b-}^{\xi_k, \chi} \mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^1[a, b]$, therefore from 3.3.6 it follows that

$${}_k \mathcal{I}_{b-}^{\theta(k-p), \chi} R(\gamma, \mathbf{y}(\gamma)) \in C_{1-\frac{\xi_k}{k}, \chi}^1[a, b]. \quad (3.3.7)$$

Since $p < k$, $\theta \in [0, 1]$ and $0 < 1 - \frac{p}{k} < 1$, we obtain $\frac{\theta(k-p)}{k} = \theta(1 - \frac{p}{k}) < 1$, therefore

$$\left[\theta \left(1 - \frac{p}{k} \right) \right] + 1 = 1.$$

In this case the definition of right (k, χ) -Riemann-Liouville derivative reduces to

$${}_{k, H} \mathfrak{D}_{b^-}^{\theta(k-p), \chi} R(\gamma, \mathbf{y}(\gamma)) = \left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right] {}_k \mathfrak{I}_{b^-}^{k-\theta(k-p), \chi} R(\gamma, \mathbf{y}(\gamma)). \quad (3.3.8)$$

Clearly, by 3.3.7 and 3.3.8, we obtain

$$\left[-\frac{k}{\chi'(\gamma)} \frac{d}{d\gamma} \right] {}_k \mathfrak{I}_{b^-}^{k-\theta(k-p), \chi} R(\gamma, \mathbf{y}(\gamma)) \in C_{1-\frac{\xi_k}{k}, \chi}^1[a, b]. \quad (3.3.9)$$

Since $\xi_k = p + \theta k - \theta p > \theta(k-p)$, we have $k - \xi_k < k - \theta k - \theta p$.

Since $R(\cdot, \mathbf{y}(\cdot)) \in C_{1-\frac{\xi_k}{k}, \chi}^1[a, b]$, by applying Theorem 3.2.1 with $l = k - \theta(k-p)$, we get

$${}_k \mathfrak{I}_{b^-}^{k-\theta(k-p), \chi} R(\cdot, \mathbf{y}(\cdot)) \in C[a, b]. \quad (3.3.10)$$

Using the definition of weighted space $C_{1-\frac{\xi_k}{k}, \chi}^q[a, b]$ from Equation 3.3.9 and 3.3.10, it follows that

$${}_k \mathfrak{I}_{b^-}^{k-\theta k - \theta p, \chi} R(\cdot, \mathbf{y}(\cdot)) \in C_{1-\frac{\xi_k}{k}, \chi}^1[a, b].$$

By applying ${}_k \mathfrak{I}_{b^-}^{\theta(k-p), \chi}$ on both sides of Equation 3.3.6 and using Theorem 3.2.2

with $\sigma = k - \xi_k$, $l = \theta k - \theta p$ and $q = 1$, we have

$$\begin{aligned} {}_k \mathfrak{I}_{b^-}^{\theta(k-p), \chi} {}_{k, H} \mathfrak{D}_{b^-}^{\xi_k, \chi} \mathbf{y}(\gamma) &= {}_k \mathfrak{I}_{b^-}^{\theta(k-p), \chi} {}_{k, H} \mathfrak{D}_{b^-}^{\theta(k-p), \chi} R(\gamma, \mathbf{y}(\gamma)) \\ &= R(\gamma, \mathbf{y}(\gamma)) - \frac{\mathfrak{A}_{\chi', -}^{\frac{k-p}{k}}(b, \gamma)}{\Gamma_k(\theta(k-p))} \left[{}_k \mathfrak{I}_{b^-}^{k-\theta(k-p), \chi} R(\gamma, \mathbf{y}(\gamma)) \right]_{\gamma=b}. \end{aligned} \quad (3.3.11)$$

Using the theorem 3.2.4 with $l = k - \theta(k-p)$, we obtain

$$\left[{}_k \mathfrak{I}_{b^-}^{k-\theta(k-p), \chi} R(\gamma, \mathbf{y}(\gamma)) \right]_{\gamma=b} = 0. \quad (3.3.12)$$

Using the definition of right (k, χ) -Hilfer fractional derivative and Equation 3.3.12, from Equation 3.3.11, we get

$${}_{k, H} \mathfrak{D}_{b^-}^{p, \theta, \chi} \mathbf{y}(\gamma) = R(\gamma, \mathbf{y}(\gamma)), \quad \gamma \in [a, b].$$

Consequently, Equation 3.1 is verified.

On the other hand it remains verify initial condition 3.1.

Firstly, taking ${}_k b^- \overset{k-\xi_k, \chi}{\mathcal{I}} \mathbf{y}(\gamma)$ on both sides of Equation 3.3.1, using the theorem 3.2.5, we get

$$\begin{aligned} {}_k b^- \overset{k-\xi_k, \chi}{\mathcal{I}} \mathbf{y}(\gamma) &= \frac{\mathbf{y}_b}{\Gamma_k(\xi_k)} {}_k b^- \overset{k-\xi_k, \chi}{\mathcal{I}} \mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \gamma) + {}_k b^- \overset{k-\xi_k+p, \chi}{\mathcal{I}} R(\gamma, \mathbf{y}(\gamma)) \\ &= \frac{\mathbf{y}_b}{\Gamma_k(k)} + {}_k b^- \overset{k-\theta(k-p), \chi}{\mathcal{I}} R(\gamma, \mathbf{y}(\gamma)). \end{aligned}$$

Using the definition 3.2.1, from above equation, we get

$$\left[{}_k b^- \overset{k-\xi_k, \chi}{\mathcal{I}} \mathbf{y}(\gamma) \right]_{\gamma=b} = \mathbf{y}_b + \left[{}_k b^- \overset{k-\theta(k-p), \chi}{\mathcal{I}} R(\gamma, \mathbf{y}(\gamma)) \right]_{\gamma=b}.$$

Using the equation 3.3.12 en above equation, we get

$$\left[{}_k b^- \overset{k-\xi_k, \chi}{\mathcal{I}} \mathbf{y}(\gamma) \right]_{\gamma=b} = \mathbf{y}_b$$

This proves the initial condition 3.1 is verified. \square

Lemma 3.3.2. Let $\bar{h} : [a, b) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function such that $\bar{h} \in C_{1-\frac{\xi_k}{k}, \chi}[a, b]$, for any $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}[a, b]$. A function $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}[a, b]$ is a solution of the problem

$$(\mathbb{P}\mathbb{R}\tau) \begin{cases} {}_{k, H} \mathfrak{D}_{b^-}^{p, \theta, \chi} \mathbf{y}(\gamma) = \bar{h}(\gamma, \mathbf{y}(\gamma), \mathbf{y}(\tau\gamma), {}_{k, H} \mathfrak{D}_{b^-}^{p, \theta, \chi} \mathbf{y}(\tau\gamma)), & \gamma \in I := [a, b), \\ {}_k b^- \overset{k-\xi_k, \chi}{\mathcal{I}} \mathbf{y}(b) = \sum_{i=1}^m \lambda_i {}_k b^- \overset{\theta, \chi}{\mathcal{I}} \mathbf{y}(\mu_i), & \mu_i \in]a, b], \end{cases}$$

if and only if \mathbf{y} satisfies the mixed-type integral

$$\begin{aligned} \mathbf{y}(\gamma) &= \frac{\Gamma_k(\theta + \xi_k) \mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \gamma)}{\Gamma_k(\xi_k) Z} \left[\sum_{i=1}^m \frac{\lambda_i}{k \Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p+\theta}{k}}(t, \mu_i) \wp_{\mathbf{y}}(t) dt \right] \\ &\quad + \frac{1}{k \Gamma_k(p)} \int_{\gamma}^b \mathfrak{A}_{\chi', -}^{\frac{p}{k}}(t, \gamma) \wp_{\mathbf{y}}(t) dt \end{aligned} \quad (3.3.13)$$

where

$${}_{k, H} \mathfrak{D}_{b^-}^{p, \theta, \chi} \mathbf{y}(\gamma) = \bar{h}(\gamma, \mathbf{y}(\gamma), \mathbf{y}(\tau\gamma), {}_{k, H} \mathfrak{D}_{b^-}^{p, \theta, \chi} \mathbf{y}(\tau\gamma)) = \wp_{\mathbf{y}}(\gamma),$$

$$Z = \Gamma_k(\xi_k + \theta) - \sum_{i=1}^m \lambda_i \mathfrak{A}_{\chi, -}^{\frac{\xi_k + \theta}{k}}(b, \mu_i) = 0.$$

Proof. According to Lemma 3.3.1 a solution of problem $(\mathbb{P}\mathbb{R}\tau)$ can be expressed by

$$\mathbf{y}(\gamma) = \frac{\mathfrak{A}_{\chi,-}^{\frac{\xi_k}{k}}(b, \gamma)}{\Gamma_k(\xi_k)} \mathbf{y}_b + \frac{1}{k\Gamma_k(p)} \int_{\gamma}^b \mathfrak{A}_{\chi,-}^{\frac{p}{k}}(t, \gamma) \wp_{\mathbf{y}}(t) dt.$$

Suppose $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^{\xi_k}[a, b]$ be a solution to the problem $(\mathbb{P}\mathbb{R}\tau)$. We prove that \mathbf{y} is also a solution of fractional integral equation 3.3.13.

Since $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^{\xi_k}[a, b]$, we have $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}[a, b]$ and

$$\left[-\frac{k}{\chi'(\gamma)} \frac{d}{(d\gamma)} \right] {}_k \mathcal{I}_{b-}^{k-\xi_k, \chi} \mathbf{y} = {}_{k, RL} \mathfrak{D}_{b-}^{\xi_k, \chi} \mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}[a, b]. \quad (3.3.14)$$

Further, by applying Theorem 3.2.1 with $l = k - \xi_k$, we get

$${}_k \mathcal{I}_{b-}^{k-\xi_k, \chi} \mathbf{y} \in C[a, b]. \quad (3.3.15)$$

Consequently, from Equation 3.3.14 and 3.3.15, and using the definition of the espace $C_{1-\frac{\xi_k}{k}, \chi}^q[a, b]$, we get

$${}_k \mathcal{I}_{b-}^{k-\xi_k, \chi} \mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^1[a, b].$$

Since $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}[a, b]$ and ${}_k \mathcal{I}_{b-}^{k-\xi_k, \chi} \mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^1[a, b]$, by applying the Theorem 3.2.2 with $\sigma = k - \xi_k$, $l = \xi_k$ and $q = \left[\frac{\xi_k}{k} \right] = 1$, we get

$${}_k \mathcal{I}_{b-}^{\xi_k, \chi} {}_{k, RL} \mathfrak{D}_{b-}^{\xi_k, \chi} \mathbf{y}(\gamma) = \mathbf{y}(\gamma) - \frac{\mathfrak{A}_{\chi,-}^{\frac{\xi_k}{k}}(b, \gamma)}{\Gamma_k(\xi_k)} [{}_k \mathcal{I}_{b-}^{k-\xi_k, \chi} \mathbf{y}(\gamma)]_{\gamma=b}. \quad (3.3.16)$$

By hypothesis, ${}_{k, RL} \mathfrak{D}_{b-}^{\xi_k, \chi} \mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}[a, b]$, using Theorem 3.2.3 and Equation 3.1, we have

$${}_k \mathcal{I}_{b-}^{\xi_k, \chi} {}_{k, RL} \mathfrak{D}_{b-}^{\xi_k, \chi} \mathbf{y}(\gamma) = {}_k \mathcal{I}_{b-}^{p, \chi} {}_{k, H} \mathfrak{D}_{b-}^{p, \theta, \chi} \mathbf{y}(\gamma) = {}_k \mathcal{I}_{b-}^{p, \chi} \wp_{\mathbf{y}}(\gamma). \quad (3.3.17)$$

Comparing Equation 3.3.16 and 3.3.17, we see that

$$\mathbf{y}(\gamma) = \frac{\mathfrak{A}_{\chi,-}^{\frac{\xi_k}{k}}(b, \gamma)}{\Gamma_k(\xi_k)} [{}_k \mathcal{I}_{b-}^{k-\xi_k, \chi} \mathbf{y}(\gamma)]_{\gamma=b} + {}_k \mathcal{I}_{b-}^{p, \chi} \wp_{\mathbf{y}}(\gamma). \quad (3.3.18)$$

Now, we substitute $\gamma = \mu_i$ in 3.3.18 and multiply by λ_i we can write

$$\lambda_i \mathbf{y}(\mu_i) = \lambda_i \frac{\mathfrak{A}_{\chi,-}^{\frac{\xi_k}{k}}(b, \mu_i)}{\Gamma_k(\xi_k)} [{}_k \mathcal{I}_{b-}^{k-\xi_k, \chi} \mathbf{y}(\gamma)]_{\gamma=b} + \lambda_i {}_k \mathcal{I}_{b-}^{p, \chi} \wp_{\mathbf{y}}(\mu_i). \quad (3.3.19)$$

Next, by applying ${}_k \mathcal{I}_{b^-}^{\theta, \chi}$ to both sides of Equation 3.3.19 and Theorem 3.2.5, we get

$${}_k \mathcal{I}_{b^-}^{\theta, \chi} \lambda_i \mathbf{y}(\mu_i) = \lambda_i \frac{\mathfrak{A}_{\chi, -}^{\frac{\xi_k + \theta}{k}}(b, \mu_i)}{\Gamma_k(\xi_k + \theta)} [{}_k \mathcal{I}_{b^-}^{k - \xi_k, \chi} \mathbf{y}(\gamma)]_{\gamma=b} + \lambda_i {}_k \mathcal{I}_{b^-}^{p + \theta, \chi} \wp_{\mathbf{y}}(\mu_i). \quad (3.3.20)$$

This implies that

$$\sum_{i=1}^m {}_k \mathcal{I}_{b^-}^{\theta, \chi} \lambda_i \mathbf{y}(\mu_i) = \sum_{i=1}^m \lambda_i \frac{\mathfrak{A}_{\chi, -}^{\frac{\xi_k + \theta}{k}}(b, \mu_i)}{\Gamma_k(\xi_k + \theta)} [{}_k \mathcal{I}_{b^-}^{k - \xi_k, \chi} \mathbf{y}(b)] + \sum_{i=1}^m \lambda_i {}_k \mathcal{I}_{b^-}^{p + \theta, \chi} \wp_{\mathbf{y}}(\mu_i). \quad (3.3.21)$$

Inserting the initial condition ${}_k \mathcal{I}_{b^-}^{k - \xi_k, \chi} \mathbf{y}(b) = \sum_{i=1}^m {}_k \mathcal{I}_{b^-}^{\theta, \chi} \lambda_i \mathbf{y}(\mu_i)$ in equation 3.1 we have

$${}_k \mathcal{I}_{b^-}^{k - \xi_k, \chi} \mathbf{y}(b) = \sum_{i=1}^m \lambda_i \frac{\mathfrak{A}_{\chi, -}^{\frac{\xi_k + \theta}{k}}(b, \mu_i)}{\Gamma_k(\xi_k + \theta)} [{}_k \mathcal{I}_{b^-}^{k - \xi_k, \chi} \mathbf{y}(\gamma)]_{\gamma=b} + \sum_{i=1}^m \lambda_i {}_k \mathcal{I}_{b^-}^{p + \theta, \chi} \wp_{\mathbf{y}}(\mu_i), \quad (3.3.22)$$

which implies that

$$\begin{aligned} \sum_{i=1}^m \lambda_i {}_k \mathcal{I}_{b^-}^{p + \theta, \chi} \wp_{\mathbf{y}}(\mu_i) &= \left(1 - \sum_{i=1}^m \lambda_i \frac{\mathfrak{A}_{\chi, -}^{\frac{\xi_k + \theta}{k}}(b, \mu_i)}{\Gamma_k(\xi_k + \theta)} \right) [{}_k \mathcal{I}_{b^-}^{k - \xi_k, \chi} \mathbf{y}(\gamma)]_{\gamma=b} \\ &= \left(\frac{\Gamma_k(\theta + \xi_k)}{\Gamma_k(\theta + \xi_k)} - \sum_{i=1}^m \lambda_i \frac{\mathfrak{A}_{\chi, -}^{\frac{\xi_k + \theta}{k}}(b, \mu_i)}{\Gamma_k(\xi_k + \theta)} \right) [{}_k \mathcal{I}_{b^-}^{k - \xi_k, \chi} \mathbf{y}(\gamma)]_{\gamma=b} \\ &= \frac{Z}{\Gamma_k(\theta + \xi_k)} [{}_k \mathcal{I}_{b^-}^{k - \xi_k, \chi} \mathbf{y}(\gamma)]_{\gamma=b}. \end{aligned}$$

Thus,

$${}_k \mathcal{I}_{b^-}^{k - \xi_k, \chi} \mathbf{y}(b) = \frac{\Gamma_k(\xi_k + \theta) \sum_{i=1}^m \lambda_i {}_k \mathcal{I}_{b^-}^{p + \theta, \chi} \wp_{\mathbf{y}}(\mu_i)}{Z}. \quad (3.3.23)$$

Hence, the result follows by putting Equation 3.3.23 in Equation 3.3.18. This implies that $\mathbf{y}(\gamma)$ satisfies Equation 3.3.13.

Conversely, suppose that $\mathbf{y} \in C_{1 - \frac{\xi_k}{k}, \chi}^{\xi_k}[a, b]$ satisfying Equation 3.3.13. Then,

$$\begin{aligned} \mathbf{y}(\gamma) &= \frac{\Gamma_k(\theta + \xi_k)}{\Gamma_k(\xi_k)} \frac{\mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \gamma)}{Z} \left[\sum_{i=1}^m \frac{\lambda_i}{k \Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p + \theta}{k}}(t, \mu_i) \wp_{\mathbf{y}}(t) dt \right] \\ &\quad + \frac{1}{k \Gamma_k(p)} \int_{\gamma}^b \mathfrak{A}_{\chi', -}^{\frac{p}{k}}(t, \gamma) \wp_{\mathbf{y}}(t) dt. \end{aligned}$$

Inserting ${}_{k,RL}\mathfrak{D}_{b^-}^{\xi_k, \chi}$ on both sides of above equation, we get

$$\begin{aligned}
& {}_{k,RL}\mathfrak{D}_{b^-}^{\xi_k, \chi} \mathbf{y}(\gamma) = \\
& {}_{k,RL}\mathfrak{D}_{b^-}^{\xi_k, \chi} \left(\frac{\Gamma_k(\theta + \xi_k)}{\Gamma_k(\xi_k)} \frac{\mathfrak{A}_{\chi, -}^{\xi_k}(b, \gamma)}{\mathcal{Z}} \left[\sum_{i=1}^m \frac{\lambda_i}{k\Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p+\theta}{k}}(t, \mu_i) \wp_{\mathbf{y}}(t) dt \right] \right) \\
& + {}_{k,RL}\mathfrak{D}_{b^-}^{\xi_k, \chi} \left(\frac{1}{k\Gamma_k(p)} \int_{\gamma}^b \mathfrak{A}_{\chi', -}^{\frac{p}{k}}(t, \gamma) \wp_{\mathbf{y}}(t) dt \right) \\
& = {}_{k,RL}\mathfrak{D}_{b^-}^{\theta(k-p), \chi} \bar{h}(\gamma, \mathbf{y}(\gamma), \mathbf{y}(\tau\gamma), {}_{k,H}\mathfrak{D}_{b^-}^{p, \theta, \chi} \mathbf{y}(\tau\gamma)).
\end{aligned} \tag{3.3.24}$$

Since $\mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}^{\xi_k}[a, b]$, and by Definition of $C_{1-\frac{\xi_k}{k}, \chi}^{\xi_k}[a, b]$, we have

$${}_{k,RL}\mathfrak{D}_{b^-}^{\xi_k, \chi} \mathbf{y} \in C_{1-\frac{\xi_k}{k}, \chi}[a, b].$$

Therefore, from 3.3.24 it follows that

$${}_{k,RL}\mathfrak{D}_{b^-}^{\theta(k-p), \chi} \bar{h}(\gamma, \mathbf{y}(\gamma), \mathbf{y}(\tau\gamma), {}_{k,H}\mathfrak{D}_{b^-}^{p, \theta, \chi} \mathbf{y}(\tau\gamma)) \in C_{1-\frac{\xi_k}{k}, \chi}[a, b]. \tag{3.3.25}$$

Since $p < k$, $\theta \in [0, 1]$ and $0 < 1 - \frac{p}{k} < 1$, we obtain $\frac{\theta(k-p)}{k} = \theta(1 - \frac{p}{k}) < 1$. Therefore

$$\left[\theta(1 - \frac{p}{k}) \right] + 1 = 1.$$

In this case the definition of (k, χ) -Riemann-Liouville derivative reduce to

$$\begin{aligned}
& {}_{k,RL}\mathfrak{D}_{b^-}^{\theta(k-p), \chi} \bar{h}(\gamma, \mathbf{y}(\gamma), \mathbf{y}(\tau\gamma), {}_{k,H}\mathfrak{D}_{b^-}^{p, \theta, \chi} \mathbf{y}(\tau\gamma)) = \\
& \left(-\frac{k}{\chi'(\gamma)} \frac{d}{(d\gamma)} \right) {}_k {}_{b^-}^{k-\theta(k-p), \chi} \bar{h}(\gamma, \mathbf{y}(\gamma), \mathbf{y}(\tau\gamma), {}_{k,H}\mathfrak{D}_{b^-}^{p, \theta, \chi} \mathbf{y}(\tau\gamma)).
\end{aligned} \tag{3.3.26}$$

Clearly, by 3.3.25 and 3.3.26, we obtain

$$\left(-\frac{k}{\chi'(\gamma)} \frac{d}{(d\gamma)} \right) {}_k {}_{b^-}^{k-\theta(k-p), \chi} \bar{h}(\gamma, \mathbf{y}(\gamma), \mathbf{y}(\tau\gamma), {}_{k,H}\mathfrak{D}_{b^-}^{p, \theta, \chi} \mathbf{y}(\tau\gamma)) \in C_{1-\frac{\xi_k}{k}, \chi}[a, b]. \tag{3.3.27}$$

Since $\xi_k = p + \theta k - \theta p > \theta(k - p)$, we have $k - \xi_k < k - \theta k + \theta p$.

Since $\wp_{\mathbf{y}}(\cdot) \in C_{1-\frac{\xi_k}{k}, \chi}[a, b]$, by applying Theorem 3.2.1, we get

$${}_k {}_{b^-}^{k-\theta(k-p), \chi} \wp_{\mathbf{y}}(\cdot) \in C[a, b]. \tag{3.3.28}$$

Using the definition of the espace $C_{1-\frac{\xi_k}{k},\chi}^q[a,b]$, from Equation 3.3.27 and 3.3.28, it follows that

$${}_k \underset{b^-}{k-\theta k+\theta p,\chi} \wp_{\mathbf{y}}(\cdot) \in C_{1-\frac{\xi_k}{k},\chi}^1[a,b].$$

By applying ${}_k \underset{b^-}{\theta(k-p),\chi}$ on both sides of Equation 3.3.24 and using Theorem 3.2.2 with $l = \theta k - \theta p$ and $q = 1$, we have

$$\begin{aligned} {}_k \underset{b^-}{\theta(k-p),\chi} {}_{k,RL} \mathfrak{D}_{b^-}^{\xi_k,\chi} \mathbf{y}(\gamma) &= {}_k \underset{b^-}{\theta(k-p),\chi} {}_{k,RL} \mathfrak{D}_{b^-}^{\theta(k-p),\chi} \wp_{\mathbf{y}}(t) \\ &= \wp_{\mathbf{y}}(\gamma) - \frac{\mathfrak{A}_{\chi',-}^{\frac{k-p}{k}}(b,\gamma)}{\Gamma_k(\theta(k-p))} \left[{}_k \underset{b^-}{k-\theta(k-p),\chi} \wp_{\mathbf{y}}(b) \right]. \end{aligned} \quad (3.3.29)$$

Using the Theorem 3.2.4 with $l = k - \theta(k - p)$, we obtain

$$\left[{}_k \underset{b^-}{k-\theta(k-p),\chi} \wp_{\mathbf{y}}(b) \right] = 0. \quad (3.3.30)$$

Comparing the last Equality with 3.1, we get

$${}_{k,H} \mathfrak{D}_{b^-}^{p,\theta,\chi} \mathbf{y}(\gamma) = \wp_{\mathbf{y}}(\gamma), \quad \gamma \in [a,b],$$

which means that 3.1 holds. Next, we show that if $\mathbf{y} \in C_{1-\frac{\xi_k}{k},\chi}^{\xi_k}[a,b]$ satisfies 3.3.13, it also satisfies the condition 3.1. To this end, we multiply both sides of 3.3.13 by ${}_k \underset{b^-}{k-\xi_k,\chi}$ and use Theorem 3.2.5, we have

$$\begin{aligned} &{}_k \underset{b^-}{k-\xi_k,\chi} \mathbf{y}(\gamma) \\ &= {}_k \underset{b^-}{k-\xi_k,\chi} \left(\frac{\Gamma_k(\theta + \xi_k)}{\Gamma_k(\xi_k)} \frac{\mathfrak{A}_{\chi',-}^{\frac{\xi_k}{k}}(b,\gamma)}{\mathcal{Z}} \left[\sum_{i=1}^m \frac{\lambda_i}{k\Gamma_k(p+\theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi',-}^{\frac{p+\theta}{k}}(t,\mu_i) \wp_{\mathbf{y}}(t) dt \right] \right) \\ &+ {}_k \underset{b^-}{k-\xi_k,\chi} \left(\frac{1}{k\Gamma_k(p)} \int_{\gamma}^b \mathfrak{A}_{\chi',-}^{\frac{p}{k}}(t,\gamma) \wp_{\mathbf{y}}(t) dt \right) \\ &= \frac{\Gamma_k(\theta + \xi_k)}{\mathcal{Z}} \left[\sum_{i=1}^m \frac{\lambda_i}{k\Gamma_k(p+\theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi',-}^{\frac{p+\theta}{k}}(t,\mu_i) \wp_{\mathbf{y}}(t) dt \right] + {}_k \underset{b^-}{k-\xi_k+p,\chi} \wp_{\mathbf{y}}(\gamma). \end{aligned} \quad (3.3.31)$$

Using Theorem 3.2.4 and the fact that $k - \xi_k < k - \theta(k - p)$, then taking limits as $\gamma \rightarrow b$, in Equation 3.3.31 yields

$${}_k \underset{b^-}{k-\xi_k,\chi} \mathbf{y}(b) = \frac{\Gamma_k(\theta + \xi_k)}{\mathcal{Z}} \left[\sum_{i=1}^m \frac{\lambda_i}{k\Gamma_k(p+\theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi',-}^{\frac{p+\theta}{k}}(t,\mu_i) \wp_{\mathbf{y}}(t) dt \right]. \quad (3.3.32)$$

Substituting $\gamma = \mu_i$ into 3.3.13, we have

$$\mathbf{y}(\mu_i) = \frac{\Gamma_k(\theta + \xi_k) \mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \mu_i)}{\Gamma_k(\xi_k) Z} \left[\sum_{i=1}^m \frac{\lambda_i}{k\Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p+\theta}{k}}(t, \mu_i) \wp_{\mathbf{y}}(t) dt \right] + \frac{1}{k\Gamma_k(p)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p}{k}}(t, \mu_i) \wp_{\mathbf{y}}(t) dt.$$

Multiplying throughout by λ_i , we get

$$\lambda_i \mathbf{y}(\mu_i) = \frac{\Gamma_k(\theta + \xi_k) \lambda_i \mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \mu_i)}{\Gamma_k(\xi_k) Z} \left[\sum_{i=1}^m \frac{\lambda_i}{k\Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p+\theta}{k}}(t, \mu_i) \wp_{\mathbf{y}}(t) dt \right] + \frac{\lambda_i}{k\Gamma_k(p)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p}{k}}(t, \mu_i) \wp_{\mathbf{y}}(t) dt. \quad (3.3.33)$$

Applying ${}_k \theta, \chi_{b-}$ to both sides of Equation 3.3.33, we obtain

$${}_k \theta, \chi_{b-} \lambda_i \mathbf{y}(\mu_i) = \frac{\lambda_i \mathfrak{A}_{\chi, -}^{\frac{\xi_k + \theta}{k}}(b, \mu_i)}{Z} \left[\sum_{i=1}^m \frac{\lambda_i}{k\Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p+\theta}{k}}(t, \mu_i) \wp_{\mathbf{y}}(t) dt \right] + \frac{\lambda_i}{k\Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p+\theta}{k}}(t, \mu_i) \wp_{\mathbf{y}}(t) dt.$$

Then, we drive

$$\begin{aligned} & \sum_{i=1}^m \lambda_i {}_k \theta, \chi_{b-} \mathbf{y}(\mu_i) \\ &= \frac{\sum_{i=1}^m \lambda_i \mathfrak{A}_{\chi, -}^{\frac{\xi_k + \theta}{k}}(b, \mu_i)}{Z} \left[\sum_{i=1}^m \frac{\lambda_i}{k\Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p+\theta}{k}}(t, \mu_i) \wp_{\mathbf{y}}(t) dt \right] \\ &+ \frac{\sum_{i=1}^m \lambda_i}{k\Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p+\theta}{k}}(t, \mu_i) \wp_{\mathbf{y}}(t) dt \\ &= \left[\sum_{i=1}^m \frac{\lambda_i}{k\Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p+\theta}{k}}(t, \mu_i) \wp_{\mathbf{y}}(t) dt \right] \left(1 - \frac{\sum_{i=1}^m \lambda_i \mathfrak{A}_{\chi, -}^{\frac{\xi_k + \theta}{k}}(b, \mu_i)}{Z} \right), \end{aligned}$$

and

$${}_k {}^{k-\xi_k, \chi}_{b-} \mathbf{y}(b) = \frac{\Gamma_k(\theta + \xi_k)}{Z} \left[\sum_{i=1}^m \frac{\lambda_i}{k\Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p+\theta}{k}}(t, \mu_i) \wp_{\mathbf{y}}(t) dt \right]. \quad (3.3.34)$$

Now, it's clear that 3.3.32 and 3.3.34 3.1, hence the proof

$${}_k {}^{k-\xi_k, \chi}_{b-} \mathbf{y}(b) = \sum_{i=1}^m \lambda_i {}_k \theta, \chi_{b-} \mathbf{y}(\mu_i).$$

□

3.3.2 Existence and uniqueness results for problem $(\mathbb{P}\mathbb{R}\tau)$

We establish the existence and uniqueness of solutions to the problem $(\mathbb{P}\mathbb{R}\tau)$.

Theorem 3.3.1. *Let $A : \chi - \chi$ be a completely continuous operator. Suppose that the set $\xi(A) = \{P - \chi : P = QAP, \text{ for some } Q \in [0, 1]\}$ is bounded, then A has a fixed point.*

Thus we need the following assumptions :

(H₁) $\bar{h} : I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function such that $\bar{h} \in C_{1-\frac{\xi_k}{k}, \chi}[a, b]$ for any $y \in C_{1-\frac{\xi_k}{k}, \chi}[a, b]$.

(H₂) There exist $h, l, m, n \in C_{1-\frac{\xi_k}{k}, \chi}[a, b]$ with $h^* = \sup_{\gamma \in I} |h(\gamma)| < 1$ such that

$$|\bar{h}(\gamma, u, v, w)| \leq h(\gamma) + l(\gamma)|u| + m(\gamma)|v| + n(\gamma)|w|, \text{ for } \gamma \in [a, b] \text{ and } u, v, w \in \mathbb{R}. \quad (3.3.35)$$

Then, the problem $(\mathbb{P}\mathbb{R}\tau)$ has at least one solution in $C_{1-\frac{\xi_k}{k}, \chi}^{\xi_k}[a, b] \cap C_{1-\frac{\xi_k}{k}, \chi}^{p, \theta}[a, b]$.

Proof. Consider the operator $T : C_{1-\frac{\xi_k}{k}, \chi}[a, b] \rightarrow C_{1-\frac{\xi_k}{k}, \chi}[a, b]$ given by

$$\begin{aligned} (T\mathbf{y})(\gamma) &= \frac{\Gamma_k(\theta + \xi_k)}{\Gamma_k(\xi_k)} \frac{\mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \gamma)}{Z} \left[\sum_{i=1}^m \frac{\lambda_i}{k\Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p+\theta}{k}}(t, \mu_i) \wp_{\mathbf{y}}(t) dt \right] \\ &\quad + \frac{1}{k\Gamma_k(p)} \int_{\gamma}^b \mathfrak{A}_{\chi', -}^{\frac{p}{k}}(t, \gamma) \wp_{\mathbf{y}}(t) dt. \end{aligned} \quad (3.3.36)$$

It is obvious that the operator T is well defined.

Claim 1 : Now, we show that The operator T is continuous.

let \mathbf{y}_n be a sequence such that $\mathbf{y}_n \rightarrow \mathbf{y}$ in $C_{1-\frac{\xi_k}{k}, \chi}$. Then for each $\gamma \in I$, we have

$$\begin{aligned} &|((T\mathbf{y}_n)(\gamma) - (T\mathbf{y})(\gamma))\mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \gamma)| \\ &\leq \frac{\Gamma_k(\xi_k + \theta)}{\Gamma_k(\xi_k)|Z|} \sum_{i=1}^m \frac{\lambda_i}{k\Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p+\theta}{k}}(t, \mu_i) |\wp_{\mathbf{y}_n}(t) - \wp_{\mathbf{y}}(t)| dt \\ &\quad + \frac{\mathfrak{A}_{\chi, -}^{\frac{\xi_k}{k}}(b, \gamma)}{k\Gamma_k(p)} \int_{\gamma}^b \mathfrak{A}_{\chi', -}^{\frac{p}{k}}(t, \gamma) |\wp_{\mathbf{y}_n}(t) - \wp_{\mathbf{y}}(t)| dt \\ &\quad + \frac{\Gamma_k(\theta + \xi_k)B_k(\xi_k, p + \theta)}{\Gamma_k(\xi_k)|Z|} \sum_{i=1}^m \frac{\lambda_i}{\Gamma_k(p + \theta)} \mathfrak{A}_{\chi, -}^{\frac{p+\xi_k+\theta}{k}}(b, \mu_i) |\wp_{\mathbf{y}_n}(\cdot) - \wp_{\mathbf{y}}(\cdot)|_{C_{1-\frac{\xi_k}{k}, \chi}[a, b]} \\ &\quad + \frac{\mathfrak{A}_{\chi, -}^{\frac{p}{k}+1}(a, b) B_k(\xi_k, p)}{\Gamma_k(p)} |\wp_{\mathbf{y}_n}(\cdot) - \wp_{\mathbf{y}}(\cdot)|_{C_{1-\frac{\xi_k}{k}, \chi}[a, b]} \end{aligned}$$

$$\left(\frac{\Gamma_k(\theta + \xi_k) B_k(\xi_k, p + \theta)}{\Gamma_k(\xi_k) \Gamma_k(p + \theta)} \sum_{i=1}^m \frac{\lambda_i \mathfrak{A}_{\chi, -}^{\frac{p + \xi_k + \theta}{k}}(b, \mu_i)}{|Z|} + \frac{B_k(\xi_k, p)}{\Gamma_k(p)} \mathfrak{A}_{\chi, -}^{\frac{p}{k}}(a, b) \right) \\ \times \varphi_{\mathbf{y}_n}(\cdot) - \varphi_{\mathbf{y}}(\cdot) \quad C_{1 - \frac{\xi_k}{k}, \chi}^{[a, b]}.$$

Since $\varphi_{\mathbf{y}}$ is continuous (i.e. \bar{h} is continuous), then we have

$$T\mathbf{y}_n - T\mathbf{y} \quad C_{1 - \frac{\xi_k}{k}, \chi}^{[a, b]} \quad 0 \text{ as } n \rightarrow +\infty.$$

Claim 2 : The operator T maps bounded sets into bounded sets in $C_{1 - \frac{\xi_k}{k}, \chi}^{[a, b]}$.

Indeed, it is enough to show that for $r > 0$, there exists a positive constant l such that $\mathbf{y} \in B_r = \left\{ \mathbf{y} \in C_{1 - \frac{\xi_k}{k}, \chi}^{[a, b]} : \|\mathbf{y}\|_{C_{1 - \frac{\xi_k}{k}, \chi}^{[a, b]}} \leq r \right\}$, we have $T(\mathbf{y}) \in C_{1 - \frac{\xi_k}{k}, \chi}^{[a, b]}$ with $\|\mathbf{y}\|_{C_{1 - \frac{\xi_k}{k}, \chi}^{[a, b]}} \leq l$.

$$\begin{aligned} \|(T\mathbf{y})(\gamma) \mathfrak{A}_{\chi, -}^{-\frac{\xi_k}{k}}(b, \gamma)\| &= \left\| \frac{\Gamma_k(\theta + \xi_k)}{|Z| \Gamma_k(\xi_k)} \left[\sum_{i=1}^m \frac{\lambda_i}{k \Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p + \theta}{k}}(t, \mu_i) \|\varphi_{\mathbf{y}}(t)\| dt \right] \right. \\ &\quad \left. + \frac{\mathfrak{A}_{\chi, -}^{-\frac{\xi_k}{k}}(b, \gamma)}{k \Gamma_k(p)} \int_{\gamma}^b \mathfrak{A}_{\chi', -}^{\frac{p}{k}}(t, \gamma) \|\varphi_{\mathbf{y}}(t)\| dt \right\} \\ &= A_1 + A_2. \end{aligned} \tag{3.3.37}$$

where

$$\begin{aligned} A_1 &= \frac{\Gamma_k(\theta + \xi_k)}{|Z| \Gamma_k(\xi_k)} \left[\sum_{i=1}^m \frac{\lambda_i}{k \Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p + \theta}{k}}(t, \mu_i) \|\varphi_{\mathbf{y}}(t)\| dt \right], \\ A_2 &= \frac{\mathfrak{A}_{\chi, -}^{-\frac{\xi_k}{k}}(b, \gamma)}{k \Gamma_k(p)} \int_{\gamma}^b \mathfrak{A}_{\chi', -}^{\frac{p}{k}}(t, \gamma) \|\varphi_{\mathbf{y}}(t)\| dt. \end{aligned}$$

It follows from the assumption (H_2) , we obtain

$$\|\varphi_{\mathbf{y}}(\gamma)\| \leq \bar{h}(\gamma, \mathbf{y}(\gamma), \mathbf{y}(\tau\gamma), \varphi_{\mathbf{y}}(\gamma)) \leq h(\gamma) + l(\gamma) \|\mathbf{y}\| + m(\gamma) \|\mathbf{y}\| + n(\gamma) \|\varphi_{\mathbf{y}}(\gamma)\| \\ \leq \frac{h^* + (l^* + m^*) \|\mathbf{y}(\gamma)\|}{1 - n^*}.$$

By estimating A_1, A_2 terms separately, we get

$$\begin{aligned} A_1 &= \frac{\Gamma_k(\theta + \xi_k)}{\Gamma_k(\xi_k) |Z| (1 - \eta^*)} \sum_{i=1}^m \lambda_i \left(\frac{h^* \mathfrak{A}_{\chi, -}^{\frac{p + \theta}{k} + 1}(b, \mu_i)}{k \Gamma_k(p + \theta + k)} + \frac{(l^* + m^*) \mathfrak{A}_{\chi, -}^{\frac{p + \theta + \xi_k}{k}}(b, \mu_i) B_k(\xi_k, p + \theta)}{\Gamma_k(p + \theta)} \right) \\ &\quad \times \|\mathbf{y}\|_{C_{1 - \frac{\xi_k}{k}, \chi}^{[a, b]}} \end{aligned} \tag{3.3.38}$$

$$A_2 = \frac{1}{1-\eta^*} \left(h^* \frac{\mathfrak{A}_{\chi,-}^{\frac{p-\xi_k+\theta}{k}+2}(b,\gamma)}{k\Gamma_k(p+k)} + (l^* + m^*) \frac{\mathfrak{A}_{\chi,-}^{\frac{p}{k}+1}(b,\gamma)}{\Gamma_k(p)} B_k(\xi_k,p) \mathbf{y} \ C_{1-\frac{\xi_k}{k},\chi}[a,b] \right). \quad (3.3.39)$$

Brining inequalities A_1 and A_2 into 3.3.37 we get

$$\begin{aligned} & / (T\mathbf{y})(\gamma) \mathfrak{A}_{\chi,-}^{\frac{\xi_k}{k}}(b,\gamma) / \\ & \frac{h^*}{(1-\eta^*)} \left(\frac{\Gamma_k(\theta+\xi_k)}{\Gamma_k(p+\theta+k)\Gamma_k(\xi_k)} /Z/ \sum_{i=1}^m \lambda_i \mathfrak{A}_{\chi,-}^{\frac{p+\theta}{k}+1}(b,\mu_i) + \frac{\mathfrak{A}_{\chi,-}^{\frac{p-\xi_k+\theta}{k}+2}(b,\gamma)}{k\Gamma_k(p+k)} \right) \\ & + \frac{(l^* + m^*)}{1-\eta^*} \left(\frac{\mathfrak{A}_{\chi,-}^{\frac{p}{k}+1}(b,\gamma)}{\Gamma_k(p)} B_k(\xi_k,p) \right. \\ & \left. + \frac{\Gamma_k(\theta+\xi_k)}{\Gamma_k(\xi_k)\Gamma_k(p+\theta)} /Z/ \mathfrak{A}_{\chi,-}^{\frac{p+\theta+\xi_k}{k}}(b,\mu_i) B_k(\xi_k,p+\theta) \right) \mathbf{y} \ C_{1-\frac{\xi_k}{k},\chi}[a,b] = l. \end{aligned}$$

Claim 3 : T maps bounded sets into equicontinuous set of $C_{1-\frac{\xi_k}{k},\chi}[a,b]$.

let $\gamma_1, \gamma_2 \in I, \gamma_2 \in \gamma_1, B_r$ be a bounded set of $C_{1-\frac{\xi_k}{k},\chi}[a,b]$, as in **Claim 2**, and let $\mathbf{y} \in B_r$. Then,

$$\begin{aligned} & / \mathfrak{A}_{\chi,-}^{\frac{\xi_k}{k}}(b,\gamma_1) T\mathbf{y}(\gamma_1) - \mathfrak{A}_{\chi,-}^{\frac{\xi_k}{k}}(b,\gamma_2) T\mathbf{y}(\gamma_2) / \\ & \left| \frac{\mathfrak{A}_{\chi,-}^{\frac{\xi_k}{k}}(b,\gamma_1)}{k\Gamma_k(p)} \int_{\gamma_1}^b \mathfrak{A}_{\chi',-}^{\frac{p}{k}}(t,\gamma_1) \wp_{\mathbf{y}}(t) dt - \frac{\mathfrak{A}_{\chi,-}^{\frac{\xi_k}{k}}(b,\gamma_2)}{k\Gamma_k(p)} \int_{\gamma_2}^b \mathfrak{A}_{\chi',-}^{\frac{p}{k}}(t,\gamma_2) \wp_{\mathbf{y}}(t) dt \right| \\ & \left| \frac{1}{k\Gamma_k(p)} \int_{\gamma_1}^b \left[\mathfrak{A}_{\chi,-}^{\frac{\xi_k}{k}}(b,\gamma_1) \mathfrak{A}_{\chi',-}^{\frac{p}{k}}(t,\gamma_1) - \mathfrak{A}_{\chi,-}^{\frac{\xi_k}{k}}(b,\gamma_2) \mathfrak{A}_{\chi',-}^{\frac{p}{k}}(t,\gamma_2) \right] \wp_{\mathbf{y}}(t) dt \right| \\ & + \left| \frac{\mathfrak{A}_{\chi,-}^{\frac{\xi_k}{k}}(b,\gamma_2)}{k\Gamma_k(p)} \int_{\gamma_1}^{\gamma_2} \mathfrak{A}_{\chi',-}^{\frac{p}{k}}(t,\gamma_2) \wp_{\mathbf{y}}(t) dt \right|. \end{aligned}$$

As, $\gamma_1 \rightarrow \gamma_2$, the right hand side of the above inequality tends to zero.

As a consequence of **Claim 1-3**, together with Arzela-Ascoli Theorem, we can conclude that

$T : C_{1-\frac{\xi_k}{k},\chi}[a,b] \rightarrow C_{1-\frac{\xi_k}{k},\chi}[a,b]$ is completely continuous.

Claim 4 : A priori bounds.

Now it remains to show that the set

$$w = \left\{ \mathbf{y} \in C_{1-\frac{\xi_k}{k},\chi}[a,b], \mathbf{y} = \delta(T\mathbf{y}), 0 < \delta < 1 \right\} \text{ is bounded set.}$$

Let $\mathbf{y} \in w$, $\mathbf{y} = \delta(T\mathbf{y})$ for some $0 < \delta < 1$. So for each $\gamma \in I$, we get

$$\begin{aligned} \mathbf{y}(\gamma) &= \delta \left[\frac{\Gamma_k(\theta+\xi_k)}{\Gamma_k(\theta+p)} \frac{\mathfrak{A}_{\chi,-}^{\frac{\xi_k}{k}}(b,\gamma)}{/Z/} \sum_{i=1}^m \frac{\lambda_i}{k\Gamma_k(\xi_k)} \int_{\mu_i}^b \mathfrak{A}_{\chi',-}^{\frac{p+\theta}{k}}(t,\mu_i) \wp_{\mathbf{y}}(t) dt \right. \\ & \left. + \frac{1}{k\Gamma_k(p)} \int_{\gamma}^b \mathfrak{A}_{\chi',-}^{\frac{p}{k}}(t,\gamma) \wp_{\mathbf{y}}(t) dt \right]. \end{aligned}$$

This implies by (H_2) that for each $\gamma \in I$, we have

$$\begin{aligned}
& \|\mathbf{y}(\gamma)\mathfrak{A}_{\chi,-}^{-\frac{\xi_k}{k}}(b,\gamma) - (T\mathbf{y})(\gamma)\mathfrak{A}_{\chi,-}^{-\frac{\xi_k}{k}}(b,\gamma)\| \\
& \leq \frac{h^*}{(1-\eta^*)} \left(\frac{\Gamma_k(\theta+\xi_k)\sum_{i=1}^m \lambda_i}{\Gamma_k(\xi_k)/Z/k\Gamma_k(p+\theta+k)} \mathfrak{A}_{\chi,-}^{\frac{p+\theta}{k}+1}(b,\mu_i) + \frac{\mathfrak{A}_{\chi,-}^{\frac{p-\xi_k+\theta}{k}+2}(b,\gamma)}{\Gamma_k(p+k)} \right) \\
& + \frac{(l^*+m^*)}{1-\eta^*} \left(\frac{B_k(\xi_k,p)\mathfrak{A}_{\chi,-}^{\frac{p}{k}+1}(b,\gamma)}{\Gamma_k(p)} + \frac{\Gamma_k(\theta+\xi_k)\sum_{i=1}^m \lambda_i \mathfrak{A}_{\chi,-}^{\frac{p+\theta+\xi_k}{k}}(b,\mu_i) B_k(\xi_k,p+\theta)}{|Z/\Gamma_k(\xi_k)\Gamma_k(p+\theta)} \right) \\
& \leq \mathbf{y} C_{1-\frac{\xi_k}{k},\chi} [a,b] < \infty.
\end{aligned}$$

This shows that the set w is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that T has a fixed point which is a solution of problem $(\mathbb{P}\mathbb{R}\tau)$. \square

The subsequent results rely on the Banach contraction principle.

Theorem 3.3.2. *Suppose that the following assumption is satisfied:*

(H_3) Let $\bar{h}: [a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $\bar{h} \in C^{\theta(k-p)}[a,b]$ for any \mathbf{y} in $C_{1-\frac{\xi_k}{k},\chi}^{\xi_k}[a,b]$ and there exist positive constants $M > 0$ and $L > 0$ such that

$$\begin{aligned}
& \|\bar{h}(\gamma,u,v,w) - \bar{h}(\gamma,u_1,v_1,w_1)\| \leq M(|u-u_1|+|v-v_1|) + L|w-w_1|, \text{ for any } \gamma \in [a,b] \\
& \text{and } u,v,w,u_1,v_1,w_1 \in \mathbb{R}.
\end{aligned}$$

If

$$\left(\frac{2M}{1-L} \right) \left[\frac{B_k(\xi_k,p+\theta)\Gamma_k(\xi_k+\theta)}{\Gamma_k(\xi_k)\Gamma_k(p+\theta)/Z} \sum_{i=1}^m \lambda_i \mathfrak{A}_{\chi,-}^{\frac{p+\xi_k+\theta}{k}}(b,\mu_i) + \frac{B_k(\xi_k,\theta)\mathfrak{A}_{\chi,-}^{\frac{p}{k}+1}(a,b)}{\Gamma_k(p)} \right] < 1, \quad (3.3.40)$$

are satisfied. Then the problem $(\mathbb{P}\mathbb{R}\tau)$ has a unique solution.

Proof. Let the operator $T: C_{1-\frac{\xi_k}{k},\chi}[a,b] \rightarrow C_{1-\frac{\xi_k}{k},\chi}[a,b]$ defined as in Equation 3.3.36. In view of Theorem 3.3.1, we know that the fixed point of T are solutions of problem $(\mathbb{P}\mathbb{R}\tau)$. Now, we prove that T has a unique fixed point, which is a solution of problem $(\mathbb{P}\mathbb{R}\tau)$.

Let $\mathbf{y}, x \in C_{1-\frac{\xi_k}{k}, \chi}[a, b]$ and $\gamma \in [a, b]$, then we get

$$\begin{aligned}
& \left| \mathfrak{A}_{\chi, -}^{-\frac{\xi_k}{k}}(b, \gamma) T \mathbf{y}(\gamma) - \mathfrak{A}_{\chi, -}^{-\frac{\xi_k}{k}}(b, \gamma) T x(\gamma) \right| \\
& \quad \sum_{i=1}^m \frac{\lambda_i}{\Gamma_k(\xi_k) / Z} \frac{\Gamma_k(\theta + \xi_k)}{k \Gamma_k(p + \theta)} \int_{\mu_i}^b \mathfrak{A}_{\chi', -}^{\frac{p+\theta}{k}}(t, \mu_i) \left| \wp_{\mathbf{y}}(t) - \wp_x(t) \right| dt \\
& \quad + \frac{\mathfrak{A}_{\chi, -}^{-\frac{\xi_k}{k}}(b, \gamma)}{k \Gamma_k(p)} \int_{\gamma}^b \mathfrak{A}_{\chi', -}^{\frac{p}{k}}(t, \gamma) \left| \wp_{\mathbf{y}}(t) - \wp_x(t) \right| dt,
\end{aligned} \tag{3.3.41}$$

and

$$\begin{aligned}
& \left| \wp_{\mathbf{y}}(\gamma) - \wp_x(\gamma) \right| = \left| \bar{h}(\gamma, \mathbf{y}(\gamma), \mathbf{y}(\tau\gamma), \wp_{\mathbf{y}}(\gamma)) - \bar{h}(\gamma, x(\gamma), x(\tau\gamma), \wp_x(\gamma)) \right| \\
& \quad M \left(\left| \mathbf{y}(\gamma) - x(\gamma) \right| + \left| \mathbf{y}(\tau\gamma) - x(\tau\gamma) \right| \right) + L \left| \wp_{\mathbf{y}}(\gamma) - \wp_x(\gamma) \right| \\
& \quad \frac{2M}{1-L} \left| \mathbf{y}(\gamma) - x(\gamma) \right|.
\end{aligned} \tag{3.3.42}$$

By replacing 3.3.42 in the inequality 3.3.41, we get

$$\begin{aligned}
& \left| \mathfrak{A}_{\chi, -}^{-\frac{\xi_k}{k}}(\gamma, a) T \mathbf{y}(\gamma) - \mathfrak{A}_{\chi, -}^{-\frac{\xi_k}{k}}(b, \gamma) T x(\gamma) \right| \\
& \quad \left(\frac{2M}{1-L} \right) \left[\frac{B_k(\xi_k, p + \theta)}{\Gamma_k(p + \theta)} \frac{\Gamma_k(\theta + \xi_k)}{\Gamma_k(\xi_k) / Z} \sum_{i=1}^m \lambda_i \mathfrak{A}_{\chi, -}^{\frac{p + \xi_k + \theta}{k}}(b, \mu_i) + \frac{B_k(\xi_k, p)}{\Gamma_k(p)} \mathfrak{A}_{\chi, -}^{\frac{p}{k}}(a, b) \right] \\
& \quad \times \left| \mathbf{y} - x \right|_{C_{1-\frac{\xi_k}{k}, \chi}}.
\end{aligned}$$

This gives,

$$\begin{aligned}
& \left| T \mathbf{y} - T x \right|_{C_{1-\frac{\xi_k}{k}, \chi}} \\
& \quad \left(\frac{2M}{1-L} \right) \left[\frac{B_k(\xi_k, p + \theta)}{\Gamma_k(p + \theta)} \frac{\Gamma_k(\theta + \xi_k)}{\Gamma_k(\xi_k) / Z} \sum_{i=1}^m \lambda_i \mathfrak{A}_{\chi, -}^{\frac{p + \xi_k + \theta}{k}}(b, \mu_i) + \frac{B_k(\xi_k, p)}{\Gamma_k(p)} \mathfrak{A}_{\chi, -}^{\frac{p}{k}}(a, b) \right] \\
& \quad \times \left| \mathbf{y} - x \right|_{C_{1-\frac{\xi_k}{k}, \chi}}.
\end{aligned}$$

From 3.3.42, it follows that T has a unique fixed point, which is a solution of problem $(\mathbb{P}\mathbb{R}\tau)$. \square

3.4 Application to fractional differential equations involving the (k, χ) -Hilfer fractional derivative

Consider the following fractional differential equation involving the (k, χ) -Hilfer derivativ:

$$(\tau') \begin{cases} {}_{1,H}\mathfrak{D}_{1-}^{\frac{3}{6}, \frac{3}{9}, \gamma} \mathbf{y}(\gamma) = \frac{30+|\mathbf{y}(\gamma)|+|\mathbf{y}(\frac{3}{24}\gamma)|+{}_{1,H}\mathfrak{D}_{1-}^{\frac{3}{6}, \frac{3}{9}, \gamma} \mathbf{y}(\frac{3}{24}\gamma)}{(89^\gamma) \left[1+|\mathbf{y}(\gamma)|+|\mathbf{y}(\frac{3}{24}\gamma)|+{}_{1,H}\mathfrak{D}_{1-}^{\frac{3}{6}, \frac{3}{9}, \gamma} \mathbf{y}(\frac{3}{24}\gamma) \right]}, & \gamma \in [0, 1), \\ {}_1 \mathbf{y}(1) = \frac{6}{15} \mathbf{y}(\frac{6}{9}), & \xi_1 = \frac{3}{6} + (\frac{3}{9} \times 1) - (\frac{3}{9} \times \frac{3}{6}) = \frac{6}{9}, \end{cases}$$

By comparing the problem (τ') and the problem $(\mathbb{P}\mathbb{R}\tau)$, we obtain : $a = 0, b = 1, k = 1, p = 3/6, \theta = 3/9, 0 < \tau < 1$, and $\chi(\gamma) = \gamma$. Also from the initial condition we can easily see that $\lambda_1 = \frac{6}{15}, \mu_1 = \frac{6}{9}$ since $\xi_1 = 6/9, 1 - \xi_1 = 3/6$ and $\hbar : [a, b) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is function defined by :

$$\hbar(\gamma, \mathbf{y}(\gamma), \mathbf{y}(\tau\gamma), {}_{k,H}\mathfrak{D}_{b-}^{p,\theta,\chi} \mathbf{y}(\tau\gamma)) = \frac{30+|\mathbf{y}(\gamma)|+|\mathbf{y}(\frac{3}{24}s)|+{}_{1,H}\mathfrak{D}_{1-}^{\frac{3}{6}, \frac{3}{9}, \gamma} \mathbf{y}(\frac{3}{24}\gamma)}{(89^\gamma) \left[1+|\mathbf{y}(\gamma)|+|\mathbf{y}(\frac{3}{24}s)|+{}_{1,H}\mathfrak{D}_{1-}^{\frac{3}{6}, \frac{3}{9}, \gamma} \mathbf{y}(\frac{3}{24}\gamma) \right]}.$$

Obviously, \hbar is continuous for all $u, \theta, w, u_1, \theta_1, w_1 \in \mathbb{R}^+$ and $\gamma \in [0, 1)$, we have

$$|\hbar(\gamma, u, \theta, w) - \hbar(\gamma, u_1, \theta_1, w_1)| \leq \frac{1}{89} (|u - u_1| + |\theta - \theta_1| + |w - w_1|).$$

Thus, it follows that condition (H_2) is true with $M = L = \frac{1}{89}$. Furthermore,

$$|\hbar(\gamma, u, \theta, w)| \leq \frac{1}{89} (30 + |u| + |\theta| + |w|).$$

The above implies that (H_1) is true with $h(\gamma) = \frac{30}{89^\gamma}, l(\gamma) = m(\gamma) = \eta(\gamma) = \frac{1}{89^\gamma}$, and $h^* = \frac{30}{89}, l^* = m^* = \eta^* = \frac{1}{89} < 1$

Moreover, by calculation, we get $Z = \Gamma_1(\xi_1) - \lambda_1 \mathfrak{A}_{\chi, -}^{\xi_1}(b, \mu_1) = \Gamma(\frac{6}{9}) - \frac{6}{15} (\frac{6}{9})^{-\frac{3}{9}} = 0.9 = 0$.

Therefore, all the hypotheses of Theorem 3.3.1 are satisfied, which means that problem (τ') has at least one solution on $[a, b)$, we obtain

$$\begin{aligned} & \left(\frac{2M}{1-L} \right) \left[\frac{B_k(\xi_k, p+\theta)\Gamma_k(\theta+\xi_k)}{k\Gamma_k(\xi_k)\Gamma_k(p+\theta)} \frac{1}{|Z|} \sum_{i=1}^m \lambda_i \mathfrak{A}_{\chi, -}^{\frac{p+\xi_k+\theta}{k}}(b, \mu_i) + \frac{B_k(\xi_k, \theta)}{k\Gamma_k(p)} \mathfrak{A}_{\chi, -}^{\frac{p}{k}+1}(a, b) \right] \\ &= \left(\frac{\frac{2}{89}}{1-\frac{1}{89}} \right) \left[\frac{B_1(\frac{6}{9}, \frac{3}{6}+\frac{3}{9})\Gamma_1(\frac{3}{9}+\frac{6}{9})}{\Gamma_1(\frac{6}{9})\Gamma_1(\frac{3}{6}+\frac{3}{9})} \frac{1}{/0.9/9} \mathfrak{A}_{\gamma, -}^{\frac{\frac{3}{6}+\frac{6}{9}+\frac{3}{9}}{1}}(1, \frac{6}{9}) + \frac{B_1(\frac{6}{9}, \frac{3}{9})}{\Gamma_1(\frac{3}{6})} \mathfrak{A}_{\gamma, -}^{\frac{3}{9}+1}(0, 1) \right] \\ & 0.022 \left[0.7 \frac{B_1(\frac{6}{9}, \frac{15}{18})}{\Gamma_1(\frac{6}{9})\Gamma_1(\frac{15}{18})} + \frac{B_1(\frac{6}{9}, \frac{3}{9})}{\Gamma_1(\frac{3}{6})} \right] < 1. \end{aligned}$$

Since the assumptions of Theorem 3.3.2 are satisfied. Then the problem (τ') has a unique solution on $[0, 1)$.

Chapter 4

Nonlocal integral boundary value problems for sequential differential equation involving fractional mixed derivatives

4.1 Introduction

We study the existence and compactness of the solution set for a sequential fractional differential equation subject to nonlocal integral boundary conditions. More precisely, we consider the following problem [7]:

$${}^H\mathfrak{D}_{a^+}^{\rho,\theta,\chi}\left(D^\chi\mathbf{y}(\gamma) - \kappa\mathbf{y}(\gamma)\right) = \tilde{h}\left(\gamma, \mathbf{y}(\gamma), {}^C\mathfrak{D}_{a^+}^{\xi,\chi}\mathbf{y}(\gamma)\right), \quad \gamma \in (a, b], \quad (4.1.1)$$

$${}^{1-\xi,\chi}D^\chi\mathbf{y}(a^+) = \sum_{i=1}^n \zeta_i D^\chi\mathbf{y}(\gamma_i), \quad \mathbf{y}(a) = 0, \quad (4.1.2)$$

where:

- κ is a real number,
- ${}^H\mathfrak{D}_{a^+}^{\rho,\theta,\chi}$ denotes the χ -Hilfer fractional derivative of order ρ and parameter θ such that $0 < \rho < 1$

and $0 < \theta < 1$,

- ${}^C \mathcal{D}_{a^+}^{\xi, \chi}$ is the χ -Caputo fractional derivative of order $\xi = \rho + \theta - \theta\rho$,
- $\zeta_i, i = 1, \dots, m$ are real numbers,
- E is a Banach space and $h : (a, b] \times E^2 \rightarrow E$ is a function,
- $\chi \in C^1([a, b], \mathbb{R})$ such that $\chi'(\gamma) > 0$ for all $\gamma \in [a, b]$,
- $D^\chi = \frac{1}{\chi'(\gamma)} \frac{d}{d\gamma}$, $a, b \in \mathbb{R}_+^*$ with $a < b$ and $\gamma_i \in (a, b), i = 1, \dots, n$ such that

$$\Gamma(\xi) = \sum_{i=1}^n \zeta_i (\chi(\gamma_i) - \chi(a))^{\xi-1}.$$

For the study of the problems, we need to recall some spaces.

Throughout this chapter, we denote by $C([a, b])$ (resp. by $L^1([a, b])$) the space of E -valued continuous functions (resp. the space of E-Bochner's integrable functions) with the following norm

$$\|g\|_\infty = \sup \left\{ \|g(\gamma)\|, \gamma \in [a, b] \right\} \quad \left(\text{resp. } \|g\|_{L^1} = \int_a^b \|g(\gamma)\| d\gamma \right).$$

Let $C_{1-\xi}([a, b])$ be the Banach spaces of functions from $(a, b]$ into E which is defined as:

$$C_{1-\xi, \chi}([a, b]) = \left\{ g \in C([a, b]) : (\chi(\cdot) - \chi(a))^{1-\xi} g(\cdot) \in C([a, b], E) \right\}.$$

with his norm $\|g\|_{\xi, \chi}$, that is given by

$$\|g\|_{\xi, \chi} = \sup_{\gamma \in (a, b]} (\chi(\gamma) - \chi(a))^{1-\xi} \|g(\gamma)\|.$$

Next, we denote by $C_{1-\xi, \chi}^1([a, b])$ the space of functions (ξ, χ) -continuously differentiable defined as follows

$$C_{1-\xi, \chi}^1([a, b]) = \left\{ g : (a, b] \rightarrow E : g(\cdot) \in C([a, b]) \text{ and } D^\chi g(\cdot) \in C_{1-\xi, \chi}([a, b]) \right\}.$$

we note that the space $C_{1-\xi, \chi}^1([a, b])$ with the norm $\|g\|_{\xi, \chi}^1 = \|g\|_\infty + \|D^\chi g\|_{\xi, \chi}$ is a Banach space.

In the following, for all $\eta > -1$, we put $\Psi^\eta(r, s) = (\chi(r) - \chi(s))^\eta$, for all $s, r \in [a, b]$ with $r > s$ and $\Psi_*^\eta = (\chi(b) - \chi(a))^\eta$.

We consider the following auxiliary spaces

$$\begin{aligned} \mathcal{C}_{1-\xi,\chi}^\xi([a,b]) &= \left\{ g : (a,b) \rightarrow E/g \in \mathcal{C}_{1-\xi,\chi}([a,b]), {}^{RL}\mathfrak{D}_{a^+}^{\xi,\chi} g \in \mathcal{C}_{1-\xi,\chi}([a,b]) \right\}, \\ \mathcal{C}_{1-\xi,\chi}^{1,\xi}([a,b]) &= \left\{ g : (a,b) \rightarrow E/g \in C([a,b]), D^\chi g \in \mathcal{C}_{1-\xi,\chi}^\xi([a,b]) \right\} \text{ and} \\ \mathcal{C}_{1-\xi,\chi}^{1,\rho,\theta}([a,b]) &= \left\{ g : (a,b) \rightarrow E/g \in C([a,b]), D^\chi g, {}^H\mathfrak{D}_{a^+}^{\rho,\theta,\chi} D^\chi g \in \mathcal{C}_{1-\xi,\chi}([a,b]) \right\}, \end{aligned}$$

it is clear to see that $\mathcal{C}_{1-\xi,\chi}^{1,\xi}([a,b]) \subset \mathcal{C}_{1-\xi,\chi}^{1,\rho,\theta}([a,b])$.

4.2 Integral Equation

In the content of Lemma below, we will illustrate the equivalence between the problem at hand (4.1.1)-(4.1.2) and the following integral equation

$$\begin{aligned} \mathbf{y}(\gamma) &= \frac{\sum_{i=1}^n \zeta_i \left[\kappa \mathbf{y}(\gamma_i) + \frac{\rho}{a^+} \bar{h}(\gamma_i, \mathbf{y}(\gamma_i), {}^C\mathfrak{D}_{a^+}^{\xi,\chi} \mathbf{y}(\gamma_i)) \right]}{\Gamma(\xi+1) - \xi \sum_{i=1}^n \zeta_i \Psi^{\xi-1}(\gamma_i, a)} \Psi^\xi(\gamma, a) + \kappa \int_a^\gamma \chi'(s) \mathbf{y}(s) ds \\ &+ \frac{1}{\Gamma(\rho+1)} \int_a^\gamma \chi'(s) \Psi^\rho(\gamma, s) \bar{h}(s, \mathbf{y}(s), {}^C\mathfrak{D}_{a^+}^{\xi,\chi} \mathbf{y}(s)) ds. \end{aligned} \quad (4.2.1)$$

Lemma 4.2.1. *Let $\xi = \rho + \theta - \rho\theta$ with $0 < \rho < 1$ and $0 < \theta < 1$, we assume that the function $\bar{h} : (a,b] \times E^2 \rightarrow E$ satisfies $\bar{h}(\cdot, \mathbf{y}(\cdot), {}^C\mathfrak{D}_{a^+}^{\xi,\chi} \mathbf{y}(\cdot)) \in \mathcal{C}_{1-\xi,\chi}([a,b])$, for all $\mathbf{y}(\cdot) \in \mathcal{C}_{1-\xi,\chi}^{1,\xi}([a,b])$. If $\mathbf{y} \in \mathcal{C}_{1-\xi,\chi}^{1,\xi}([a,b])$. Then, \mathbf{y} is a solution of the problem (4.1.1)-(4.1.2) if and only if \mathbf{y} satisfies the integral equation (4.2.1).*

Proof. Let $\mathbf{y} \in \mathcal{C}_{1-\xi,\chi}^{1,\xi}([a,b])$ be a solution of the problem (4.1.1)-(4.1.2), since $\bar{h}(\cdot, \mathbf{y}(\cdot), {}^C\mathfrak{D}_{a^+}^{\xi,\chi} \mathbf{y}(\cdot)) \in \mathcal{C}_{1-\xi,\chi}([a,b])$, from Lemma (2.2.1) we have

$$D^\chi \mathbf{y}(\gamma) = \frac{{}^{1-\xi,\chi}D^\chi \mathbf{y}(a^+)}{\Gamma(\xi)} \Psi^{\xi-1}(\gamma, a) + \kappa \mathbf{y}(\gamma) + \frac{\rho,\chi}{a^+} \bar{h}(\gamma, \mathbf{y}(\gamma), {}^C\mathfrak{D}_{a^+}^{\xi,\chi} \mathbf{y}(\gamma)). \quad (4.2.2)$$

Next, we substitute γ by γ_i into the above equation, we get

$$D^\chi \mathbf{y}(\gamma_i) = \frac{{}^{1-\xi,\chi}D^\chi \mathbf{y}(a^+)}{\Gamma(\xi)} \Psi^{\xi-1}(\gamma_i, a) + \kappa \mathbf{y}(\gamma_i) + \frac{\rho,\chi}{a^+} \bar{h}(\gamma_i, \mathbf{y}(\gamma_i), {}^C\mathfrak{D}_{a^+}^{\xi,\chi} \mathbf{y}(\gamma_i)).$$

By utilizing the second condition (4.1.2), we obtain

$$\begin{aligned} {}^{1-\xi,\chi}D^\chi \mathbf{y}(a^+) &= \frac{{}^{1-\xi,\chi}D^\chi \mathbf{y}(a^+)}{\Gamma(\xi)} \sum_{i=1}^n \zeta_i \Psi^{\xi-1}(\gamma_i, a) \\ &+ \sum_{i=1}^n \zeta_i \left[\kappa \mathbf{y}(\gamma_i) + \frac{\rho,\chi}{a^+} \bar{h}(\gamma_i, \mathbf{y}(\gamma_i), {}^C\mathfrak{D}_{a^+}^{\xi,\chi} \mathbf{y}(\gamma_i)) \right], \end{aligned}$$

this implies

$${}_{a^+}^{1-\xi, \chi} D^\chi \mathbf{y}(a^+) = \frac{\Gamma(\xi) \sum_{i=1}^n \zeta_i \left[\kappa \mathbf{y}(\gamma_i) + {}_{a^+}^{\rho, \chi} \bar{h}(\gamma_i, \mathbf{y}(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(\gamma_i)) \right]}{\Gamma(\xi) - \sum_{i=1}^n \zeta_i \Psi^{\xi-1}(\gamma_i, a)}. \quad (4.2.3)$$

By substituting (4.2.3) to (4.2.2), we deduce that

$$\begin{aligned} D^\chi \mathbf{y}(\gamma) &= \frac{\sum_{i=1}^n \zeta_i \left[\kappa \mathbf{y}(\gamma_i) + {}_{a^+}^{\rho, \chi} \bar{h}(\gamma_i, \mathbf{y}(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(\gamma_i)) \right]}{\Gamma(\xi) - \sum_{i=1}^n \zeta_i \Psi^{\xi-1}(\gamma_i, a)} \Psi^{\xi-1}(\gamma, a) + \kappa \mathbf{y}(\gamma) \\ &\quad + \frac{1}{\Gamma(\rho)} \int_a^\gamma \chi'(s) \Psi^{\rho-1}(\gamma, s) \bar{h}(s, \mathbf{y}(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(s)) ds. \end{aligned} \quad (4.2.4)$$

Next, applying $I_{a^+}^\chi$ to both sides of (4.2.4), we obtain

$$\begin{aligned} \mathbf{y}(\gamma) &= \frac{\sum_{i=1}^n \zeta_i \left[\kappa \mathbf{y}(\gamma_i) + {}_{a^+}^{\rho, \chi} \bar{h}(\gamma_i, \mathbf{y}(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(\gamma_i)) \right]}{\Gamma(\xi+1) - \xi \sum_{i=1}^n \zeta_i \Psi^{\xi-1}(\gamma_i, a)} \Psi^\xi(\gamma, a) + \kappa \int_a^\gamma \chi'(s) \mathbf{y}(s) ds \\ &\quad + \frac{1}{\Gamma(\rho+1)} \int_a^\gamma \chi'(s) \Psi^\rho(\gamma, s) \bar{h}(s, \mathbf{y}(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(s)) ds. \end{aligned}$$

Conversely, let $\mathbf{y} \in C_{1-\xi, \chi}^{1, \xi}([a, b])$ be a function verifies the equation (4.2.1), it is clear that $\mathbf{y}(0) = 0$. By applying D^χ to both sides of (4.2.1), we obtain the equation (4.2.4), using Lemma 2.2.1, we can easily establish that the function \mathbf{y} satisfies the second condition (4.1.2). \square

4.3 Existence and compactness

We prove that the solution set (denoted \mathbf{SS}) of the problem (4.1.1) – (4.1.2) is nonempty and compact, we necessarily assume the following hypotheses

(H₁) Suppose that the function $\bar{h} : (a, b] \times E^2 \rightarrow E$ verifies $\bar{h}(\cdot, g(\cdot), v(\cdot)) \in C_{1-\xi, \chi}^{\theta(1-\rho)}([a, b])$, for all $g(\cdot), v(\cdot) \in C([a, b])$, $\bar{h}(\cdot, 0, 0) \in C([a, b], E)$ and there exists $\alpha, \beta \in \mathbb{R}_+$ such that

(H₁₋₁) For all $g, v, \bar{g}, \bar{v} \in E$:

$$\bar{h}(\gamma, g, v) - \bar{h}(\gamma, \bar{g}, \bar{v}) \leq \alpha |g - \bar{g}| + \beta |v - \bar{v}|.$$

(H₁₋₂) For each nonempty, bounded set $\Omega \in C_{1-\xi, \chi}^1([a, b])$, for all $\gamma \in (a, b]$, we have

$$\vartheta \left(\bar{h}(\gamma, \Omega(\gamma), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \Omega(\gamma)) \right) \leq \alpha \vartheta(\Omega(\gamma)) + \beta \vartheta \left({}^C \mathfrak{D}_{a^+}^{\xi, \chi} \Omega(\gamma) \right),$$

where

$$\Omega(\gamma) = \left\{ \mathbf{y}(\gamma), \mathbf{y} \in C_{1-\xi, \chi}^1([a, b]) \right\} \text{ and } {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \Omega(\gamma) = \left\{ {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(\gamma), \mathbf{y} \in C_{1-\xi, \chi}^1([a, b]) \right\}.$$

(H₂)

$$\left(\kappa \Gamma(\rho + 2) + (\rho + 1) A_0 \right) \left(|T| / \zeta^* n (\Psi_*^\xi + \xi) + \Psi_*^{1-\xi} \right) + \left(A_0 + \kappa \Gamma(\rho + 2) \right) \Psi_*^1 < \frac{\Gamma(\rho + 2)}{2},$$

where

$$T = \frac{1}{\Gamma(\xi + 1) - \xi \sum_{i=1}^n \zeta_i \Psi^{\xi-1}(\gamma_i, a)} \text{ and } A_0 = \left(\alpha + \beta \Gamma(\xi) \right) \Psi_*^\rho.$$

Define the operator $\Xi : C_{1-\xi, \chi}^1([a, b]) \times C_{1-\xi, \chi}^1([a, b])$ by

$$\begin{aligned} \Xi \mathbf{y}(\gamma) &= T \sum_{i=1}^n \zeta_i \left[\kappa \mathbf{y}(\gamma_i) + \frac{\rho, \chi}{a^+} \mathfrak{h} \left(\gamma_i, \mathbf{y}(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(\gamma_i) \right) \right] \Psi^\xi(\gamma, a) + \kappa \int_a^\gamma \chi'(s) \mathbf{y}(s) ds \\ &\quad + \frac{1}{\Gamma(\rho + 1)} \int_a^\gamma \chi'(s) \Psi^\rho(\gamma, s) \mathfrak{h} \left(s, \mathbf{y}(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(s) \right) ds. \end{aligned}$$

and the operator $D^X \Xi : C_{1-\xi, \chi}^1([a, b]) \times C_{1-\xi, \chi}^1([a, b])$ by

$$\begin{aligned} D^X \Xi \mathbf{y}(\gamma) &= \xi T \sum_{i=1}^n \zeta_i \left[\kappa \mathbf{y}(\gamma_i) + \frac{\rho, \chi}{a^+} \mathfrak{h} \left(\gamma_i, \mathbf{y}(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(\gamma_i) \right) \right] \Psi^{\xi-1}(\gamma, a) + \kappa \mathbf{y}(\gamma) \\ &\quad + \frac{1}{\Gamma(\rho)} \int_a^\gamma \chi'(s) \Psi^{\rho-1}(\gamma, s) \mathfrak{h} \left(s, \mathbf{y}(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(s) \right) ds. \end{aligned}$$

In this part, we will present the result concerning the existence of solutions of the problem (4.1.1) – (4.1.2). First, we will give some useful lemmas to demonstrate this result.

Lemma 4.3.1. *We assume the hypotheses (H₁) and (H₁₋₁) hold. Then*

(1) Ξ is bounded and continuous.

(2) $\Xi(B)$ is equicontinuous for all bounded subset B of $C_{1-\xi, \chi}^1([a, b])$.

Proof. Let us show the condition (1), we begin to prove that Ξ is bounded operator. Let $\mathbf{y} \in C_{1-\xi, \chi}^1([a, b])$, it is clear to see that $\Xi \mathbf{y} \in C_{1-\xi, \chi}^1([a, b])$. Using (H₁) and (H₁₋₁), for all $\mathbf{y} \in B_r =$

$\{\mathbf{y} \in C_{1-\xi, \chi}^1([a, b]) : \mathbf{y} \in \mathcal{B}_{1-\xi, \chi} < r\}$ and $\gamma \in (a, b]$, we have

$$\begin{aligned} \Xi \mathbf{y}(\gamma) &= |T| \sum_{i=1}^n |\zeta_i| \left[\kappa \mathbf{y}(\gamma_i) + \frac{\rho, \chi}{a^+} \bar{h}(\gamma_i, \mathbf{y}(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(\gamma_i)) \right] \Psi^\xi(\gamma, a) \\ &\quad + \kappa \int_a^\gamma \chi'(s) \mathbf{y}(s) ds \\ &\quad + \frac{1}{\Gamma(\rho+1)} \int_a^\gamma \chi'(s) \Psi^\rho(\gamma, s) \bar{h}(s, \mathbf{y}(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(s)) ds, \\ &= |T| |\zeta^* n \Psi_*^\xi \left[\kappa r + \frac{\bar{h}^* \Psi_*^\rho}{\Gamma(\rho+1)} + \frac{r\alpha \Psi_*^\rho}{\Gamma(\rho+1)} + \frac{r\beta \Gamma(\xi) \Psi_*^\rho}{\Gamma(\rho+1)} \right] + \kappa r \Psi_*^1 \\ &\quad + \frac{\bar{h}^* \Psi_*^{\rho+1}}{\Gamma(\rho+2)} + \frac{r\alpha \Psi_*^{\rho+1}}{\Gamma(\rho+2)} + \frac{r\beta \Gamma(\xi) \Psi_*^{\rho+1}}{\Gamma(\rho+2)}. \end{aligned}$$

where $\bar{h}^* = \sup_{\gamma \in [a, b]} \bar{h}(\gamma, 0, 0)$, we also have, for each $\gamma \in (a, b]$

$$\begin{aligned} \Psi^{1-\xi}(\gamma, a) D^\chi \Xi \mathbf{y}(\gamma) &= \xi |T| \sum_{i=1}^n \zeta_i \left[\kappa \mathbf{y}(\gamma_i) + \frac{\rho, \chi}{a^+} \bar{h}(\gamma_i, \mathbf{y}(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(\gamma_i)) \right] \\ &\quad + \kappa \Psi^{1-\xi}(\gamma, a) \mathbf{y}(\gamma) \\ &\quad + \frac{\Psi^{1-\xi}(\gamma, a)}{\Gamma(\rho)} \int_a^\gamma \chi'(s) \Psi^{\rho-1}(\gamma, s) \bar{h}(s, \mathbf{y}(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(s)) ds \\ &= (\xi |T| |\zeta^* n + \Psi_*^{1-\xi}) \left[\kappa r + \frac{\bar{h}^* \Psi_*^\rho}{\Gamma(\rho+1)} + \frac{r\alpha \Psi_*^\rho}{\Gamma(\rho+1)} + \frac{r\beta \Gamma(\xi) \Psi_*^\rho}{\Gamma(\rho+1)} \right]. \end{aligned}$$

So,

$$\begin{aligned} \Xi \mathbf{y} \infty + D^\chi \Xi \mathbf{y} \in \xi, \chi &= \left(|T| |\zeta^* n (\xi + \Psi_*^\xi) + \Psi_*^{1-\xi} \right) \left[\kappa r + \frac{\bar{h}^* \Psi_*^\rho}{\Gamma(\rho+1)} + \frac{r\alpha \Psi_*^\rho}{\Gamma(\rho+1)} \right. \\ &\quad \left. + \frac{r\beta \Gamma(\xi) \Psi_*^\rho}{\Gamma(\rho+1)} \right] + \kappa r \Psi_*^1 + \frac{\bar{h}^* \Psi_*^{\rho+1}}{\Gamma(\rho+2)} + \frac{r\alpha \Psi_*^{\rho+1}}{\Gamma(\rho+2)} \\ &\quad + \frac{r\beta \Gamma(\xi) \Psi_*^{\rho+1}}{\Gamma(\rho+2)}. \end{aligned}$$

Now we will show that Ξ is continuous. Let $\{\mathbf{y}_n\}_{n \in \mathbb{N}} \subset \mathbf{y}$ in $C_{1-\xi, \chi}^1([a, b])$, from (\mathbf{H}_{1-1}) and Lemma 1.2.1 we can easily prove that $\Xi \mathbf{y}_n(\cdot) \rightarrow \Xi \mathbf{y}(\cdot)$ in $C([a, b])$ and $D^\chi \Xi \mathbf{y}_n(\cdot) \rightarrow D^\chi \Xi \mathbf{y}(\cdot)$ in $C_{1-\xi, \chi}([a, b])$, that implies $\Xi \mathbf{y}_n(\cdot) \rightarrow \Xi \mathbf{y}(\cdot)$ in $C_{1-\xi, \chi}^1([a, b])$, then Ξ is continuous.

Let us show the second condition (2), it is enough to show that $\Xi(B_r)$ (resp. $D^\chi \Xi(B_r)$) is equicontinuous in $C([a, b])$ (resp. in $C_{1-\xi, \chi}([a, b])$). Let $\mathbf{y} \in B_r$ and $\gamma_1, \gamma_2 \in (a, b]$ with $\gamma_1 < \gamma_2$, from

(\mathbf{H}_{1-1}), we have

$$\begin{aligned}
\Xi \mathbf{y}(\gamma_2) - \Xi \mathbf{y}(\gamma_1) & \left[\kappa r + \frac{\hbar^* \Psi_*^\rho}{\Gamma(\rho+1)} + \frac{r\alpha \Psi_*^\rho}{\Gamma(\rho+1)} + \frac{r\beta \Gamma(\xi) \Psi_*^\rho}{\Gamma(\rho+1)} \right] \\
& \times \left(\Psi^\xi(\gamma_2, a) - \Psi^\xi(\gamma_1, a) \right) \\
& + \frac{1}{\Gamma(\rho+1)} \int_a^{\gamma_1} \chi'(s) [\Psi^\rho(\gamma_2, s) - \Psi^\rho(\gamma_1, s)] \hbar(s, \mathbf{y}(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(s)) ds \\
& + \frac{1}{\Gamma(\rho+1)} \int_{\gamma_1}^{\gamma_2} \chi'(s) \Psi^\rho(\gamma_2, s) \hbar(s, \mathbf{y}(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(s)) ds + \kappa \int_{\gamma_1}^{\gamma_2} \chi'(s) \mathbf{y}(s) ds \\
& \left[\kappa r + \frac{\hbar^* \Psi_*^\rho}{\Gamma(\rho+1)} + \frac{r\alpha \Psi_*^\rho}{\Gamma(\rho+1)} + \frac{r\beta \Gamma(\xi) \Psi_*^\rho}{\Gamma(\rho+1)} \right] \left(\Psi^\xi(\gamma_2, a) - \Psi^\xi(\gamma_1, a) \right) \\
& + \frac{\hbar^* + r(\alpha + \beta \Gamma(\xi))}{\Gamma(\rho+2)} \left[\Psi^{\rho+1}(\gamma_2, a) - \Psi^{\rho+1}(\gamma_1, a) + \Psi^{\rho+1}(\gamma_2, \gamma_1) \right] \\
& + \frac{\hbar^* + r(\alpha + \beta \Gamma(\xi))}{\Gamma(\rho+2)} \Psi^{\rho+1}(\gamma_2, \gamma_1) + \kappa r \Psi^1(\gamma_2, \gamma_1).
\end{aligned}$$

As γ_2 tends to γ_1 , the right-hand side of the last inequality tends to 0. Therefore $\Xi(B_r)$ is equicontinuous in $\mathcal{C}([a, b])$.

And, we also have

$$\begin{aligned}
& \Psi^{1-\xi}(\gamma_2, a) D^\chi \Xi \mathbf{y}(\gamma_2) - \Psi^{1-\xi}(\gamma_1, a) D^\chi \Xi \mathbf{y}(\gamma_1) \\
& \kappa \Psi^{1-\xi}(\gamma_2, a) \mathbf{y}(\gamma_2) - \Psi^{1-\xi}(\gamma_1, a) \mathbf{y}(\gamma_1) \\
& + \left\| \frac{\Psi^{1-\xi}(\gamma_2, a)}{\Gamma(\rho)} \int_a^{\gamma_2} \chi'(s) \Psi^{\rho-1}(\gamma_2, s) \hbar(s, \mathbf{y}(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(s)) ds \right. \\
& \left. - \frac{\Psi^{1-\xi}(\gamma_1, a)}{\Gamma(\rho-1)} \int_a^{\gamma_1} \chi'(s) \Psi^\rho(\gamma_1, s) \hbar(s, \mathbf{y}(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(s)) ds \right\| \\
& \kappa \left(\Psi^{1-\xi}(\gamma_2, a) - \Psi^{1-\xi}(\gamma_1, a) \right) \mathbf{y}(\gamma_2) + \kappa \Psi^{1-\xi}(\gamma_1, a) \mathbf{y}(\gamma_2) - \mathbf{y}(\gamma_1) \\
& + \frac{\Psi^{1-\xi}(\gamma_1, a)}{\Gamma(\rho)} \int_a^{\gamma_1} \chi'(s) \left[\Psi^{\rho-1}(\gamma_1, s) - \Psi^{\rho-1}(\gamma_2, s) \right] \hbar(s, \mathbf{y}(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(s)) ds \\
& + \frac{\Psi^{1-\xi}(\gamma_2, a) - \Psi^{1-\xi}(\gamma_1, a)}{\Gamma(\alpha)} \int_a^{\gamma_1} \chi'(s) \Psi^{\rho-1}(\gamma_2, s) \hbar(s, \mathbf{y}(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(s)) ds \\
& + \frac{\Psi^{1-\xi}(\gamma_2, a)}{\Gamma(\rho)} \int_{\gamma_1}^{\gamma_2} \chi'(s) \Psi^{\rho-1}(\gamma_2, s) \hbar(s, \mathbf{y}(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(s)) ds \\
& \kappa \left(\Psi^{1-\xi}(\gamma_2, a) - \Psi^{1-\xi}(\gamma_1, a) \right) r + \frac{r\kappa \Psi_*^{1-\xi}}{\xi} \Psi^\xi(\gamma_2, \gamma_1) \\
& + \frac{\left(\hbar^* + r[\alpha + \beta \Gamma(\xi)] \right) \Psi^{1-\xi}(\gamma_1, a)}{\Gamma(\rho)} \int_a^{\gamma_1} \chi'(s) \left[\Psi^{\rho-1}(\gamma_1, s) - \Psi^{\rho-1}(\gamma_2, s) \right] ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{(\hbar^* + r[\alpha + \beta\Gamma(\xi)])(\Psi^{1-\xi}(\gamma_2, a) - \Psi^{1-\xi}(\gamma_1, a))}{\Gamma(\rho)} \int_a^{\gamma_1} \chi'(s) \Psi^{\rho-1}(\gamma_2, s) ds \\
& + \frac{(\hbar^* + r[\alpha + \beta\Gamma(\xi)])\Psi^{1-\xi}(\gamma_2, a)}{\Gamma(\rho)} \int_{\gamma_1}^{\gamma_2} \chi'(s) \Psi^{\rho-1}(\gamma_2, s) ds \\
& \kappa(\Psi^{1-\xi}(\gamma_2, a) - \Psi^{1-\xi}(\gamma_1, a))r + \frac{r\kappa\Psi_*^{1-\xi}}{\xi} \Psi^\xi(\gamma_2, \gamma_1) \\
& + \frac{(\hbar^* + r[\alpha + \beta\Gamma(\xi)])\Psi_*^{1-\xi}}{\Gamma(\rho+1)} [\Psi^\rho(\gamma_2, a) - \Psi^\rho(\gamma_1, a) + 2\Psi^\rho(\gamma_2, \gamma_1)] \\
& + \frac{(\hbar^* + r[\alpha + \beta\Gamma(\xi)])\Psi_*^\rho}{\Gamma(\rho)} [\Psi^{1-\xi}(\gamma_2, a) - \Psi^{1-\xi}(\gamma_1, a)].
\end{aligned}$$

By taking γ_2 tends to γ_1 , the right-hand side of the last inequality tends to 0, and hence $D^X \Xi(B_r)$ is equicontinuous in $C_{1-\xi, \chi}([a, b])$, thus, $\Xi(B_r)$ is equicontinuous in $C_{1-\xi, \chi}^1([a, b])$. \square

We denote by $\vartheta_C, \vartheta_\xi$ and ϑ_ξ^1 the Kuratowski noncompactness measure defined respectively on $C([a, b])$, $C_{1-\xi, \chi}([a, b])$ and $C_{1-\xi, \chi}^1([a, b])$.

Lemma 4.3.2. *Let B be a bounded subset of $C_{1-\xi, \chi}^1([a, b])$, we have*

$$\vartheta_\xi^1(B) = \vartheta(B) + \vartheta_\xi(D^X B) = 2\vartheta_\xi^1(B). \quad (4.3.1)$$

Proof. Let B be a bounded subset of $C_{1-\xi, \chi}^1([a, b])$ and let ϵ be a strictly positive real number. So, there exists a finite partition B_i , $i = 1, \dots, m$, such that

$$\text{Diam}_\xi^1(B_i) = \epsilon + \vartheta_\xi^1(B), \quad i = 1, \dots, m.$$

Then for all $\mathbf{y}_1, \mathbf{y}_2$ in B_i and $\gamma \in (a, b]$, we have

$$\|\mathbf{y}_2(\gamma) - \mathbf{y}_1(\gamma)\| = \epsilon + \vartheta_\xi^1(B) \text{ and } \|D^X \mathbf{y}_2(\gamma) - D^X \mathbf{y}_1(\gamma)\| = \epsilon + \vartheta_\xi^1(B), \quad i = 1, \dots, m.$$

So,

$$\text{Diam}(B_i) = \epsilon + \vartheta_\xi^1(B) \text{ and } \text{Diam}_\xi(D^X B_i) = \epsilon + \vartheta_\xi^1(B), \quad i = 1, \dots, m.$$

Thus,

$$\vartheta(B) + \vartheta_\xi(D^X B) = 2\epsilon + 2\vartheta_\xi^1(B).$$

Since ϵ is arbitrary, this means that we arrive at

$$\vartheta(B) + \vartheta_\xi(D^\chi B) \leq 2\vartheta_\xi^1(B). \quad (4.3.2)$$

Conversely, we want to prove that $\vartheta_\xi^1(B) \leq \vartheta(B) + \vartheta_\xi(D^\chi B)$, from the definition of Kuratowski non-compactness measure, we have, for each $\epsilon > 0$, there are a finite partitions $\{B_i\}_{i=1, \dots, m_1}$ of B and $\{D_j\}_{j=1, \dots, m_2}$ of $D^\chi B$ such that

$$\text{Diam}(B_i) \leq \epsilon + \vartheta(B), \quad \text{and} \quad \text{Diam}_\xi(D_j) \leq \epsilon + \vartheta_\xi(D^\chi B),$$

it is clear that the partition $\{B_i \cup_{a^+}^\chi D_j\}_{i,j}$ belongs to $C_{1-\xi, \chi}^1([a, b])$ and verifies the following inequality

$$\text{Diam}(B_i \cup_{a^+}^\chi D_j) + \text{Diam}_\xi(D^\chi(B_i \cup_{a^+}^\chi D_j)) \leq 2\epsilon + \vartheta(B) + \vartheta_\xi(D^\chi B).$$

As ϵ is arbitrary, we obtain

$$\vartheta_\xi^1(B) \leq \vartheta(B) + \vartheta_\xi(D^\chi B). \quad (4.3.3)$$

From (4.3.2)-(4.3.3), we get

$$\vartheta_\xi^1(B) \leq \vartheta(B) + \vartheta_\xi(D^\chi B) \leq 2\vartheta_\xi^1(B).$$

□

From Lemma 1.3.1 and Lemma 4.3.2, we easily show the following inequality

$$\vartheta_\xi^1(D) \leq \sup_{\gamma \in [a, b]} \vartheta(D(\gamma)) + \sup_{\gamma \in [a, b]} \vartheta(\Psi^{1-\xi}(\gamma, a)D^\chi D(\gamma)) \leq 2\vartheta_\xi^1(D), \quad (4.3.4)$$

where D is a bounded and equicontinuous subset of $C_{1-\xi, \chi}^1([a, b])$, $D(\gamma) = \{\mathbf{y} \in D\}$ and $D^\chi D(\gamma) = \{D^\chi \mathbf{y}(\gamma) : \mathbf{y} \in D\}$.

Let

$$B_R = \left\{ \mathbf{y} \in C_{1-\xi, \chi}^1([a, b]) : \mathbf{y} \in_{\xi, \chi}^1 B_R \right\}.$$

We are about to present our main result which is as follows.

Theorem 4.3.1. *Assume that the hypotheses $(\mathbf{H}_1) - (\mathbf{H}_2)$ are satisfied and that R verifies the following inequality*

$$\frac{1}{R} < \frac{\Gamma(\rho+2) - (\kappa\Gamma(\rho+2) + (\rho+1)A_0) \left(|T| |\zeta^* n(\Psi_*^\xi + \xi) + \Psi_*^{1-\xi}| \right)}{(\rho+1) \left(|T| |\zeta^* n(\Psi_*^\xi + \xi) + \Psi_*^{1-\xi}| \right) \Psi_*^\rho \bar{h}^* + \Psi_*^{\rho-1} \bar{h}^*} \quad (4.3.5)$$

$$- \frac{(A_0 + \kappa\Gamma(\rho+2)) \Psi_*^1}{(\rho+1) \left(|T| |\zeta^* n(\Psi_*^\xi + \xi) + \Psi_*^{1-\xi}| \right) \Psi_*^\rho \bar{h}^* + \Psi_*^{\rho+1} \bar{h}^*}$$

Then, the problem (4.1.1) – (4.1.2) has at least one solution in $C_{1-\xi, \chi}^{1, \xi}([a, b])$. In addition, the solution set \mathbf{SS} of the problem (4.1.1) – (4.1.2) is compact in $C_{1-\xi, \chi}^1([a, b])$.

Proof. From the definition of Ξ and Lemma 4.2.1, it is clear that the solutions of (4.1.1) – (4.1.2) is equivalent to the fixed point of Ξ . For this reason, we want to verify that Ξ satisfies the assumptions of Mönch fixed point theorem. First, we will prove that Ξ is well defined from B_R to B_R , indeed, let $\mathbf{y} \in B_R$. By using the condition (\mathbf{H}_{1-1}) and after some calculations, for each $\gamma \in (a, b]$ and $\mathbf{y} \in B_R$, we get

$$\begin{aligned} \Xi \mathbf{y}(\gamma) &= \Psi^{1-\xi}(\gamma, a) D^\chi \Xi \mathbf{y}(\gamma) - |T| \left| \sum_{i=1}^n \zeta_i \left[\kappa \mathbf{y}(\gamma_i) \right. \right. \\ &\quad \left. \left. + {}^{\rho, \chi} \bar{h}(\gamma_i, \mathbf{y}(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(\gamma_i)) \right] \Psi_*^\xi \right. \\ &\quad \left. + \kappa \int_a^\gamma \chi'(s) \mathbf{y}(s) ds + \frac{1}{\Gamma(\rho+1)} \int_a^\gamma \chi'(s) \Psi^\rho(\gamma, s) \bar{h}(s, \mathbf{y}(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(s)) ds, \right. \\ &\quad \left. + \xi |T| \left| \sum_{i=1}^n \zeta_i \left[\kappa \mathbf{y}(\gamma_i) + {}^{\rho, \chi} \bar{h}(\gamma_i, \mathbf{y}(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(\gamma_i)) \right] \right. \right. \\ &\quad \left. \left. + \frac{\Psi_*^{1-\xi}}{\Gamma(\rho)} \int_a^\gamma \chi'(s) \Psi^{\rho-1}(\gamma, s) \bar{h}(s, \mathbf{y}(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(s)) ds \right. \right. \\ &\quad \left. \left. \frac{(\rho+1) \left(|T| |\zeta^* n(\Psi_*^\xi + \xi) + \Psi_*^{1-\xi}| \right) \Psi_*^\rho \bar{h}^* + \Psi_*^{\rho+1} \bar{h}^*}{\Gamma(\rho+2)} \right. \right. \\ &\quad \left. \left. + \frac{(\kappa\Gamma(\rho+2) + (\rho+1)A_0) \left(|T| |\zeta^* n(\Psi_*^\xi + \xi) + \Psi_*^{1-\xi}| \right)}{\Gamma(\rho+2)} R \right. \right. \\ &\quad \left. \left. + \frac{(A_0 + \kappa\Gamma(\rho+2)) \Psi_*^1}{\Gamma(\rho+2)} R. \right. \right. \end{aligned}$$

From **(H₂)** and the inequality (4.3.5), we obtain

$$\mathbf{y} \in B_R : \Xi \mathbf{y} \in C_{1-\xi, \chi}^1 < R.$$

Note that B_R is bounded, convex and closed subset of $C_{1-\xi, \chi}^1([a, b])$ and Ξ is continuous on B_R . Next, it is enough to show the following implication

$$V \in \overline{\text{conv}}\{N(V) \setminus \{0\}\} \implies \vartheta_\xi^1(V) = 0, \text{ for any } V \in B_R.$$

Let V be a subset of B_R such that $V \in \overline{\text{conv}}\{N(V) \setminus \{0\}\}$. By using Definition 1.3.2 and Lemma 1.3.1, we obtain

$$\begin{aligned} & \vartheta(\Xi V(\gamma)) + \vartheta\left(\Psi^{1-\xi}(\gamma, a) D^\chi \Xi(V(\gamma))\right) - |T| \sum_{i=1}^n \zeta^* \left[\kappa \vartheta(V(\gamma_i)) + \right. \\ & \left. + \frac{\rho, \chi}{a^+} \vartheta\left(\tilde{h}\left(\gamma_i, V(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} V(\gamma_i)\right)\right)\right] \Psi_*^\xi \\ & + \kappa \int_a^\gamma \chi'(s) \vartheta(V(s)) ds + \frac{1}{\Gamma(\rho+1)} \int_a^\gamma \chi'(s) \Psi^\rho(\gamma, s) \vartheta\left(\tilde{h}\left(s, V(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} V(s)\right)\right) ds \\ & + \xi |T| \sum_{i=1}^n \zeta^* \left[\kappa \vartheta(V(\gamma_i)) + \frac{\rho, \chi}{a^+} \vartheta\left(\tilde{h}\left(\gamma_i, V(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} V(\gamma_i)\right)\right)\right] + \kappa \Psi_*^{1-\xi} \vartheta(V(\gamma)) \\ & + \frac{\Psi_*^{1-\xi}}{\Gamma(\rho)} \int_a^\gamma \chi'(s) \Psi^{\rho-1}(\gamma, s) \vartheta\left(\tilde{h}\left(s, V(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} V(s)\right)\right) ds. \end{aligned}$$

From 1.3.1, 4.3.1 and 4.3.2 and the hypotheses **(H₁₋₂)** – **(H₂)** and inequality (4.3.4), we arrive at

$$\begin{aligned} & \vartheta_\xi^1(\Xi V) \leq \sup_{\gamma \in [a, b]} \vartheta(\Xi V(\gamma)) + \sup_{\gamma \in [a, b]} \vartheta\left(\Psi^{1-\xi}(\gamma, a) D^\chi \Xi(V(\gamma))\right) \\ & \frac{2\left(\kappa \Gamma(\rho+2) + (\rho+1) A_0\right) \left(|T| \zeta^* n (\Psi_*^\xi + \xi) + \Psi_*^{1-\xi}\right)}{\Gamma(\rho+2)} \vartheta_\xi^1(\Xi V) \\ & + \frac{2\left(A_0 + \kappa \Gamma(\rho+2)\right) \Psi_*^1}{\Gamma(\rho+2)} \vartheta_\xi^1(\Xi V). \end{aligned}$$

By the condition **(H₂)**, we get $\vartheta_\xi^1(\Xi V) = 0$, that means $\vartheta_\xi^1(V) = 0$. From Theorem 1.4.4, the operator Ξ has at least one fixed point $\mathbf{y} \in B_R$. By using Lemma 4.2.1, we conclude that the problem (4.1.1) – (4.1.2) has at least one solution. Let us prove that solution set **SS** of (4.1.1) – (4.1.2) is included in $C_{1-\xi, \chi}^{1, \xi}([a, b])$, Let $w \in \{g \in C_{1-\xi, \chi}^1([a, b]) : \Xi g = g \text{ and } D^\chi \Xi g = D^\chi g\}$, we need to show that $D^\chi w$

$\mathcal{C}_{1-\xi, \chi}^\xi([a, b])$, so, for all $\gamma \in (a, b]$, we have

$$\begin{aligned} D^\chi w(\gamma) &= \xi T \sum_{i=1}^n \zeta_i \left[\kappa w(\gamma_i) + \frac{\rho, \chi}{a^+} \bar{h}(\gamma_i, w(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} w(\gamma_i)) \right] \Psi^{\xi-1}(\gamma, a) + \kappa w(\gamma) \\ &\quad + \frac{1}{\Gamma(\rho)} \int_a^\gamma \chi'(s) \Psi^{\rho-1}(\gamma, s) \bar{h}(s, w(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} w(s)) ds. \end{aligned}$$

By using ${}^{RL} \mathfrak{D}_{a^+}^\xi$ on both sides the last inequality, from Lemmas 1.2.1 and Theorem 1.2.4, we obtain

$$\begin{aligned} (1 - \kappa) {}^{RL} \mathfrak{D}_{a^+}^\xi D^\chi w(t) &= {}^{RL} \mathfrak{D}_{a^+}^\xi \frac{\rho, \chi}{a^+} \bar{h}(s, w(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} w(s)) \\ &= {}^{RL} \mathfrak{D}_{a^+}^{\theta(1-\rho)} \bar{h}(\gamma, w(\gamma), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} w(\gamma)). \end{aligned}$$

So, from (\mathbf{H}_1) , we have ${}^{RL} \mathfrak{D}_{a^+}^\xi D^\chi w(t) \in \mathcal{C}_{1-\xi}^\xi([a, b])$, that means $w \in \mathcal{C}_{1-\xi, \chi}^{\xi, 1}([a, b])$. Finally, the solution set \mathbf{SS} of problem (4.1.1) – (4.1.2) is included in $\mathcal{C}_{1-\xi, \chi}^{\xi, 1}([a, b])$.

We show now that the solution set \mathbf{SS} of the problem (4.1.1) – (4.1.2) is compact subset of $\mathcal{C}_{1-\xi, \chi}^{\xi, 1}([a, b])$. Let $\{\mathbf{y}_n\}_{n \in \mathbb{N}}$ be a sequence of the solution set, as $\mathcal{C}_{1-\xi, \chi}^{\xi, 1}([a, b])$ is compact space, there exists a subsequence of $\{\mathbf{y}_n\}_{n \in \mathbb{N}}$ (still denoted $\{\mathbf{y}_n\}_{n \in \mathbb{N}}$) converges to \mathbf{y}^* , it is enough to demonstrate that \mathbf{y}^* is a solution of (4.1.1) – (4.1.2), for each $\gamma \in (a, b]$, we have

$$\begin{aligned} \mathbf{y}_n(\gamma) &= T \sum_{i=1}^n \zeta_i \left[\kappa \mathbf{y}_n(\gamma_i) + \frac{\rho, \chi}{a^+} \bar{h}(\gamma_i, \mathbf{y}_n(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}_n(\gamma_i)) \right] \Psi^\xi(\gamma, a) \\ &\quad + \kappa \int_a^\gamma \chi'(s) \mathbf{y}_n(s) ds + \frac{1}{\Gamma(\rho+1)} \int_a^\gamma \chi'(s) \Psi^\rho(\gamma, s) \bar{h}(s, \mathbf{y}_n(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}_n(s)) ds. \end{aligned}$$

and

$$\begin{aligned} D^\chi \mathbf{y}_n(\gamma) &= \xi T \sum_{i=1}^n \zeta_i \left[\kappa \mathbf{y}_n(\gamma_i) + \frac{\rho, \chi}{a^+} \bar{h}(\gamma_i, \mathbf{y}_n(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}_n(\gamma_i)) \right] \Psi^{\xi-1}(\gamma, a) + \kappa \mathbf{y}_n(\gamma) \\ &\quad + \frac{1}{\Gamma(\rho)} \int_a^\gamma \chi'(s) \Psi^{\rho-1}(\gamma, s) \bar{h}(s, \mathbf{y}_n(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}_n(s)) ds. \end{aligned}$$

From (\mathbf{H}_1) , we have $\bar{h}(\cdot, \mathbf{y}_n(\cdot), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}_n(\cdot))$ converges to $\bar{h}(\cdot, \mathbf{y}^*(\cdot), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}^*(\cdot))$ as $n \rightarrow +\infty$, let $\gamma \in (a, b]$, from (\mathbf{H}_{1-1}) , for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \chi'(s) \Psi^\rho(\gamma, s) \bar{h}(s, \mathbf{y}_n(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}_n(s)) &\rightarrow (\bar{h}^* + (\alpha + \beta \Gamma(\xi)) M) \chi'(s) \Psi^\rho(\gamma, s) \text{ and} \\ \chi'(s) \Psi^{\rho-1}(\gamma, s) \bar{h}(s, \mathbf{y}_n(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}_n(s)) &\rightarrow (\bar{h}^* + (\alpha + \beta \Gamma(\xi)) M) \chi'(s) \Psi^{\rho-1}(\gamma, s). \end{aligned}$$

Using Lebesgue's dominated convergence theorem, for each $\gamma \in (a, b]$, we obtain

$$\begin{aligned} \mathbf{y}^*(\gamma) &= T \sum_{i=1}^n \zeta_i \left[\kappa \mathbf{y}^*(\gamma_i) + {}^{\rho, \chi} \mathfrak{h} \left(\gamma_i, \mathbf{y}^*(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}^*(\gamma_i) \right) \right] \Psi^\xi(\gamma, a) \\ &\quad + \kappa \int_a^\gamma \chi'(s) \mathbf{y}^*(s) ds + \frac{1}{\Gamma(\rho+1)} \int_a^\gamma \chi'(s) \Psi^\rho(\gamma, s) \mathfrak{h} \left(s, \mathbf{y}^*(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}^*(s) \right) ds. \end{aligned}$$

and

$$\begin{aligned} D^\chi \mathbf{y}^*(\gamma) &= \xi T \sum_{i=1}^n \zeta_i \left[\kappa \mathbf{y}^*(\gamma_i) + {}^{\rho, \chi} \mathfrak{h} \left(\gamma_i, \mathbf{y}^*(\gamma_i), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}^*(\gamma_i) \right) \right] \Psi^{\xi-1}(\gamma, a) + \kappa \mathbf{y}^*(\xi) \\ &\quad + \frac{1}{\Gamma(\rho)} \int_a^\gamma \chi'(s) \Psi^{\rho-1}(\gamma, s) \mathfrak{h} \left(s, \mathbf{y}^*(s), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}^*(s) \right) ds. \end{aligned}$$

So, the solution set of Problem (4.1.1) – (4.1.2) is compact subset of $C_{1-\xi, \chi}^1([a, b])$. \square

4.4 Example

We take $\Psi(t) = \frac{4 \arctan t}{10\pi}$, $a = 0, \gamma_1 = 0.5, b = 1, \theta = \rho = 0.25, \kappa = \frac{1}{40}$, E the Banach space defined by

$$E = \left\{ (\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n, \dots) : \sup_n \|\mathbf{y}_n\| < \infty \right\},$$

with the norm $\|\mathbf{y}\| = \sup_n \|\mathbf{y}_n\|$, we define the function $\mathfrak{h} : (0, 1] \times E^2 \rightarrow E$ by

$$\mathfrak{h} \left(\gamma, \mathbf{y}(\gamma), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}(\gamma) \right) = \left(\mathfrak{h}_1 \left(\gamma, \mathbf{y}_1(\gamma), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}_1(\gamma) \right), \dots, \mathfrak{h}_n \left(\gamma, \mathbf{y}_n(\gamma), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}_n(\gamma) \right), \dots \right),$$

where

$$\mathfrak{h}_n \left(\gamma, \mathbf{y}_n(\gamma), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}_n(\gamma) \right) = \frac{{}^C \mathfrak{D}_{a^+}^{\xi, \chi} \mathbf{y}_n(\gamma)}{40 + nt^2} + \frac{\mathbf{y}_n(\gamma)}{40 + t^n}, \quad \gamma \in (0, 1].$$

We easily see that $\mathfrak{h} : (0, 1] \times E^2 \rightarrow E$ is continuous and

$$\|\mathfrak{h}(\gamma, g, v) - \mathfrak{h}(\gamma, \bar{g}, \bar{v})\| \leq \frac{1}{40} \|g - \bar{g}\| + \frac{1}{40} \|v - \bar{v}\|, \quad \text{for all } \gamma \in (0, 1] \text{ and } g, v, \bar{g}, \bar{v} \in E.$$

Next, For all Ω a bounded subset of $C_{1-\xi, \chi}^1([0, 1])$, we have

$$\vartheta \left(\mathfrak{h} \left(\gamma, \Omega(\gamma), {}^C \mathfrak{D}_{a^+}^{\xi, \chi} \Omega(\gamma) \right) \right) \leq \frac{1}{40} \left(\vartheta(\Omega(\gamma)) + \vartheta({}^C \mathfrak{D}_{a^+}^{\xi, \chi} \Omega(\gamma)) \right), \quad \gamma \in (0, 1].$$

So, (\mathbf{H}_1) , (\mathbf{H}_{1-1}) and (\mathbf{H}_{1-2}) are satisfied. A quick calculation gives us

$$\left(\kappa \Gamma(\rho+2) + (\rho+1)A_0 \right) \left(\|T\| \zeta^* n (\Psi_*^\xi + \xi) + \Psi_*^{1-\xi} \right) + \left(A_0 + \kappa \Gamma(\rho+2) \right) \Psi_*^1 < \frac{\Gamma(\rho+2)}{2}.$$

So, (\mathbf{H}_2) holds. Therefore, Theorem 4.3.1 ensures that the solution set of Problem (4.1.1)-(4.1.2) is nonempty and compact.

Chapter 5

On the existence of χ -differentiable solutions for Sequential differential system involving a mixed derivative in Banach space

5.1 Introduction

We consider the following system presented in [10]:

$$(S) \quad \begin{cases} {}^{RL}\mathfrak{D}_{a+}^{\rho_1, \chi} D^\chi \mathbf{y}_1(\gamma) = \tilde{h}_1(\gamma, \mathbf{y}_1(\gamma), \mathbf{y}_2(\gamma), {}^C\mathfrak{D}_{a+}^{\theta_1, \chi} \mathbf{y}_1(\gamma), {}^C\mathfrak{D}_{a+}^{\theta_2, \chi} \mathbf{y}_2(\gamma)), \\ {}^{RL}\mathfrak{D}_{a+}^{\rho_2, \chi} D^\chi \mathbf{y}_2(\gamma) = \tilde{h}_2(\gamma, \mathbf{y}_1(\gamma), \mathbf{y}_2(\gamma), {}^C\mathfrak{D}_{a+}^{\theta_1, \chi} \mathbf{y}_1(\gamma), {}^C\mathfrak{D}_{a+}^{\theta_2, \chi} \mathbf{y}_2(\gamma)), \quad \gamma \in (a, b], \end{cases}$$

associated with the following nonlocal integral boundary conditions

$$(NIB) \quad \begin{cases} {}^{1-\rho_1, \chi} D^\chi \mathbf{y}_1(a+) = \sum_{j=1}^{n_1} \zeta_{1j} D^\chi \mathbf{y}_1(\gamma_{1j}), & \mathbf{y}_1(a) = \Lambda_1, \\ {}^{1-\rho_2, \chi} D^\chi \mathbf{y}_2(a+) = \sum_{k=1}^{n_2} \zeta_{2k} D^\chi \mathbf{y}_2(\gamma_{2k}), & \mathbf{y}_2(a) = \Lambda_2, \end{cases}$$

where ${}^{RL}\mathfrak{D}_{a+}^{\rho_i, \chi}$ denotes the χ -Riemann-Liouville fractional derivative of order $0 < \rho_i < 1$, $i = 1, 2$, ${}^C\mathfrak{D}_{a+}^{\rho_i, \chi}$ is the χ -Caputo fractional derivative of order ρ_i , $i = 1, 2$, E is a Banach space, $\tilde{h}_i : (a, b] \times E^2$

E , $i = 1, 2$ are functions satisfying some specified conditions, $\chi \in C^1([a, b], \mathbb{R}^+)$ satisfied $\chi'(\gamma) > 0$, for all $\gamma \in [a, b]$, $D^\chi = \frac{1}{\chi'(\gamma)} \frac{d}{d\gamma}$, $a, b \in \mathbb{R}_*^+$ with $a < b$ and $\gamma_{1j}, \gamma_{2k} \in (a, b), j = 1, \dots, n_1, k = 1, \dots, n_2$ with $\Gamma(\rho_i) = \sum_{k=1}^{n_i} \zeta_{ik} (\chi(\gamma) - \chi(a))^{\rho_i - 1}, i = 1, 2$.

Note that in our problem, the derivative we took is a composition of fractional and ordinary derivatives, it is clear that the relation ${}^{RL}\mathfrak{D}_{a^+}^{\rho_i, \chi} D^\chi \mathbf{y}_i = {}^{RL}\mathfrak{D}_{a^+}^{\rho_i + 1, \chi} \mathbf{y}_i$ is not correct except in the case $\mathbf{y}_i(a) = 0$. This is the second motivation to consider the **(S) – (NIB)** problem involving a mixed derivatives.

In order to start, we will give some concepts and notations about the functional spaces, fractional calculus, noncompactness measure which are used throughout this chapter. we denote by $\mathcal{C}([a, b])$ (resp. by $L^1([a, b])$) the space of E -valued continuous functions (resp. the space of E -Bochner's integrable functions) with the following norm

$$\|g\|_\infty = \sup \left\{ \|g(\gamma)\|, \gamma \in [a, b] \right\} \quad \left(\text{resp. } \|g\|_{L^1} = \int_a^b \|g(\gamma)\| d\gamma \right).$$

Let $\mathcal{C}_{1-\rho_i}([a, b])$ be the Banach spaces of functions from $(a, b]$ into E which is defined as:

$$\mathcal{C}_{1-\rho_i, \chi}([a, b]) = \left\{ g \in \mathcal{C}((a, b]) : (\chi(\cdot) - \chi(a))^{1-\rho_i} g(\cdot) \in \mathcal{C}([a, b], E) \right\}, i = 1, 2.$$

with his norm $\|g\|_{\rho_i, \chi}$, that is given by

$$\|g\|_{\rho_i, \chi} = \sup_{\gamma \in (a, b]} (\chi(\gamma) - \chi(a))^{1-\rho_i} \|g(\gamma)\|, i = 1, 2.$$

Next, we denote by $\mathcal{C}_{1-\rho_i, \chi}^1((a, b])$ the space of functions χ -continuously differentiable defined as follows

$$\mathcal{C}_{1-\rho_i, \chi}^1([a, b]) = \left\{ g : (a, b] \rightarrow E : g(\cdot) \in \mathcal{C}([a, b]) \text{ and } D^\chi g(\cdot) \in \mathcal{C}_{1-\rho_i, \chi}([a, b]) \right\}, i = 1, 2.$$

with the norm

$$\|g\|_{\rho_i, \chi}^1 = \|g\|_\infty + \|D^\chi g\|_{\rho_i, \chi}, i = 1, 2.$$

Let $\prod_{i=1}^2 \mathcal{C}_{1-\rho_i, \chi}^1([a, b])$ be the product space (will be denoted in all that follows by $\Pi_\chi^1([a, b])$), which is a Banach space with the following norm

$$\|(g_1, g_2)\|_\chi^1 = \max \left\{ \|g_1\|_{\rho_1, \chi}^1, \|g_2\|_{\rho_2, \chi}^1 \right\}.$$

In the following, for all $\eta > -1$, we pose $\Psi^\eta(r, s) = (\chi(r) - \chi(s))^\eta$, for all $s, r \in [a, b]$ with $r > s$ and for all $\eta > 0$, we put $\Psi_*^\eta = (\chi(b) - \chi(a))^\eta$.

5.2 Integral Equation

In the following Lemma, we establish the equivalence between the problem at hand **(S)** – **(NIB)** and the following system of integral equations

$$(\mathbf{S}^*) \quad \begin{cases} \mathbf{y}_1(\gamma) = \Lambda_1 + \frac{\sum_{j=1}^{n_1} \zeta_{1j} \mathfrak{S}_{a^+}^{\rho_1} \bar{h}_1(\gamma_{1j}, \mathbf{y}_1(\gamma_{1j}), \mathbf{y}_2(\gamma_{1j}), {}^C \mathfrak{D}_{a^+}^{\rho_1, \chi} \mathbf{y}_1(\gamma_{1j}), {}^C \mathfrak{D}_{a^+}^{\rho_2, \chi} \mathbf{y}_2(\gamma_{1j}))}{(\rho_1 + 1) - \rho_1 \sum_{j=1}^{n_1} \zeta_{1j} \rho_1^{-1}(\gamma_{1j}, a)} \Psi^{\rho_1}(\gamma, a) \\ + \frac{1}{(\rho_1 + 1)} \int_a^\gamma \chi'(s) \Psi^{\rho_1}(\gamma, s) \bar{h}_1(s, \mathbf{y}_1(s), \mathbf{y}_2(s), {}^C \mathfrak{D}_{a^+}^{\rho_1, \chi} \mathbf{y}_1(s), {}^C \mathfrak{D}_{a^+}^{\rho_2, \chi} \mathbf{y}_2(s)) ds, \\ \mathbf{y}_2(\gamma) = \Lambda_2 + \frac{\sum_{k=1}^{n_2} \zeta_{2k} \mathfrak{S}_{a^+}^{\rho_2} \bar{h}_2(\gamma_{2k}, \mathbf{y}_1(\gamma_{2k}), \mathbf{y}_2(\gamma_{2k}), {}^C \mathfrak{D}_{a^+}^{\rho_1, \chi} \mathbf{y}_1(\gamma_{2k}), {}^C \mathfrak{D}_{a^+}^{\rho_2, \chi} \mathbf{y}_2(\gamma_{2k}))}{(\rho_2 + 1) - \rho_2 \sum_{k=1}^{n_2} \zeta_{2k} \rho_2^{-1}(\gamma_{2k}, a)} \Psi^{\rho_2}(\gamma, a) \\ + \frac{1}{(\rho_2 + 1)} \int_a^\gamma \chi'(s) \Psi^{\rho_2}(\gamma, s) \bar{h}_2(s, \mathbf{y}_1(s), \mathbf{y}_2(s), {}^C \mathfrak{D}_{a^+}^{\rho_1, \chi} \mathbf{y}_1(s), {}^C \mathfrak{D}_{a^+}^{\rho_2, \chi} \mathbf{y}_2(s)) ds. \end{cases}$$

In all that follows, we put

$$i(\gamma, Y(\gamma)) = \bar{h}_i(\gamma, \mathbf{y}_1(\gamma), \mathbf{y}_2(\gamma), {}^C \mathfrak{D}_{a^+}^{\rho_1, \chi} \mathbf{y}_1(\gamma), {}^C \mathfrak{D}_{a^+}^{\rho_2, \chi} \mathbf{y}_2(\gamma)), \quad i = 1, 2,$$

where $Y(\cdot) = (\mathbf{y}_1(\cdot), \mathbf{y}_2(\cdot))$.

Lemma 5.2.1. *Let $0 < \rho_1, \rho_2 < 1$, we assume that $i(\cdot, Y(\cdot)) \in C([a, b])$, $i = 1, 2$, for all $(\mathbf{y}_1, \mathbf{y}_2) \in \Pi_\chi^1([a, b])$. Then, $(\mathbf{y}_1, \mathbf{y}_2)$ is a solution of the system **(S)** – **(NIB)** if and only if $(\mathbf{y}_1, \mathbf{y}_2)$ satisfies the system of integral equations **(S*)**.*

Proof. Let $(\mathbf{y}_1, \mathbf{y}_2) \in \Pi_\chi^1([a, b])$ be a solution of the system **(S)** – **(NIB)**, we want to prove that $(\mathbf{y}_1, \mathbf{y}_2)$ is a solution of **(S*)**. From Definition of $\prod_{i=1}^2 C_{1-\rho_i, \chi}^1([a, b])$, Lemma 1.2.3 and Definition of ${}_{a^+}^{1-\rho_i, \chi}$, we have ${}_{a^+}^{1-\rho_i, \chi} D^\chi \mathbf{y}_i(\cdot) \in C([a, b])$, $i = 1, 2$ and $\frac{d}{\chi'(\gamma) d\gamma} \left({}_{a^+}^{1-\rho_i, \chi} D^\chi \mathbf{y}_i(\gamma) \right) = {}^{RL} \mathfrak{D}_{a^+}^{\rho_i} D^\chi \mathbf{y}_i(\gamma) = i(\gamma, Y(\gamma)) \in C([a, b]) \in C_{1-\rho_i, \chi}([a, b])$, $i = 1, 2$.

From Theorem (1.2.1) we have

$$D^\chi \mathbf{y}_i(\gamma) = \frac{{}_{a^+}^{1-\rho_i, \chi} D^\chi \mathbf{y}_i(a^+)}{\Gamma(\rho_i)} \Psi^{\rho_i-1}(\gamma, a) + {}_{a^+}^{\rho_i, \chi} i(\gamma, Y(\gamma)), \quad i = 1, 2. \quad (5.2.1)$$

Next, we substitute γ by γ_{1j}, γ_{2k} into (5.2.1), we get

$$\begin{cases} D^\chi \mathbf{y}_1(\gamma_{1j}) = \frac{{}_{a^+}^{1-\rho_1, \chi} D^\chi \mathbf{y}_1(a^+)}{(\rho_1)} \Psi^{\rho_1-1}(\gamma_{1j}, a) + {}_{a^+}^{\rho_1, \chi} 1(\gamma_{1j}, Y(\gamma_{1j})), \quad j = 1, \dots, n_1, \\ D^\chi \mathbf{y}_2(\gamma_{2k}) = \frac{{}_{a^+}^{1-\rho_2, \chi} D^\chi \mathbf{y}_2(a^+)}{(\rho_2)} \Psi^{\rho_2-1}(\gamma_{2k}, a) + {}_{a^+}^{\rho_2, \chi} 2(\gamma_{2k}, Y(\gamma_{2k})), \quad k = 1, \dots, n_2. \end{cases}$$

By utilizing the second condition **(NIB)**, we obtain

$$\begin{cases} {}_{a+}^{1-\rho_1, \chi} D^\chi \mathbf{y}_1(a+) = \frac{(\rho_1) \sum_{j=1}^{n_1} \zeta_{1j} \mathfrak{S}_{a+}^{\rho_1, \chi} \aleph_1(\gamma_{1j}, Y(\gamma_{1j}))}{(\rho_1) - \sum_{j=1}^{n_1} \zeta_{1j} \rho_1^{-1}(\gamma_{1j}, a)}, \\ {}_{a+}^{1-\rho_2, \chi} D^\chi \mathbf{y}_2(a+) = \frac{(\rho_2) \sum_{k=1}^{n_2} \zeta_{2k} \mathfrak{S}_{a+}^{\rho_2, \chi} \aleph_2(\gamma_{2k}, Y(\gamma_{2k}))}{(\rho_2) - \sum_{k=1}^{n_2} \zeta_{2k} \rho_2^{-1}(\gamma_{2k}, a)}. \end{cases}$$

By substituting, we deduce that

$$\begin{cases} D^\chi \mathbf{y}_1(\gamma) = \frac{\sum_{j=1}^{n_1} \zeta_{1j} \mathfrak{S}_{a+}^{\rho_1, \chi} \aleph_1(\gamma_{1j}, Y(\gamma_{1j}))}{(\rho_1) - \sum_{j=1}^{n_1} \zeta_{1j} \rho_1^{-1}(\gamma_{1j}, a)} \Psi^{\rho_1-1}(\gamma, a) + {}_{a+}^{\rho_1, \chi} \mathbf{1}(\gamma, Y(\gamma)), \\ D^\chi \mathbf{y}_2(\gamma) = \frac{\sum_{k=1}^{n_2} \zeta_{2k} \mathfrak{S}_{a+}^{\rho_2, \chi} \aleph_2(\gamma_{2k}, Y(\gamma_{2k}))}{(\rho_2) - \sum_{k=1}^{n_2} \zeta_{2k} \rho_2^{-1}(\gamma_{2k}, a)} \Psi^{\rho_2-1}(\gamma, a) + {}_{a+}^{\rho_2, \chi} \mathbf{2}(\gamma, Y(\gamma)). \end{cases} \quad (5.2.2)$$

Next, applying I_{a+}^χ to both sides of each equation of (5.2.2), we obtain

$$\begin{cases} \mathbf{y}_1(\gamma) = \Lambda_1 + \frac{\left(\sum_{j=1}^{n_1} \zeta_{1j} \mathfrak{S}_{a+}^{\rho_1} \aleph_1(\gamma_{1j}, Y(\gamma_{1j})) \right) \rho_1(\gamma_{1j}, a)}{(\rho_1+1) - \rho_1 \sum_{j=1}^{n_1} \zeta_{1j} \rho_1^{-1}(\gamma_{1j}, a)} + \frac{1}{(\rho_1+1)} \int_a^\gamma \chi'(s) \Psi^{\rho_1}(\gamma, s) \mathbf{1}(s, Y(s)) ds, \\ \mathbf{y}_2(\gamma) = \Lambda_2 + \frac{\left(\sum_{k=1}^{n_2} \zeta_{2k} \mathfrak{S}_{a+}^{\rho_2} \aleph_2(\gamma_{2k}, Y(\gamma_{2k})) \right) \rho_2(\gamma_{2k}, a)}{(\rho_2+1) - \rho_2 \sum_{k=1}^{n_2} \zeta_{2k} \rho_2^{-1}(\gamma_{2k}, a)} + \frac{1}{(\rho_2+1)} \int_a^\gamma \chi'(s) \Psi^{\rho_2}(\gamma, s) \mathbf{2}(s, Y(s)) ds. \end{cases}$$

Conversely, let $(\mathbf{y}_1, \mathbf{y}_2) \in \Pi_\chi^1([a, b])$ be a solution of the system **(S*)**, it is clear that $\mathbf{y}_1(a) = \Lambda_1$ and $\mathbf{y}_2(a) = \Lambda_2$. By applying D_{a+}^χ to both sides of each equation of **(S*)**, we obtain the system of equations (5.2.2), applying ${}_{a+}^{1-\rho_i, \chi}$ to both sides of the equation

$$D^\chi \mathbf{y}_i(\gamma) = \frac{\sum_{j=1}^{n_i} \zeta_{ij} \mathfrak{S}_{a+}^{\rho_i, \chi} \aleph_i(\gamma_{ij}, Y(\gamma_{ij}))}{\Gamma(\rho_i) - \sum_{j=1}^{n_i} \zeta_{ij} \Psi^{\rho_i-1}(\gamma_{ij}, a)} \Psi^{\rho_i-1}(\gamma, a) + {}_{a+}^{\rho_i, \chi} \mathbf{i}(\gamma, Y(\gamma)), \quad i = 1, 2, \quad (5.2.3)$$

and utilizing Lemma 1.2.1, we get

$${}_{a+}^{1-\rho_i, \chi} D^\chi \mathbf{y}_i(\gamma) = \frac{\Gamma(\rho_i) \sum_{k=1}^{n_i} \zeta_{ik} \mathfrak{S}_{a+}^{\rho_i, \chi} \aleph_i(\gamma_{ik}, Y(\gamma_{ik}))}{\Gamma(\rho_i) - \sum_{k=1}^{n_i} \zeta_{ik} \Psi^{\rho_i-1}(\gamma_{ik}, a)} + {}_{a+}^{1, \chi} \mathbf{i}(\gamma, Y(\gamma)), \quad i = 1, 2$$

Taking $\gamma = a^+$, we get

$${}_{a+}^{1-\rho_i, \chi} D^\chi \mathbf{y}_i(a+) = \frac{\Gamma(\rho_i) \sum_{k=1}^{n_i} \zeta_{ik} \mathfrak{S}_{a+}^{\rho_i, \chi} \aleph_i(\gamma_{ik}, Y(\gamma_{ik}))}{\Gamma(\rho_i) - \sum_{k=1}^{n_i} \zeta_{ik} \Psi^{\rho_i-1}(\gamma_{ik}, a)}, \quad i = 1, 2. \quad (5.2.4)$$

Substituting and adding side to side in the equation (5.2.3), we find,

$$\sum_{j=1}^{n_i} \zeta_{ij} D^\chi \mathbf{y}_i(\gamma_j) = \frac{\sum_{j=1}^{n_i} \zeta_{ij} \mathfrak{S}_{a+}^{\rho_i, \chi} \aleph_i(\gamma_{ij}, Y(\gamma_{ij}))}{\Gamma(\rho_i) - \sum_{j=1}^{n_i} \zeta_{ij} \Psi^{\rho_i-1}(\gamma_{ij}, a)} \sum_{j=1}^{n_i} \zeta_{ij} \Psi^{\rho_i-1}(\gamma_{ij}, a) + \sum_{j=1}^{n_i} \zeta_{ij} \mathfrak{S}_{a+}^{\rho_i, \chi} \aleph_i(\gamma_{ij}, Y(\gamma_{ij})), \quad i = 1, 2. \quad (5.2.5)$$

From (5.2.4) and (5.2.5), we have

$${}_{a+}^{1-\rho_i, \chi} D^\chi \mathbf{y}_i(a+) = \sum_{j=1}^{n_i} \zeta_{ij} D^\chi \mathbf{y}_i(\gamma_j), \quad i = 1, 2.$$

□

5.3 Main Results

We prove the existence of χ -differential solutions of the system **(S)** – **(NIB)**, also the compactness of its solution set. We necessarily assume the following hypotheses

(H₁) Suppose that $\varphi_i(\cdot, Y(\cdot)) \in C([a, b])$, for all $(\mathbf{y}_1, \mathbf{y}_2) \in \Pi_\chi^1([a, b])$, and there exists $\alpha_i, \beta_i, \xi_i, \delta_i \in \mathbb{R}_+$, $i = 1, 2$, such that

(H₁₋₁) For all $(\mathbf{y}_1, \mathbf{y}_2), (\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2) \in \Pi_\chi^1([a, b])$, $\gamma \in [a, b]$:

$$\begin{aligned} & \varphi_i(\gamma, Y(\gamma)) - \varphi_i(\gamma, \bar{Y}(\gamma)) \leq \alpha_i \|\mathbf{y}_1(\gamma) - \bar{\mathbf{y}}_1(\gamma)\| + \beta_i \|\mathbf{y}_2(\gamma) - \bar{\mathbf{y}}_2(\gamma)\| \\ & + \xi_i {}^C \mathfrak{D}^{\rho_1, \chi}(\mathbf{y}_1(\gamma) - \bar{\mathbf{y}}_1(\gamma)) + \delta_i {}^C \mathfrak{D}^{\rho_2, \chi}(\mathbf{y}_2(\gamma) - \bar{\mathbf{y}}_2(\gamma)), \quad i = 1, 2. \end{aligned}$$

(H₁₋₂) For each nonempty, bounded set $\Omega_i \in \mathcal{C}_{1-\rho_i, \chi}^1([a, b])$, for all $\gamma \in (a, b]$, we have

$$\begin{aligned} & \vartheta \left(\varphi_i(\gamma, Y(\gamma)) \right) \leq \alpha \vartheta \left(\Omega_1(\gamma) \right) + \beta_i \vartheta \left(\Omega_2(\gamma) \right) \\ & + \xi_i \vartheta \left({}^C \mathfrak{D}_{a^+}^{\rho_1, \chi} \Omega_1(\gamma) \right) + \delta_i \vartheta \left({}^C \mathfrak{D}_{a^+}^{\rho_2, \chi} \Omega_2(\gamma) \right), \quad i = 1, 2. \end{aligned}$$

where

$$\begin{aligned} \Omega_i(\gamma) &= \left\{ \mathbf{y}_i(\gamma), \mathbf{y}_i \in \mathcal{C}_{1-\rho_i, \chi}^1([a, b]) \right\} \text{ and} \\ {}^C \mathfrak{D}_{a^+}^{\rho_i, \chi} \Omega_i(\gamma) &= \left\{ {}^C \mathfrak{D}_{a^+}^{\rho_i, \chi} \mathbf{y}_i(\gamma), \mathbf{y}_i \in \mathcal{C}_{1-\rho_i, \chi}^1([a, b]) \right\}, \quad i = 1, 2. \end{aligned}$$

(H₂)

$$\left[1 + (\rho_i + 1) \left(T \zeta_i^* n_i (\Psi_*^{\rho_i} + \rho_i) + \Psi_*^{1-\rho_i} \right) \right] A_0 < \frac{\Gamma(\rho_i + 2)}{2}, \quad i = 1, 2.$$

where

$$\begin{aligned} T &= \max \left\{ \frac{1}{|\Gamma(\rho_i + 1) - \rho_i \sum_{j=1}^{n_i} \zeta_j \Psi^{\rho_i-1}(\gamma_{ij}, a)|}, i = 1, 2 \right\}, \\ A_0 &= \max \left\{ \left(\alpha_i + \beta_i + \xi_i \Gamma(\rho_1) + \delta_i \Gamma(\rho_2) \right) \Psi_*^{\rho_i}, i = 1, 2 \right\} \text{ and } \zeta_i^* = \max_{j=1, \dots, n_i} \{ \zeta_{ij} \}. \end{aligned}$$

Define the operator $\Xi : \Pi_\chi^1([a, b]) \rightarrow \Pi_\chi^1([a, b])$ by

$$\Xi(\mathbf{y}_1, \mathbf{y}_2) = \begin{cases} \Xi_1(\mathbf{y}_1, \mathbf{y}_2), \\ \Xi_2(\mathbf{y}_1, \mathbf{y}_2), \end{cases}$$

where, for $i = 1, 2$, we have

$$\Xi_i(\mathbf{y}_1, \mathbf{y}_2)(\gamma) = \Lambda_i + \frac{\Psi^{\rho_i}(\gamma, a) \sum_{j=1}^{n_i} \zeta_{ij} \frac{\rho_i}{a^+} i(\gamma_{ij}, Y(\gamma_{ij}))}{\Gamma(\rho_i + 1) - \rho_i \sum_{j=1}^{n_i} \zeta_{ij} \Psi^{\rho_i-1}(\gamma_{ij}, a)} + \frac{1}{\Gamma(\rho_i + 1)} \int_a^\gamma \chi'(s) \Psi^{\rho_i}(\gamma, s) i(s, Y(s)) ds.$$

We note that, for all $\gamma \in (a, b]$ we have

$$D^X \Xi(\mathbf{y}_1, \mathbf{y}_2)(\gamma) = \begin{cases} D^X \Xi_1(\mathbf{y}_1, \mathbf{y}_2)(\gamma), \\ D^X \Xi_2(\mathbf{y}_1, \mathbf{y}_2)(\gamma), \end{cases}$$

where, for $i = 1, 2$, we have

$$D^X \Xi_i(\mathbf{y}_1, \mathbf{y}_2)(\gamma) = \frac{\Psi^{\rho_i-1}(\gamma, a) \sum_{j=1}^{n_i} \zeta_{ij} \frac{\rho_i}{a^+} i(\gamma_{ij}, Y(\gamma_{ij}))}{\Gamma(\rho_i) - \sum_{j=1}^{n_i} \zeta_{ij} \Psi^{\rho_i-1}(\gamma_{ij}, a)} + \frac{1}{\Gamma(\rho_i)} \int_a^\gamma \chi'(s) \Psi^{\rho_i-1}(\gamma, s) i(s, Y(s)) ds.$$

5.3.1 Existence results

We present the existence result for χ -continuously differentiable solutions of the problem **(S)** – **(NIB)**. We begin by introducing several useful lemmas that will be used to establish this result.

Lemma 5.3.1. *We assume the hypotheses **(H₁)** and **(H₁₋₁)** are hold. Then*

(1) Ξ is bounded and continuous.

(2) $\Xi(B)$ is equicontinuous for all bounded subset B of $\Pi_\chi^1([a, b])$.

Proof. Let us show the axiom (1), we begin to prove that Ξ is bounded operator. Let $(\mathbf{y}_1, \mathbf{y}_2) \in \Pi_\chi^1([a, b], \gamma \in [a, b])$, it is clear to see that $\Xi(\mathbf{y}_1, \mathbf{y}_2) \in \Pi_\chi^1([a, b], \gamma \in [a, b])$. Using **(H₁)** and **(H₁₋₁)**, for all $\mathbf{y} \in B_r = \{(\mathbf{y}_1, \mathbf{y}_2) \in \Pi_\chi^1([a, b]) : (\mathbf{y}_1, \mathbf{y}_2) \frac{1}{\chi} < r\}$, $i = 1, 2$ and $\gamma \in (a, b]$, we have

$$\begin{aligned} \Xi_i(\mathbf{y}_1, \mathbf{y}_2)(\gamma) &= \Lambda_i + T \frac{\Psi^{\rho_i}(\gamma, a) \sum_{j=1}^{n_i} |\zeta_{ij}| \frac{\rho_i}{a^+} i(\gamma_{ij}, Y(\gamma_{ij}))}{\Gamma(\rho_i + 1) - \rho_i \sum_{j=1}^{n_i} \zeta_{ij} \Psi^{\rho_i-1}(\gamma_{ij}, a)} + \frac{1}{\Gamma(\rho_i + 1)} \int_a^\gamma \chi'(s) \Psi^{\rho_i}(\gamma, s) i(s, Y(s)) ds \\ &= \Lambda_i + T \zeta_i^* n_i \Psi_*^{\rho_i} \left[\frac{\bar{h}_i^* \Psi_*^{\rho_i}}{\Gamma(\rho_i + 1)} + \frac{r A_0}{\Gamma(\rho_i + 1)} \right] + \frac{\bar{h}_i^* \Psi_*^{\rho_i+1}}{\Gamma(\rho_i + 2)} + \frac{r A_0}{\Gamma(\rho_i + 2)}, \end{aligned}$$

where $\bar{h}_i^* = \sup_{\gamma \in [a, b]} \bar{h}_i(\gamma, 0, 0, 0, 0)$, $i = 1, 2$, we also have, for all $\gamma \in (a, b]$

$$\begin{aligned} \Psi^{1-\rho_i}(\gamma, a) D^X \Xi_i(\mathbf{y}_1, \mathbf{y}_2)(\gamma) &= \rho_i T \frac{\sum_{j=1}^{n_i} |\zeta_{ij}| \frac{\rho_i}{a^+} i(\gamma_{ij}, Y(\gamma_{ij}))}{\Gamma(\rho_i) - \sum_{j=1}^{n_i} \zeta_{ij} \Psi^{\rho_i-1}(\gamma_{ij}, a)} + \frac{1}{\Gamma(\rho_i)} \int_a^\gamma \chi'(s) \Psi^{\rho_i-1}(\gamma, s) i(s, Y(s)) ds \\ &= \left(\rho_i T \zeta_i^* n_i + \Psi_*^{1-\rho_i} \right) \left[\frac{\bar{h}_i^* \Psi_*^{\rho_i}}{\Gamma(\rho_i + 1)} + \frac{r A_0}{\Gamma(\rho_i + 1)} \right]. \end{aligned}$$

So,

$$\begin{aligned} \Xi_i(\mathbf{y}_1, \mathbf{y}_2) \infty + D^\chi \Xi_i(\mathbf{y}_1, \mathbf{y}_2) \rho_i, \chi & \left(T \zeta_i^* n_i (\rho_i + \Psi_*^{\rho_i}) + \Psi_*^{1-\rho_i} \right) \left[\frac{\hbar_i^* \Psi_*^{\rho_i}}{\Gamma(\rho_i + 1)} + \frac{r A_0}{\Gamma(\rho_i + 2)} \right] \\ & + \frac{\hbar_i^* \Psi_*^{\rho_i + 1}}{\Gamma(\rho_i + 2)} + \frac{r A_0}{\Gamma(\rho_i + 2)} = \mathbf{M}_i. \end{aligned}$$

Thus, $\Xi(\mathbf{y}_1, \mathbf{y}_2) \stackrel{1}{\chi} \max\{\mathbf{M}_1, \mathbf{M}_2\}$.

Now we will show that Ξ is continuous. Let $\{(\mathbf{y}_{1n}(\cdot), \mathbf{y}_{2n}(\cdot))\}_{n \in \mathbb{N}}$ be a sequence converges to $(\mathbf{y}_1^*(\cdot), \mathbf{y}_2^*(\cdot))$ in $\Pi_\chi^1([a, b])$, it enough to prove $\Xi_i(\mathbf{y}_{1n}, \mathbf{y}_{2n})(\cdot) \rightarrow \Xi_i(\mathbf{y}_1^*, \mathbf{y}_2^*)(\cdot)$ as $n \rightarrow \infty$ in $C_{1-\rho_i, \chi}^1([a, b])$, $i = 1, 2$, from (\mathbf{H}_{1-1}) and Lemma 1.2.1 we can easily prove that $\Xi_i(\mathbf{y}_{1n}, \mathbf{y}_{2n})(\cdot) \rightarrow \Xi_i(\mathbf{y}_1^*, \mathbf{y}_2^*)(\cdot)$ in $C([a, b])$ and $D^\chi \Xi_i(\mathbf{y}_{1n}, \mathbf{y}_{2n})(\cdot) \rightarrow D^\chi \Xi_i(\mathbf{y}_1^*, \mathbf{y}_2^*)(\cdot)$ in $C_{1-\rho_i, \chi}([a, b])$, $i = 1, 2$, that implies $\Xi(\mathbf{y}_{1n}, \mathbf{y}_{2n})(\cdot) \rightarrow \Xi(\mathbf{y}_1^*, \mathbf{y}_2^*)(\cdot)$ in $C_{1-\rho_i, \chi}^1([a, b])$, then Ξ is continuous.

Let us show the second axiom (2), it is enough to show that $\Xi_i(B_r)$ (resp. $D^\chi \Xi_i(B_r)$) is equicontinuous on $C([a, b])$ (resp. on $C_{1-\rho_i, \chi}([a, b])$), $i = 1, 2$. Let $(\mathbf{y}_1, \mathbf{y}_2) \in B_r$ and $\gamma_1, \gamma_2 \in [a, b]$ with $\gamma_1 < \gamma_2$, from (\mathbf{H}_{1-1}) , we have

$$\begin{aligned} \Xi_i(\mathbf{y}_1, \mathbf{y}_2)(\gamma_2) - \Xi_i(\mathbf{y}_1, \mathbf{y}_2)(\gamma_1) & \leq T \zeta_i^* n_i \Psi_*^{\rho_i} \left[\frac{r A_0}{\Gamma(\rho_i + 1)} + \frac{\hbar_i^* \Psi_*^{\rho_i}}{\Gamma(\rho_i + 1)} \right] \left(\Psi^{\rho_i}(\gamma_2, a) - \Psi^{\rho_i}(\gamma_1, a) \right) \\ & + \frac{1}{\Gamma(\rho_i + 1)} \int_a^{\gamma_1} \chi'(s) [\Psi^{\rho_i}(\gamma_2, s) - \Psi^{\rho_i}(\gamma_1, s)] \chi_i(s, Y(s)) ds \\ & + \frac{1}{\Gamma(\rho_i + 1)} \int_{\gamma_1}^{\gamma_2} \chi'(s) \Psi^{\rho_i}(\gamma_2, s) \chi_i(s, Y(s)) ds \\ & \leq T \zeta_i^* n_i \Psi_*^{\rho_i} \left[\frac{r A_0}{\Gamma(\rho_i + 1)} + \frac{\hbar_i^* \Psi_*^{\rho_i}}{\Gamma(\rho_i + 1)} \right] \left(\Psi^{\rho_i}(\gamma_2, a) - \Psi^{\rho_i}(\gamma_1, a) \right) \\ & + \frac{\hbar_i^* + r \Psi_*^{-\rho_i} A_0}{\Gamma(\rho_i + 2)} \left[\Psi^{\rho_i + 1}(\gamma_2, a) - \Psi^{\rho_i + 1}(\gamma_1, a) + 2 \Psi^{\rho_i + 1}(\gamma_2, \gamma_1) \right]. \end{aligned}$$

As γ_2 tends to γ_1 , the right-hand side of the last inequality tends to 0. Therefore $\Xi_i(B_r)$, $i = 1, 2$ is equicontinuous on $C([a, b])$. And, we also have

$$\begin{aligned} & \Psi^{1-\rho_i}(\gamma_2, a) D^\chi \Xi(\mathbf{y}_1, \mathbf{y}_2)(\gamma_2) - \Psi^{1-\rho_i}(\gamma_1, a) D^\chi \Xi(\mathbf{y}_1, \mathbf{y}_2)(\gamma_1) \\ & \left\| \frac{\Psi^{1-\rho_i}(\gamma_2, a)}{\Gamma(\rho_i)} \int_a^{\gamma_2} \chi'(s) \Psi^{\rho_i - 1}(\gamma_2, s) \chi_i(s, Y(s)) ds \right. \\ & \left. - \frac{\Psi^{1-\rho_i}(\gamma_1, a)}{\Gamma(\rho_i)} \int_a^{\gamma_1} \chi'(s) \Psi^{\rho_i}(\gamma_1, s) \chi_i(s, Y(s)) ds \right\| \\ & \leq \frac{\Psi^{1-\rho_i}(\gamma_1, a)}{\Gamma(\rho_i)} \int_a^{\gamma_1} \chi'(s) \left[\Psi^{\rho_i - 1}(\gamma_1, s) - \Psi^{\rho_i - 1}(\gamma_2, s) \right] \chi_i(s, Y(s)) ds \end{aligned}$$

$$\begin{aligned}
& + \frac{\Psi^{1-\rho_i}(\gamma_2, a) - \Psi^{1-\rho_i}(\gamma_1, a)}{\Gamma(\rho_i)} \int_a^{\gamma_1} \chi'(s) \Psi^{\rho_i-1}(\gamma_2, s) \vartheta_i(s, Y(s)) ds \\
& + \frac{\Psi^{1-\rho_i}(\gamma_2, a)}{\Gamma(\rho_i)} \int_{\gamma_1}^{\gamma_2} \chi'(s) \Psi^{\rho_i-1}(\gamma_2, s) \vartheta_i(s, Y(s)) ds \\
& + \frac{(\hbar_i^* + r\Psi_*^{-\rho_i} A_0) \Psi^{1-\rho_i}(\gamma_1, a)}{\Gamma(\rho_i)} \int_a^{\gamma_1} \chi'(s) [\Psi^{\rho_i-1}(\gamma_1, s) - \Psi^{\rho_i-1}(\gamma_2, s)] ds \\
& + \frac{(\hbar_i^* + r\Psi_*^{-\rho_i} A_0) (\Psi^{1-\rho_i}(\gamma_2, a) - \Psi^{1-\rho_i}(\gamma_1, a))}{\Gamma(\rho_i)} \int_a^{\gamma_1} \chi'(s) \Psi^{\rho_i-1}(\gamma_2, s) ds \\
& + \frac{(\hbar_i^* + r\Psi_*^{-\rho_i} A_0) \Psi^{1-\rho_i}(\gamma_2, a)}{\Gamma(\rho_i)} \int_{\gamma_1}^{\gamma_2} \chi'(s) \Psi^{\rho_i-1}(\gamma_2, s) ds \\
& + \frac{(\hbar_i^* + r\Psi_*^{-\rho_i} A_0) \Psi_*^{1-\rho_i}}{\Gamma(\rho_i + 1)} [\Psi^{\rho_i}(\gamma_2, a) - \Psi^{\rho_i}(\gamma_1, a) + 2\Psi^{\rho_i}(\gamma_2, \gamma_1)] \\
& + \frac{(\hbar_i^* + r\Psi_*^{-\rho_i} A_0) \Psi_*^{\rho_i}}{\Gamma(\rho_i)} [\Psi^{1-\rho_i}(\gamma_2, a) - \Psi^{1-\rho_i}(\gamma_1, a)].
\end{aligned}$$

By taking γ_2 tends to γ_1 , the right-hand side of the last inequality tends to 0, and hence $D^X \Xi_i(B_r)$, $i = 1, 2$ is equicontinuous on $C_{1-\rho_i, \chi}([a, b])$, thus, $\Xi(B_r)$ is equicontinuous on $\Pi_\chi^1([a, b])$. \square

We denote by ϑ_C , ϑ_{ρ_i} , $\vartheta_{\rho_i}^1$ and ϑ_χ^1 the Kuratowski measure of noncompactness defined respectively on $C([a, b])$, $C_{1-\rho_i, \chi}([a, b])$, $C_{1-\rho_i, \chi}^1([a, b])$ and $\Pi_\chi^1([a, b])$.

Lemma 5.3.2. *Let $B = B_1 \times B_2$ be a bounded subset of $\Pi_\chi^1([a, b])$, we have*

$$(i) \vartheta_{\rho_i}^1(B_i) = \vartheta(B_i) + \vartheta_{\rho_i}(D^X B_i) = 2\vartheta_{\rho_i}^1(B_i), \quad i = 1, 2.$$

$$(ii) \vartheta_\chi^1(B) = \max \{ \vartheta_{\rho_1}^1(B_1), \vartheta_{\rho_2}^1(B_2) \}.$$

Proof. Let $B = B_1 \times B_2$ be a bounded subset of $\Pi_\chi^1([a, b])$. Let us show the axiom (i), we have B_i , $i = 1, 2$ is a bounded subset of $C_{1-\rho_i, \chi}^1([a, b])$, let ϵ be a strictly positive real number. So, for $i = 1, 2$. there exists a finite partition B_i^j , $j = 1, \dots, m$, such that

$$\text{Diam}(B_i^j) \leq \epsilon + \vartheta_{\rho_i}^1(B_i), \quad j = 1, \dots, m.$$

Then for all $\mathbf{y}_1, \mathbf{y}_2$ in B_i^j , $j = 1, \dots, m$ and $\gamma \in (a, b]$, we have

$$\|\mathbf{y}_2(\gamma) - \mathbf{y}_1(\gamma)\| \leq \epsilon + \vartheta_{\rho_i}^1(B_i) \quad \text{and} \quad \|D^X \mathbf{y}_2(\gamma) - D^X \mathbf{y}_1(\gamma)\| \leq \epsilon + \vartheta_{\rho_i}^1(B_i).$$

So,

$$\text{Diam}(B_i^j) \leq \epsilon + \vartheta_{\rho_i}^1(B_i) \text{ and } \text{Diam}(D^\times B_i^j) \leq \epsilon + \vartheta_{\rho_i}^1(B_i), \quad j = 1, \dots, m.$$

Thus,

$$\vartheta(B_i) + \vartheta_{\rho_i}(D^\times B_i) \leq 2\epsilon + 2\vartheta_{\rho_i}^1(B_i).$$

Since ϵ is arbitrary, this means that we arrive at

$$\vartheta(B_i) + \vartheta_{\rho_i}(D^\times B_i) \leq 2\vartheta_{\rho_i}^1(B_i), \quad i = 1, 2. \quad (5.3.1)$$

Conversely, we want to prove that $\vartheta_{\rho_i}^1(B) \leq \vartheta(B) + \vartheta_{\rho_i}(D^\times B)$, from the definition of Kuratowski measure of noncompactness, we have, for each $\epsilon > 0$, there are a finite partitions $\{B_i^j\}_{j=1, \dots, m_1}$ of B_i and $\{D_i^k\}_{k=1, \dots, m_2}$ of $D^\times B_i$ such that

$$\text{Diam}(B_i^j) \leq \epsilon + \vartheta(B_i), \text{ and } \text{Diam}(D_i^k) \leq \epsilon + \vartheta_{\rho_i}(D^\times B_i),$$

it is clear that the partition $\{B_i^j \cup I_{a^+}^\times D_i^k\}_{j,k}$ belongs to $C_{1-\rho_i, \chi}^1([a, b])$ and verifies the following inequality

$$\text{Diam}(B_i^j \cup I_{a^+}^\times D_i^k) + \text{Diam}(D^\times(B_i^j \cup I_{a^+}^\times D_i^k)) \leq 2\epsilon + \vartheta(B_i) + \vartheta_{\rho_i}(D^\times B_i).$$

As ϵ is arbitrary, we obtain

$$\vartheta_{\rho_i}^1(B_i) \leq \vartheta(B_i) + \vartheta_{\rho_i}(D^\times B_i), \quad i = 1, 2. \quad (5.3.2)$$

From (5.3.1)-(5.3.2), we get

$$\vartheta_{\rho_i}^1(B) \leq \vartheta(B) + \vartheta_{\rho_i}(D^\times B) \leq 2\vartheta_{\rho_i}^1(B).$$

Let us prove the second axiom (ii), Let $B = B_1 \times B_2$ be a bounded subset of $\Pi_\chi^1([a, b])$, for all $\epsilon > 0$, there exists a finite partition B^j , $j = 1, \dots, m$, such that,

$$\text{Diam}(B^j) \leq \epsilon + \vartheta_\chi^1(B), \quad j = 1, \dots, m.$$

Then for all $(x_1, \mathbf{y}_1), (x_2, \mathbf{y}_2)$ in $B^j = B_1^j \times B_2^j$, $j = 1, \dots, m$ and $\gamma \in (a, b]$, we have

$$\max \left\{ |x_2(\gamma) - x_1(\gamma)|, \|\mathbf{y}_2(\gamma) - \mathbf{y}_1(\gamma)\| \right\} \leq \epsilon + \vartheta_\chi^1(B),$$

that implies

$$x_2(\gamma) - x_1(\gamma) \leq \epsilon + \vartheta_\chi^1(B) \text{ and } x_2(\gamma) - x_1(\gamma) \leq \epsilon + \vartheta_\chi^1(B).$$

So,

$$\text{Diam}(B_1^j) \leq \epsilon + \vartheta_\chi^1(B) \text{ and } \text{Diam}(B_2^j) \leq \epsilon + \vartheta_\chi^1(B), j = 1, \dots, m.$$

As $B_i = \bigcup_j B_i^j$, $i = 1, 2$, we have

$$\max \left\{ \vartheta_{\rho_i}^1(B_i), i = 1, 2 \right\} \leq \epsilon + \vartheta_\chi^1(B).$$

As ϵ is arbitrary, we obtain

$$\max \left\{ \vartheta_{\rho_i}^1(B_i), i = 1, 2 \right\} \leq \vartheta_\chi^1(B). \quad (5.3.3)$$

Conversely, for all $\epsilon > 0$, there are a finite partitions $\{B_1^j\}_{j=1, \dots, m_1}$ of B_1 and $\{B_2^k\}_{k=1, \dots, m_2}$ of B_2 such that

$$\text{Diam}(B_1^j) \leq \epsilon + \vartheta_{\rho_1}^1(B_1), \text{ and } \text{Diam}(B_2^k) \leq \epsilon + \vartheta_{\rho_2}^1(B_2),$$

it is clear that the partition $\bigcup_{j,k} B_1^j \times B_2^k$ belongs to $\Pi_\chi^1([a, b])$ and verifies the following inequality

$$\text{Diam}(B_1^j \times B_2^k) \leq \epsilon + \max \left\{ \vartheta_{\rho_i}^1(B_i), i = 1, 2 \right\}.$$

Since ϵ is arbitrary, we get

$$\vartheta_\chi^1(B) \leq \max \left\{ \vartheta_{\rho_i}^1(B_i), i = 1, 2 \right\}. \quad (5.3.4)$$

From (5.3.3)-(5.3.4), we have

$$\vartheta_\chi^1(B) = \max \left\{ \vartheta_{\rho_i}^1(B_i), i = 1, 2 \right\}.$$

□

From Lemma 1.3.1 and Lemma 5.3.2, we easily show the following inequality

$$\vartheta_{\rho_i}^1(D) \leq \sup_{\gamma \in [a, b]} \vartheta(D(\gamma)) + \sup_{\gamma \in [a, b]} \vartheta(\Psi^{1-\rho_i}(\gamma, a) D^\times D(\gamma)) \leq 2\vartheta_{\rho_i}^1(D), i = 1, 2, \quad (5.3.5)$$

where D is a bounded and equicontinuous subset of $C_{1-\rho_i, \chi}^1([a, b])$, $D(\gamma) = \{\mathbf{y}(\gamma) : \mathbf{y} \in D\}$ and $D^\times D(\gamma) = \{D^\times \mathbf{y}(\gamma) : \mathbf{y} \in D\}$.

Let

$$B_R = \left\{ (\mathbf{y}_1, \mathbf{y}_2) \in \Pi_\chi^1([a, b]) : (\mathbf{y}_1, \mathbf{y}_2) \leq_\chi R \right\}.$$

We are about to present our main result which is as follows.

Theorem 5.3.1. Assume that the hypotheses $(\mathbf{H}_1) - (\mathbf{H}_2)$ are satisfied and that R verifies the following inequality

$$R > \max \left\{ \frac{(\rho_i + 1) \left(T \zeta_i^* n_i (\Psi_*^{\rho_i} + \rho_i) + \Psi_*^{1-\rho_i} \right) \Psi_*^{\rho_i} \bar{h}_i^* + \Psi_*^{\rho_i+1} \bar{h}_i^* + \Lambda_i \Gamma(\rho_i + 2)}{\Gamma(\rho_i + 2) - \left[1 + (\rho_i + 1) \left(T \zeta_i^* n_i (\Psi_*^{\rho_i} + \rho_i) + \Psi_*^{1-\rho_i} \right) \right] A_0}, i = 1, 2 \right\}. \quad (5.3.6)$$

Then, Problem $(\mathbf{S}) - (\mathbf{NIB})$ has at least one solution in $\Pi_\chi^1([a, b])$. In addition, the solution set \mathbf{SS} of the problem $(\mathbf{S}) - (\mathbf{NIB})$ is compact.

Proof. By using Lemma 5.2.1, it is clear that the fixed points of the operator Ξ are solutions of the problem $(\mathbf{S}) - (\mathbf{NIB})$. We want to verify that Ξ satisfies the assumptions of Mönch fixed point theorem. First, we will prove that Ξ is well defined from B_R to B_R , indeed, let $(\mathbf{y}_1, \mathbf{y}_2) \in B_R$. By using the condition (\mathbf{H}_{1-1}) and after some calculations, for each $\gamma \in (a, b]$, $i = 1, 2$ and $(\mathbf{y}_1, \mathbf{y}_2) \in B_R$, we get

$$\begin{aligned} \Xi_i(\mathbf{y}_1, \mathbf{y}_2)(\gamma) &= \Psi^{1-\rho_i}(\gamma, a) D_i^\chi \Xi(\mathbf{y}_1, \mathbf{y}_2)(\gamma) - \Lambda_i + T \Psi^{\rho_i}(\gamma, a) \sum_{j=1}^{n_i} |\zeta_{ij}| \frac{\rho_i}{a_+} i(\gamma_{ij}, Y(\gamma_{ij})) \\ &+ \frac{1}{\Gamma(\rho_i + 1)} \int_a^\gamma \chi'(s) \Psi^{\rho_i}(\gamma, s) i(s, Y(s)) ds + \rho_i T \sum_{j=1}^{n_i} |\zeta_{ij}| \frac{\rho_i}{a_+} i(\gamma_{ij}, Y(\gamma_{ij})) \\ &+ \frac{1}{\Gamma(\rho_i)} \int_a^\gamma \chi'(s) \Psi^{\rho_i-1}(\gamma, s) i(s, Y(s)) ds \\ &= \frac{(\rho_i + 1) \left(T \zeta_i^* n_i (\Psi_*^{\rho_i} + \rho_i) + \Psi_*^{1-\rho_i} \right) \Psi_*^{\rho_i} \bar{h}_i^* + \Psi_*^{\rho_i+1} \bar{h}_i^* + \Lambda_i \Gamma(\rho_i + 2)}{\Gamma(\rho_i + 2)} \\ &+ \frac{\left[1 + (\rho_i + 1) \left(T \zeta_i^* n_i (\Psi_*^{\rho_i} + \rho_i) + \Psi_*^{1-\rho_i} \right) \right] A_0}{\Gamma(\rho_i + 2)} R. \end{aligned}$$

From (\mathbf{H}_2) and the inequality (5.3.6), we obtain

$$(\mathbf{y}_1, \mathbf{y}_2) \in B_R : \Xi(\mathbf{y}_1, \mathbf{y}_2) \Big|_\chi < R.$$

Note that B_R is bounded, convex and closed subset of $\Pi_\chi^1([a, b])$ and Ξ is continuous on B_R . Next, it is enough to show the following implication

$$V \cap \overline{\text{conv}} \{ \Xi V \setminus \{0\} \} = \vartheta_\chi^1(V) = 0, \text{ for any } V \in B_R.$$

Let $V = V_1 \times V_2$ be a subset of B_R such that $V \cap \overline{\text{conv}} \{ \Xi V \setminus \{0\} \} \neq \emptyset$. By using Definition 1.3.2 and

Lemma 1.3.1, for all $\gamma \in (a, b]$, $i = 1, 2$, we obtain

$$\begin{aligned}
& \vartheta(\Xi_i V(\gamma)) + \vartheta\left(\Psi^{1-\rho_i}(\gamma, a) D^\chi \Xi_i(V(\gamma))\right) \\
& \quad \Psi_*^{\rho_i} T \left[\sum_{j=1}^{n_i} \zeta_{ij} \vartheta^{\rho_i} \left(\bar{h}_i(\gamma_{ij}, V_1(\gamma_{ij}), V_2(\gamma_{ij}), {}^C \mathfrak{D}_{a^+}^{\rho_1, \chi} V_1(\gamma_{ij}), {}^C \mathfrak{D}_{a^+}^{\rho_2, \chi} V_2(\gamma_{ij})) \right) \right] \\
& \quad + \frac{1}{\Gamma(\rho_i + 1)} \int_a^\gamma \chi'(s) \Psi^{\rho_i}(\gamma, s) \vartheta \left(\bar{h}_i(s, V_1(s), V_2(s), {}^C \mathfrak{D}_{a^+}^{\rho_1, \chi} V_1(s), {}^C \mathfrak{D}_{a^+}^{\rho_2, \chi} V_2(s)) \right) ds \\
& \quad + \rho_i T \sum_{j=1}^{n_i} |\zeta_{ij}| \vartheta^{\rho_i} \left(\bar{h}_i(\gamma_{ij}, V_1(\gamma_{ij}), V_2(\gamma_{ij}), {}^C \mathfrak{D}_{a^+}^{\rho_1, \chi} V_1(\gamma_{ij}), {}^C \mathfrak{D}_{a^+}^{\rho_2, \chi} V_2(\gamma_{ij})) \right) \\
& \quad + \frac{1}{\Gamma(\rho_i)} \int_a^\gamma \chi'(s) \Psi^{\rho_i-1}(\gamma, s) \vartheta \left(\bar{h}_i(s, V_1(s), V_2(s), {}^C \mathfrak{D}_{a^+}^{\rho_1, \chi} V_1(s), {}^C \mathfrak{D}_{a^+}^{\rho_2, \chi} V_2(s)) \right) ds.
\end{aligned}$$

From Lemmas 1.3.1, 5.3.1 and 5.3.2 and the hypotheses $(\mathbf{H}_{1-2}) - (\mathbf{H}_2)$ and inequality (5.3.5), we arrive at

$$\begin{aligned}
\vartheta_{\rho_i}^1(\Xi_i V) & \sup_{\gamma \in [a, b]} \vartheta(\Xi_i V(\gamma)) + \sup_{\gamma \in [a, b]} \vartheta\left(\Psi^{1-\rho_i}(\gamma, a) D^\chi \Xi_i(V(\gamma))\right) \\
& \frac{2 \left[1 + (\rho_i + 1) \left(T \zeta_i^* n_i (\Psi_*^{\rho_i} + \rho_i) + \Psi_*^{1-\rho_i} \right) \right] A_0}{\Gamma(\rho_i + 2)} \vartheta_{\rho_i}^1(\Xi_i V).
\end{aligned}$$

From Lemma 5.3.2, we have

$$\vartheta_\chi^1(\Xi V) \leq \max \left\{ \frac{2 \left[1 + (\rho_i + 1) \left(T \zeta_i^* n_i (\Psi_*^{\rho_i} + \rho_i) + \Psi_*^{1-\rho_i} \right) \right] A_0}{\Gamma(\rho_i + 2)}, i = 1, 2 \right\} \vartheta_\chi^1(\Xi V).$$

By the condition (\mathbf{H}_2) , we get $\vartheta_\chi^1(\Xi V) = 0$, that means $\vartheta_\chi^1(V) = 0$. From the theorem 1.4.4, the operator Ξ has at least one fixed point $(\mathbf{y}_1, \mathbf{y}_2) \in B_R$. By using Lemma 5.2.1, we conclude that the problem $(\mathbf{S}) - (\mathbf{NIB})$ has at least one solution.

5.3.2 Compactness of the Solution Set

We show that the solution set of the problem $(\mathbf{S}) - (\mathbf{NIB})$ is a compact subset of $\Pi_\chi^1([a, b])$. Let $\{(\mathbf{y}_1^n, \mathbf{y}_2^n)\}_{n \in \mathbb{N}}$ be a sequence in the solution set. Since $\Pi_\chi^1([a, b])$ is a compact space, there exists a subsequence of $\{(\mathbf{y}_1^n, \mathbf{y}_2^n)\}_{n \in \mathbb{N}}$ (still denoted $\{(\mathbf{y}_1^n, \mathbf{y}_2^n)\}_{n \in \mathbb{N}}$) converges to $(\mathbf{y}_1^*, \mathbf{y}_2^*)$, it enough to demonstrate that $(\mathbf{y}_1^*, \mathbf{y}_2^*)$ is a solution of $(\mathbf{S}) - (\mathbf{NIB})$, for each $\gamma \in (a, b]$, $i = 1, 2$, we have

$$\mathbf{y}_i^n(\gamma) = \Lambda_i + T \Psi^{\rho_i}(\gamma, a) \sum_{j=1}^{n_i} |\zeta_{ij}| \vartheta^{\rho_i} \left(\bar{h}_i(\gamma_{ij}, Y^n(\gamma_{ij})) \right) + \frac{1}{\Gamma(\rho_i + 1)} \int_a^\gamma \chi'(s) \Psi^{\rho_i}(\gamma, s) \bar{h}_i(s, Y^n(s)) ds,$$

and

$$D^X \mathbf{y}_i^n(\gamma) = \rho_i T \sum_{j=1}^{n_i} |\zeta_{ij}| \rho_{a+}^i i(\gamma_{ij}, Y^n(\gamma_{ij})) + \frac{1}{\Gamma(\rho_i)} \int_a^\gamma \chi'(s) \Psi^{\rho_i-1}(\gamma, s) i(s, Y^n(s)) ds.$$

From (\mathbf{H}_1) , we have $i(\cdot, Y^n(\cdot)) = \tilde{h}_i(\cdot, \mathbf{y}_1^n(\cdot), \mathbf{y}_2^n(\cdot), {}^C \mathfrak{D}_{a+}^{\rho_1, \chi} \mathbf{y}_1^n(\cdot), {}^C \mathfrak{D}_{a+}^{\rho_2, \chi} \mathbf{y}_2^n(\cdot))$ simply converges to $i(\cdot, Y^*(\cdot)) = \tilde{h}_i(\cdot, \mathbf{y}_1^*(\cdot), \mathbf{y}_2^*(\cdot), {}^C \mathfrak{D}_{a+}^{\rho_1, \chi} \mathbf{y}_1^*(\cdot), {}^C \mathfrak{D}_{a+}^{\rho_2, \chi} \mathbf{y}_2^*(\cdot))$ as $n \rightarrow \infty$, let $\gamma \in (a, b]$, from (\mathbf{H}_{1-1}) , for all $n \in \mathbb{N}$, $i = 1, 2$, we have

$$\begin{aligned} \chi'(s) \Psi^{\rho_i}(\gamma, s) i(s, Y^n(s)) & \leq \left(\tilde{h}_i^* + \left((\alpha_i + \beta_i + \xi_i \Gamma(\rho_1) + \delta_i \Gamma(\rho_2)) M \right) \chi'(s) \Psi^{\rho_i}(\gamma, s) \right) \text{ and} \\ \chi'(s) \Psi^{\rho_i-1}(\gamma, s) i(s, Y^n(s)) & \leq \left(\tilde{h}_i^* + \left((\alpha_i + \beta_i + \xi_i \Gamma(\rho_1) + \delta_i \Gamma(\rho_2)) M \right) \right) \chi'(s) \Psi^{\rho_i-1}(\gamma, s). \end{aligned}$$

Using Lebesgue's dominated convergence theorem, for each $\gamma \in (a, b]$, $i = 1, 2$, we obtain

$$\mathbf{y}_i^*(\gamma) = \Lambda_i + T \Psi^{\rho_i}(\gamma, a) \sum_{j=1}^{n_i} |\zeta_{ij}| \rho_{a+}^i i(\gamma_{ij}, Y^*(\gamma_{ij})) + \frac{1}{\Gamma(\rho_i + 1)} \int_a^\gamma \chi'(s) \Psi^{\rho_i}(s) i(s, Y^*(s)) ds,$$

and

$$D^X \mathbf{y}_i^*(\gamma) = \rho_i T \sum_{j=1}^{n_i} |\zeta_{ij}| \rho_{a+}^i i(\gamma_{ij}, Y^*(\gamma_{ij})) + \frac{1}{\Gamma(\rho_i)} \int_a^\gamma \chi'(s) \Psi^{\rho_i-1}(\gamma, s) i(s, Y^*(s)) ds.$$

Thus, the solution set of the problem $(\mathbf{S}) - (\mathbf{NIB})$ is compact subset of $\Pi_\chi^1([a, b])$. \square

5.4 An Example

We take $\chi(t) = \frac{4 \arctan t}{\pi}$, $a = 0$, $b = 1$, $\rho_1 = \rho_2 = 0.25$, $n_1 = n_2 = 1$, $\gamma_{11} = \gamma_{21} = 1$ and E the Banach space defined by

$$E = \left\{ (\mathbf{y}^n)_{n \in \mathbb{N}} : \sup_n \|\mathbf{y}^n\| < \infty \right\},$$

with the norm $\|\mathbf{y}\| = \sup_n \|\mathbf{y}^n\|$, we define the function $i(\cdot, Y(\cdot))$, $i = 1, 2$ by

$$i(\cdot, Y(\cdot)) = \left(i(\cdot, Y^1(\cdot)), \dots, i(\cdot, Y^n(\cdot)), \dots \right), \quad i = 1, 2$$

where

$$\begin{aligned} i_1(\gamma, Y^n(\gamma)) &= \frac{\mathbf{y}_1^n(\gamma)}{40 + \gamma^n} + \frac{\mathbf{y}_2^n(\gamma)}{39 + e^{n\gamma}} + \frac{{}^C \mathfrak{D}_{a+}^{\rho_1, \chi} \mathbf{y}_1^n(\gamma)}{40 + n\gamma^2} + \frac{{}^C \mathfrak{D}_{a+}^{\rho_2, \chi} \mathbf{y}_2^n(\gamma)}{39 + e^{n\gamma}} \text{ and} \\ i_2(\gamma, Y^n(\gamma)) &= \frac{\mathbf{y}_1^n(\gamma) + \mathbf{y}_2^n(\gamma) + {}^C \mathfrak{D}_{a+}^{\rho_1, \chi} \mathbf{y}_1^n(\gamma) + {}^C \mathfrak{D}_{a+}^{\rho_2, \chi} \mathbf{y}_2^n(\gamma)}{20 + 30e^{n\gamma}}, \quad \gamma \in (0, 1]. \end{aligned}$$

We easily see that for all $(\mathbf{y}_1, \mathbf{y}_2), (\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2) \in \Pi_{\chi}^1([a, b], \gamma) \subset [a, b]$:

$$\begin{aligned} \vartheta_1(\gamma, Y(\gamma)) - \vartheta_1(\gamma, \bar{Y}(\gamma)) &= \frac{1}{40} \left[\|\mathbf{y}_1(\gamma) - \bar{\mathbf{y}}_1(\gamma)\| + \|\mathbf{y}_2(\gamma) - \bar{\mathbf{y}}_2(\gamma)\| \right. \\ &\quad \left. + {}^C \mathfrak{D}^{\rho_1, \chi}(\mathbf{y}_1(\gamma) - \bar{\mathbf{y}}_1(\gamma)) + {}^C \mathfrak{D}^{\rho_2, \chi}(\mathbf{y}_2(\gamma) - \bar{\mathbf{y}}_2(\gamma)) \right] \text{ and} \\ \vartheta_2(\gamma, Y(\gamma)) - \vartheta_2(\gamma, \bar{Y}(\gamma)) &= \frac{1}{50} \left[\|\mathbf{y}_1(\gamma) - \bar{\mathbf{y}}_1(\gamma)\| + \|\mathbf{y}_2(\gamma) - \bar{\mathbf{y}}_2(\gamma)\| \right. \\ &\quad \left. + {}^C \mathfrak{D}^{\rho_1, \chi}(\mathbf{y}_1(\gamma) - \bar{\mathbf{y}}_1(\gamma)) + {}^C \mathfrak{D}^{\rho_2, \chi}(\mathbf{y}_2(\gamma) - \bar{\mathbf{y}}_2(\gamma)) \right]. \end{aligned}$$

Next, for all Ω_i a bounded subset of $C_{1-\rho_i, \chi}^1([0, 1])$, we have

$$\begin{aligned} \vartheta(\vartheta_i(\gamma, Y(\gamma))) &= \frac{\vartheta(\Omega_1(\gamma)) + \vartheta(\Omega_2(\gamma)) + \vartheta({}^C \mathfrak{D}_{a^+}^{\rho_1, \chi} \Omega_1(\gamma)) + \vartheta({}^C \mathfrak{D}_{a^+}^{\rho_2, \chi} \Omega_2(\gamma))}{40}, \text{ and} \\ \vartheta(\vartheta_2(\gamma, Y(\gamma))) &= \frac{\vartheta(\Omega_1(\gamma)) + \vartheta(\Omega_2(\gamma)) + \vartheta({}^C \mathfrak{D}_{a^+}^{\rho_1, \chi} \Omega_1(\gamma)) + \vartheta({}^C \mathfrak{D}_{a^+}^{\rho_2, \chi} \Omega_2(\gamma))}{50}, \quad \gamma \in (0, 1] \end{aligned}$$

So, (\mathbf{H}_1) , (\mathbf{H}_{1-1}) and (\mathbf{H}_{1-2}) are satisfied. A quick calculation gives us, for $i = 1, 2$, we have

$$\left[1 + (\rho_i + 1) \left(T \zeta_i^* n_i (\Psi_*^{\rho_i} + \rho_i) + \Psi_*^{1-\rho_i} \right) \right] A_0 < \frac{\Gamma(\rho_i + 2)}{2}.$$

So, (\mathbf{H}_2) holds. Therefore, Theorem 5.3.1 ensures that the solution set of Problem $(\mathbf{S}) - (\mathbf{NIB})$ is nonempty and compact.

Conclusion and perspectives

In this thesis, we have examined the existence and uniqueness of solutions to a certain class of differential equations of fractional order. For this, we based our work on four well-known fixed point theorems (Banach, Schaefer, Krasnoselskii) and Mönch's fixed point theorem, combined with the notion of Kuratowski's measures of noncompactness. We obtained some original results, which are presented in Chapters 2-5.

First, we explored some basics of fractional calculus by addressing different approaches to integration and differentiation. We also highlighted necessary tools for the logical progression of each chapter. After that, we presented a boundary value problem that consists of fractional differential equations with nonlocal conditions. Moreover, we applied these generalizations to a certain class of differential equations with Hilfer derivatives. The existence and uniqueness of solutions to the system have been addressed by using the Mönch fixed point theorem combined with the technique of Kuratowski's measure of noncompactness. To demonstrate the effectiveness of our results, we provided some examples.

As for future perspectives, this work opens new possibilities for exploration. For instance, there are many aspects that still need investigation, such as studying other differential equations involving different fractional derivatives like the Caputo-Hadamard fractional derivative, and employing additional techniques and theorems. We will also apply numerical methods to solve problems involving fractional equations and consider their applications in various fields of science and engineering, hopefully in our upcoming works.

As perspectives of this thesis, we propose to:

- Study fractional systems on unbonded interval.
- Examine the stability of the solutions.
- Find more accurate and comprehensive numerical methods to solve fractional equations.

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