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Existence Results for Differential Equations and Inclusions of Arbitrary Order

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Dedication

I dedicate this work

to those whose prayers have always accompanied me, whose greatest wish has been my success…

to my dear parents, the very symbols of generosity and sacrifice, who instilled in me the love of learning and the determination to reach the highest peaks,

to my wife, my children, my brothers and sisters, may God protect them all,

and to all my friends and everyone who has supported me along the way.

Abstract

In this thesis, we investigate the existence and uniqueness of solutions for certain boundary value problems generated by fractional differential equations with Caputo derivatives under nonlocal integral boundary conditions, as well as for sequential differential equations with mixed boundary conditions. The results are established using several fixed-point theorems, namely Krasnoselskii's, Schauder's, Leray–Schauder, and Banach's. Furthermore, the study is extended to fractional differential inclusions by employing Krasnoselskii's fixed-point theorem for multivalued maps and Wegrzyk's fixed-point theorem.

Keywords: Caputo fractional derivative, Hölder inequality, Boundary value problem, Existence and uniqueness, Fixed point theorem, Sequential fractional derivative, Differential inclusions, Krasnoselskii's multi-valued fixed point theorem.

Résumé

Dans cette thèse, nous étudions l'existence et l'unicité des solutions de certains problèmes aux limites engendrés par des équations différentielles fractionnaires à dérivées de Caputo, sous des conditions aux limites intégrales non locales, ainsi que des équations différentielles séquentielles à conditions aux limites mixtes. Les résultats obtenus ont été établis en appliquant plusieurs théorèmes du point fixe, à savoir ceux de Krasnoselskii, Schauder, Leray–Schauder et Banach, afin de démontrer l'existence et l'unicité. En outre, l'étude a été étendue aux inclusions différentielles fractionnaires, en s'appuyant sur le théorème du point fixe de Krasnoselskii pour les applications multivaluées et sur celui de Wegrzyk.

Mots clés: Dérivée fractionnaire de Caputo, Inégalité de Hölder, Problème aux limites, Existence et unicité, Théorème du point fixe, Dérivée fractionnaire séquentielle, Inclusions différentielles, Théorème du point fixe multivalué de Krasnoselskii.

ملخص

ناقشنا في هذه الأطروحة وجود ووحدانية الحلول لبعض المسائل الحدّية المتولّدة عن معادلات تفاضلية كسرية ذات مشتقات من نوع كابوتو، تحت شروط حدّية تكاملية غير محلية، وكذلك المعادلات التفاضلية المتسلسلة ذات الشروط الحدّية المختلطة. تمّ التوصّل إلى النتائج بالاعتماد على مبرهنات النقطة الثابتة لكلّ من كراسنوسلسكي، شاوْدر، ليرايشاوْدر، وباناخ لإثبات الوجود والوحدانية. كما تمّ توسيع الدراسة لتشمل الاحتواءات التفاضلية الكسرية، وذلك بالاعتماد على مبرهنتي كراسنوسلسكي للتطبيقات المتعددة القيم وويغرزيك للنقطة الثابتة.

الكلمات المفتاحية: مشتقة كابوتو الكسرية، متباينة هولدر، مسألة ذات شروط حدّية، الوجود و الوحدانية، مبرهنة النقطة الثابتة، المشتقة الكسرية المتسلسلة، الإحتواءات التفاضلية، مبرهنة كراسنوسلسكي للنقطة الثابتة متعددة القيم.

Contents

Introduction and Motivations

Fractional calculus is considered a natural extension of classical differentiation and integration to non-integer orders, allowing derivatives and integrals of real or even complex order. Its origins date back to the late seventeenth century, when Newton and Leibniz first raised the question of the meaning of a fractional-order derivative, such as 1/2. Subsequent contributions by mathematicians like Euler, Liouville, and Riemann established the foundations of fractional operators, notably through the Riemann–Liouville integral operator. Although these ideas have existed since the birth of mathematical analysis, their practical relevance has emerged only in recent decades. It has been demonstrated that fractional-order models provide more accurate descriptions of various natural phenomena and physical or engineering systems compared to classical approaches. Thus, fractional calculus has become a powerful analytical tool and a subject of growing interest in modern research [41, 51].

Differential inclusions, also known as multivalued differential equations, extend classical differential equations in which the derivative is allowed to take values from a set rather than a single value. This generalization makes them suitable for modeling systems involving uncertainty or multiple choices, as in economics and optimal control. When combined with non-integer order derivatives, one obtains fractional differential inclusions, which incorporate both the multivalued nature of solutions and the fractional-order dynamics, thereby providing a broader framework for both theoretical research and practical applications. For the key research findings in differential inclusion theory, we refer readers to [1, 18, 32]. Integro-differential inclusions provide a mathematical framework for describing various real-world phenomena arising in diverse areas such as economics, optimal control, and stochastic analysis. Further details and related studies can be found in the references [14, 55, 61].

Fractional differential equations extend classical models by allowing derivatives of non-integer order, making them suitable for systems with memory effects. Differential inclusions, in turn, generalize differential equations by replacing the right-hand side with a multivalued mapping, which captures uncertainty and discontinuity. Combining both concepts yields fractional differential inclusions, a broader framework that includes fractional equations as a special case and enables the analysis of more complex non-deterministic systems through multivalued fixed-point techniques.

Boundary value problems (BVPs) of fractional differential equations (FDEs) have become a central topic in recent mathematical research due to their ability to describe both linear and nonlinear phenomena relevant to science and engineering. Numerous studies have demonstrated the effectiveness of FDEs in modeling natural processes. Various types of fractional derivatives such as those of Riemann–Liouville, Caputo, Hadamard, Hilfer, Caputo–Fabrizio, Caputo–Hadamard, and Atangana–Baleanu have been investigated under different boundary conditions, highlighting the growing interest of many researchers in this field [10, 17, 19, 24, 29, 31, 34, 42, 47, 53, 60].

Fractional differential equations are widely employed to describe various phenomena in science and engineering, such as electrodynamics of complex media, aerodynamics, signal and image processing, blood flow, economics, and control theory [2, 6, 7, 9, 16, 20, 22, 25, 33, 38, 46, 50, 54, 62, 63, 67]. They provide powerful frameworks for systems with memory and nonlocal effects, and in the next section we present selected applications to illustrate their practical importance:

Fractional Financial Crisis Model

The dynamics of a financial crisis can be described using two main variables:

the healthy population $X(t)$ and the affected population $Y(t)$. Their interaction resembles a diffusion process, where the activation depends on a transmission rate and a memory effect [48]. The system is modeled as:

$$\begin{cases} \dot{X}(\mathfrak{z}) = -\mathfrak{p}X(\mathfrak{z})[Y(\mathfrak{z})]^{\alpha}, \\ \dot{Y}(\mathfrak{z}) = \mathfrak{p}X(\mathfrak{z})[Y(\mathfrak{z})]^{\alpha} - \dfrac{1}{\mathfrak{n}}Y(\mathfrak{z}), \end{cases} \qquad 0 < \alpha < 2, \;\; \mathfrak{p}, \mathfrak{n} > 0.$$

where $X(\mathfrak{z})$ = healthy units, $Y(\mathfrak{z})$ = affected units, $\mathfrak{p}$ = transmission rate, α = propagation exponent, and $\mathfrak{n}$ = average persistence time in crisis.

Fractional Schrödinger Equation and its Applications in Quantum Mechanics and Anomalous Diffusion

The fractional Schrödinger equation extends quantum mechanics by using spatial fractional derivatives, capturing anomalous diffusion and non-Gaussian processes (e.g., Lévy flights) beyond the standard model [39]. It is written as:

$$i\hbar\tfrac{\partial}{\partial\mathfrak{z}}\psi(\mathfrak{u},\mathfrak{z}) = D_{\alpha}(-\Delta)^{\alpha/2}\psi(\mathfrak{u},\mathfrak{z}) + V(\mathfrak{u})\psi(\mathfrak{u},\mathfrak{z}), \quad 1 < \alpha \leq 2.$$

Here $\psi(\mathfrak{u},\mathfrak{z})$ is the wave function, $\mathfrak{u}$ the position, $\mathfrak{z}$ time, $\hbar$ Planck's constant, $V(\mathfrak{u})$ the potential, D_{α} the fractional diffusion coefficient, $(-\Delta)^{\alpha/2}$ the fractional Laplacian, and α the order ($\alpha = 2$ gives the classical case; $1 < \alpha < 2$ anomalous transport).

Fractional Order Chaotic Circuits and Artificial Intelligence

Chua's circuit is a simple electronic oscillator that exhibits nonlinear and chaotic behavior. Classical models with integer-order derivatives fail to capture long-term memory effects. Based on [13], the incommensurate fractional-order form with disturbances is:

$$\begin{cases} D^{0.98}\mathfrak{u}_1 = a_1(\mathfrak{u}_2 - \mathfrak{u}_1 - m_1\mathfrak{u}_1 - f(\mathfrak{u}_1)) + D_1(\mathfrak{u},\mathfrak{z}), \\ D^{0.98}\mathfrak{u}_2 = \mathfrak{u}_1 - \mathfrak{u}_2 + \mathfrak{u}_3 + D_2(\mathfrak{u},\mathfrak{z}), \\ D^{0.94}\mathfrak{u}_3 = -a_2\mathfrak{u}_2 - a_3\mathfrak{u}_3 + u + D_3(\mathfrak{u},\mathfrak{z}), \\ y = \mathfrak{u}_1 \end{cases}$$

with nonlinear function

$$f(\mathfrak{u}_1) = \tfrac{1}{2}(m_0 - m_1)(|\mathfrak{u}_1 + 1| - |\mathfrak{u}_1 - 1|),$$

and external disturbances

$$D_1(\mathfrak{u}, \mathfrak{z}) = 0.005 \sin(1.5\,\mathfrak{z}) \cos(0.6\,\mathfrak{u}_1\mathfrak{u}_2),$$
$$D_2(\mathfrak{u}, \mathfrak{z}) = 0.04 \cos(1.5\,\mathfrak{z}) \cos(0.6\,\mathfrak{u}_2\mathfrak{u}_3),$$
$$D_3(\mathfrak{u}, \mathfrak{z}) = 0.04 \sin(1.5\,\mathfrak{z}) \sin(1.5\,\mathfrak{u}_1^2\mathfrak{u}_3).$$

The parameters are:

$$a_1 = 10.725,\ a_2 = 10.593,\ a_3 = 0.268,\ m_0 = -0.7872,\ m_1 = -1.1726,$$

with initial condition

$$\mathfrak{u}(0) = (1.2, -0.1, -0.9)^T.$$

Here $\mathfrak{u}_1$, $\mathfrak{u}_2$ are capacitor voltages, $\mathfrak{u}_3$ is inductor current, y is the output, u is the control input, and D^α is the fractional derivative of order α. This formulation captures memory effects and chaotic dynamics more effectively than classical models.

Fractional Modeling of Viscoelastic Materials

In mechanics, the stress–strain relation is classically modeled by Hooke's law ($\mathfrak{n}(\mathfrak{z}) = E\epsilon(\mathfrak{z})$ for elastic solids) and Newton's law ($\mathfrak{n}(\mathfrak{z}) = \eta\dot{\epsilon}(\mathfrak{z})$ for viscous fluids). However, many real materials (e.g., polymers, biological tissues, metals at certain conditions) exhibit viscoelastic behavior, which lies between purely elastic and purely viscous responses.

Fractional calculus provides a unified model, known as Nutting's law [16]:

$$\mathfrak{n}(\mathfrak{z}) = v\mathcal{D}^{\mathsf{v}}\epsilon(\mathfrak{z}), \quad 0 < \mathsf{v} < 1,$$

where $\mathfrak{n}(\mathfrak{z})$ is stress, $\epsilon(\mathfrak{z})$ is strain, v is a material constant, and $\mathcal{D}^{\mathsf{v}}$ is the fractional derivative of order v.

When $\mathsf{v} = 0$, the formula reduces to Hooke's law (elastic case), when $\mathsf{v} = 1$ it reduces to Newton's law (viscous case), and for $0 < \mathsf{v} < 1$ it naturally describes viscoelastic materials.

The objective of this dissertation is to investigate nonlinear boundary value problems for **arbitrary-order differential equations and inclusions** under different boundary conditions. The aim is to establish sufficient conditions for the existence (and sometimes uniqueness) of solutions by systematically applying **fixed point theorems** within the framework of functional analysis.

The main contribution of this work lies in its **unified treatment** of arbitrary-order problems, covering fractional Caputo equations with integral conditions, sequential equations with mixed conditions, and their extension to differential inclusions. By relying throughout on fixed point techniques, the dissertation generalizes and strengthens existing results, offering a coherent methodology and a solid theoretical foundation with potential applications in mathematics, physics, and engineering. The thesis is structured around four core chapters:

In the first chapter, we will begin by reviewing fundamental concepts of functional analysis. We will then introduce key notions and properties of calculus, encompassing special functions and foundational concepts of multivalued functions. Finally, we will present several fixed-point theorems that are ubiquitously employed throughout this thesis.

In the second chapter, we study nonlinear fractional differential equations involving the Caputo fractional derivative subject to an integral boundary condition of the form:

$$\begin{cases} {}^c\mathcal{D}^{\mathsf{v}}\mathfrak{u}(\mathfrak{z}) = f(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(\mathfrak{z})), \quad 0 < \mathfrak{n} \leq 1, 2 < \mathsf{v} \leq 3, \mathfrak{z} \in [0, T], \\ \mathfrak{u}(0) = \mathfrak{u}''(0) = 0, \\ \mathfrak{u}(T) = \lambda \int_0^{\eta} \mathfrak{u}(s)ds, \end{cases}$$

here ${}^c\mathcal{D}^{\mathsf{v}}$, ${}^c\mathcal{D}^{\mathfrak{n}}$ are the Caputo fractional derivatives of order v and $\mathfrak{n}$, respectively, f a continuous function, η and λ two parameters with $0 < \eta < T$ and $0 < \lambda < \frac{2T}{\eta^2}$. To prove the existence of solutions for the nonlinear case, we employ Schauder's and Krasnoselskii's fixed point theorems. Moreover, the uniqueness of the solution is established by using the Banach contraction principle. Finally, several illustrative examples are presented to demonstrate the applicability and effectiveness of the theoretical results.

In the third chapter, we investigate the existence of solutions for a sequential fractional differential equation involving Caputo-type derivative subject to mixed boundary conditions given as follows:

$$\begin{cases} ({}^c\mathcal{D}^{\mathsf{v}+1} + \mathtt{k}{}^c\mathcal{D}^{\mathsf{v}})\mathfrak{u}(\mathfrak{z}) = f(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c D^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z})), \quad 1 < \mathsf{v} \leq 2, \mathtt{k} > 0, \mathfrak{z} \in [0,1], \\ \mathfrak{u}(0) + \mathfrak{p}\mathfrak{u}(1) = \mathcal{I}^{\mathsf{v}-1}\mathfrak{u}(\eta) + \mathcal{I}^{\mathsf{v}}\mathfrak{u}(\eta), \quad 0 < \eta < 1, \\ \mathfrak{u}^{'}(0) + \mathfrak{q}\mathfrak{u}^{'}(1) = {}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\eta) + {}^c\mathcal{D}^{\mathsf{v}}\mathfrak{u}(\eta), \quad \mathfrak{p}, \mathfrak{q} \in \mathbb{R}, \\ \mathfrak{u}^{''}(0) = 0, \end{cases}$$

here ${}^c\mathcal{D}^{\mathsf{v}+1}$, ${}^c\mathcal{D}^{\mathsf{v}}, {}^c\mathcal{D}^{\mathsf{v}-1}$ are the Caputo fractional derivatives of order $\mathsf{v}+1$, v, and $\mathsf{v}-1$ respectively, $f : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function, and $1 + \mathfrak{q} - \frac{\eta^{2-\mathsf{v}}}{\Gamma(3-\mathsf{v})} \neq 0$, $1 + \mathfrak{p} - \frac{\mathsf{v}\eta^{\mathsf{v}-1} + \eta^{\mathsf{v}}}{\Gamma(\mathsf{v}+1)} \neq 0$.

The core results are derived by employing Krasnoselskii's fixed point theorem and the Leray-Schauder fixed point theorem. We conclude this chapter with two illustrative examples.

In the fourth chapter, we extend the problem studied in Chapter 3 to the multivalued framework. More precisely, while Chapter 3 is devoted to the existence of solutions for a sequential Caputo-type fractional differential equation with mixed boundary conditions, Chapter 4 investigates the corresponding fractional differential inclusion. The existence of solutions is established by means of Krasnoselskii's fixed point theorem for multivalued maps and Wegrzyk's fixed point theorem for generalized contractions, and the chapter concludes with numerical examples.

Chapter 1

Preliminaries

In this chapter, we will review some fundamental concepts and results related to various approaches of functional analysis, fractional calculus, multivalued analysis, and fixed-point theory, which will be essential throughout this thesis.

1.1 Some notions of functional analysis

Here, we outline essential functional analysis concepts (spaces, operators, ...) and theorems needed for our analysis.

1.1.1 Functional Spaces

Definition 1.1.1. *(**Cauchy Sequence**)[45] Let $(\mathfrak{X}, \|.\|_{\mathfrak{X}})$ be a normed vector space. We say that the sequence $(\mathscr{U}_m)_m$ of elements of $\mathfrak{X}$ is a Cauchy sequence if and only if*

$$\forall \epsilon > 0, \exists \mathsf{N} \in \mathbb{N}, \forall p, q \geq \mathsf{N} \Rightarrow \|\mathscr{U}_p - \mathscr{U}_q\|_{\mathfrak{X}} < \epsilon.$$

Definition 1.1.2. *(**Complete Space**)[45] We say that $(\mathfrak{X}, \|.\|_{\mathfrak{X}})$ is complete, if every Cauchy sequence $(\mathscr{U}_m)_m$ in $\mathfrak{X}$ is convergent.*

Definition 1.1.3. *(**Banach Space**)[45] We call a normed vector space a Banach space if it is complete.*

Let $[\mathfrak{a},\mathfrak{b}] \subset \mathbb{R}$ (where $-\infty < \mathfrak{b} < \mathfrak{a} < +\infty$) be a finite interval, and let $(\mathfrak{X}, \|.\|_{\mathfrak{X}})$ be a Banach space. We define the following important function spaces (See [5, 11, 33])

Definition 1.1.4. *(**Space of continuous functions**) The space $C([\mathfrak{a},\mathfrak{b}],\mathfrak{X})$ denotes the Banach space of all continuous functions $\mathscr{G} : [\mathfrak{a},\mathfrak{b}] \to \mathfrak{X}$ endowed with the supremum norm:*

$$\|\mathscr{G}\| = \sup_{\mathfrak{z}\in[\mathfrak{a},\mathfrak{b}]} |\mathscr{G}(\mathfrak{z})| \,.$$

Definition 1.1.5. *(**Lebesgue space**) For $1 \leq \omega < +\infty$, the Lebesgue space $L^{\omega}([\mathfrak{a},\mathfrak{b}])$ is defined as the vector space of all equivalence classes of measurable functions $\mathscr{G}$ such that:*

$$L^{\omega}([\mathfrak{a},\mathfrak{b}]) = \left\{\mathscr{G} : [\mathfrak{a},\mathfrak{b}] \to \mathbb{R} \;\; measurable, \;\; and \;\; \|\mathscr{G}\|_{\omega} < +\infty\right\},$$

where

$$\|\mathscr{G}\|_{\omega} = \left(\int_{\mathfrak{a}}^{\mathfrak{b}} |\mathscr{G}(\mathfrak{z})|^{\omega}\right)^{\frac{1}{\omega}}.$$

Definition 1.1.6. *(**Space of absolutely continuous functions**)[36] The space $AC([\mathfrak{a},\mathfrak{b}])$ consists of all absolutely continuous functions $\mathscr{G} \colon [\mathfrak{a},\mathfrak{b}] \to \mathbb{R}$ that can be expressed as:*

$$\mathscr{G} \in AC([\mathfrak{a},\mathfrak{b}]) \iff \exists \psi \in L^{1}([\mathfrak{a},\mathfrak{b}]) \;\; and \;\; \mathfrak{c} \in \mathbb{R} \;\; such\ that \;\; \mathscr{G}(\mathfrak{z}) = \mathfrak{c} + \int_{\mathfrak{a}}^{\mathfrak{z}} \psi(\mathfrak{s})\, d\mathfrak{s}, \quad \forall \mathfrak{z} \in [\mathfrak{a},\mathfrak{b}].$$

Definition 1.1.7. *[36] For $n \in \mathbb{N}^{*}$, we denote by $AC^{n}([\mathfrak{a},\mathfrak{b}])$ the space of functions $\mathscr{G} : [\mathfrak{a},\mathfrak{b}] \to \mathbb{R}$ that have continuous derivatives on $[\mathfrak{a},\mathfrak{b}]$ up to order $n-1$ and such that $\mathscr{G}^{(n-1)} \in AC([\mathfrak{a},\mathfrak{b}])$, that is:*

$$AC^{n}([\mathfrak{a},\mathfrak{b}]) = \left\{\mathscr{G} \in C^{n-1}([\mathfrak{a},\mathfrak{b}],\mathbb{R}) : \mathscr{G}^{(n-1)} \in AC([\mathfrak{a},\mathfrak{b}])\right\}.$$

1.1.2 Basic Definitions

Definition 1.1.8. *(**Contraction Mapping**)[23] Let $(\mathfrak{X}, \|.\|_{\mathfrak{X}})$ be a normed vector space. A map $\mathscr{G} : \mathfrak{X} \to \mathfrak{X}$ is called a contraction, if there exists a positive number*

$\mathcal{K} \in [0,1[$ *such that for all* $\mathscr{U}_1, \mathscr{U}_2 \in \mathfrak{X}$, *we get:*

$$\|\mathscr{G}(\mathscr{U}_1) - \mathscr{G}(\mathscr{U}_2)\|_{\mathfrak{X}} \leq \mathcal{K} \|\mathscr{U}_1 - \mathscr{U}_2\|_{\mathfrak{X}}.$$

Lemma 1.1.9. ***(Hölder inequality)***[52] *Let* $\mathfrak{f} \in L^p([\mathfrak{a}, \mathfrak{b}])$ *with* $0 < \mathfrak{a} < \mathfrak{b}$ *and* $p > 1$, $\mathfrak{g} \in L^q([\mathfrak{a}, \mathfrak{b}])$ *with* $q > 1$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $\mathfrak{f}\mathfrak{g} \in L^1([\mathfrak{a}, \mathfrak{b}])$ *and*

$$\|\mathfrak{f}\mathfrak{g}\|_1 \leq \|\mathfrak{f}\|_p \|\mathfrak{g}\|_q.$$

Let $\mathfrak{f} \in L^1([\mathfrak{a}, \mathfrak{b}])$, $\mathfrak{g} \in L^\infty([\mathfrak{a}, \mathfrak{b}])$. *Then* $\mathfrak{f}\mathfrak{g} \in L^1([\mathfrak{a}, \mathfrak{b}])$ *and*

$$\|\mathfrak{f}\mathfrak{g}\|_1 \leq \|\mathfrak{f}\|_1 \|\mathfrak{g}\|.$$

Definition 1.1.10. ***(Continuous Operator)*** *An operator A defined from a Banach space $\mathfrak{X}$ into itself is said to be* ***continuous*** *if, for every sequence $(\mathfrak{u}_n)_{n\in\mathbb{N}}$ in $\mathfrak{X}$ that converges to some $\mathfrak{u} \in \mathfrak{X}$, the sequence $(A\mathfrak{u}_n)_{n\in\mathbb{N}}$ converges to $A\mathfrak{u}$.*

Definition 1.1.11. ***(Completely Continuous Operator)***[44] *Let $\mathfrak{X}$ and $\mathfrak{E}$ two Banach spaces. The continuous operator $\mathscr{G} : \mathfrak{X} \to \mathfrak{E}$ is completely continuous if it transforms any bounded set of $\mathfrak{X}$ into a relatively compact set of $\mathfrak{E}$. In other words, the operator $\mathscr{G}$ is completely continuous, if it is compact and continuous.*

Definition 1.1.12. ***(Equicontinuous set)*** *$\mathscr{M} \subset C([\mathfrak{a}, \mathfrak{b}], \mathfrak{X})$ is said to be equicontinuous if:*

$$\forall \epsilon > 0, \exists \delta > 0 : \forall \mathfrak{z}_1, \mathfrak{z}_2 \in [\mathfrak{a}, \mathfrak{b}], \|\mathfrak{z}_1 - \mathfrak{z}_2\| \leq \delta \Rightarrow \|\mathfrak{f}(\mathfrak{z}_1) - \mathfrak{f}(\mathfrak{z}_2)\| \leq \epsilon, \quad \forall \mathfrak{f} \in \mathscr{M}.$$

Definition 1.1.13. ***(Uniformly bounded set)*** *$\mathscr{M} \subset C([\mathfrak{a}, \mathfrak{b}], \mathfrak{X})$ is said to be uniformly bounded if:*

$$\exists \mathsf{c} > 0 : \|\mathfrak{f}(\mathfrak{z})\| \leq \mathsf{c}, \quad \forall \mathfrak{z} \in [\mathfrak{a}, \mathfrak{b}], \forall \mathfrak{f} \in \mathscr{M}.$$

Theorem 1.1.14. ***(Ascoli-Arzelá Theorem)*** [56] *$\mathscr{M} \subset C([\mathfrak{a}, \mathfrak{b}], \mathfrak{X})$ is relatively compact if and only if:*

(i) $\mathscr{M}$ *is uniformly bounded;*

(ii) $\mathscr{M}$ *is equicontinuous.*

Definition 1.1.15. *(**Compact**) [66] Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be two Banach spaces, and let* A *be an operator defined from* $\mathfrak{X}$ *into* $\mathfrak{Y}$*. The operator* A *is said to be compact if:*

(i) A *is continuous in* $\mathfrak{X}$*;*

(ii) $A(\mathfrak{X})$ *is relatively compact in* $\mathfrak{Y}$*.*

Definition 1.1.16. *(**Relatively compact set**) [15]* $\mathscr{M} \subset C([\mathfrak{a}, \mathfrak{b}], \mathfrak{X})$ *is said to be relatively compact if* $\overline{\mathscr{M}}$ *(the closure of* $\mathscr{M}$*) is compact.*

Theorem 1.1.17. *(**Fubini's Theorem**) [45] If* $\mathfrak{f}(\mathfrak{u}, \mathfrak{v})$ *is a continuous function on* $\mathcal{R} = [\mathfrak{a}, \mathfrak{b}] \times [\mathfrak{c}, \mathfrak{d}]$*, then*

$$\iint_{\mathcal{R}} \mathfrak{f}(\mathfrak{u}, \mathfrak{v})\, d(\mathfrak{u}, \mathfrak{v}) = \int_{\mathfrak{c}}^{\mathfrak{d}} \left(\int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{f}(\mathfrak{u}, \mathfrak{v})\, d\mathfrak{u} \right) d\mathfrak{v} = \int_{\mathfrak{a}}^{\mathfrak{b}} \left(\int_{\mathfrak{c}}^{\mathfrak{d}} \mathfrak{f}(\mathfrak{u}, \mathfrak{v})\, d\mathfrak{v} \right) d\mathfrak{u}.$$

1.2 Elements of Fractional Calculus

1.2.1 Special Functions

Gamma Function

One of the fundamental functions in fractional calculus is the Euler Gamma function $\Gamma(\mathtt{k})$, which generalizes the factorial $n!$ and extends its domain to non-integer and even complex values of $\mathtt{k}$.

We will now recall some key results related to the Gamma function.

Definition 1.2.1. *[33] We call Eulerian Gamma function the function denoted* Γ *defined for any complex number* $\mathtt{k}$ *such that* $Re(\mathtt{k}) > 0$ *by:*

$$\Gamma(\mathtt{k}) = \int_0^{+\infty} e^{-\mathfrak{z}} \mathfrak{z}^{\mathtt{k}-1} d\mathfrak{z},$$

this integral is convergent for $Re(\mathtt{k}) > 0$*.*

Proposition 1.2.1. *[33] We have the following properties*

(i) $\Gamma(\mathtt{k}+1)=\mathtt{k}\Gamma(\mathtt{k})$ *for all* $\mathtt{k}\in\mathbb{C}$, *Re*$(\mathtt{k})>0$*;*

(ii) $\Gamma(n)=(n-1)!$ *for all* $n\in\mathbb{N}^*$.

Proof. (i) We have by definition 1.2.1

$$\Gamma(\mathtt{k}+1)=\int_0^{+\infty}e^{-\mathfrak{z}}\mathfrak{z}^{\mathtt{k}}d\mathfrak{z},$$

using integration by parts gives

$$\begin{aligned}\Gamma(\mathtt{k}+1)&=[-e^{-\mathfrak{z}}\mathfrak{z}^{\mathtt{k}}]_0^{+\infty}+\mathtt{k}\int_0^{+\infty}e^{-\mathfrak{z}}\mathfrak{z}^{\mathtt{k}-1}d\mathfrak{z}\\&=\mathtt{k}\int_0^{+\infty}e^{-\mathfrak{z}}\mathfrak{z}^{\mathtt{k}-1}d\mathfrak{z},\end{aligned}$$

so

$$\Gamma(\mathtt{k}+1)=z\Gamma(\mathtt{k}).$$

(ii) To prove that $\Gamma(n)=(n-1)!$, we use recurrence.

For n = 1, we have

$$\Gamma(1)=\int_0^{+\infty}e^{-\mathfrak{z}}\mathfrak{z}^{1-1}d\mathfrak{z}=1=0!.$$

Assume that $\Gamma(n)=(n-1)!$. Let us show that the property holds for order $n+1$,

by Property (i), we have

$$\begin{aligned}\Gamma(n+1)&=n\Gamma(n)\\&=n(n-1)!\\&=n!.\end{aligned}$$

Therefore, the property holds.

□

Beta Function

This function stands as one of the fundamental tools in fractional calculus. Defined for strictly positive numbers v and $\mathfrak{n}$, the Beta function was extensively studied by Euler and Legendre. Its nomenclature is attributed to Jacques Binet.

Definition 1.2.2. *[50] The Beta function is given by :*

$$B(\mathsf{v},\mathfrak{n}) = \int_0^1 (1-\mathfrak{z})^{\mathsf{v}-1}\mathfrak{z}^{\mathfrak{n}-1}d\mathfrak{z}, \quad Re(\mathsf{v}) > 0, \quad Re(\mathfrak{n}) > 0.$$

Relationship Between These Two Functions

The Beta and Gamma functions are related by the following identity [50]:

$$B(\mathsf{v},\mathfrak{n}) = \frac{\Gamma(\mathsf{v})\Gamma(\mathfrak{n})}{\Gamma(\mathsf{v}+\mathfrak{n})}, \quad Re(\mathsf{v}) > 0, \quad Re(\mathfrak{n}) > 0. \tag{1.2.1}$$

From (1.2.1), we also obtain:

$$B(\mathsf{v},\mathfrak{n}) = B(\mathfrak{n},\mathsf{v}).$$

1.2.2 Riemann-Liouville Fractional Integral

Like most introductory texts on fractional calculus, we will adopt the Riemann-Liouville approach to present a primary definition of the fractional integral. Alternative formulations will be discussed later.

The fractional integral arises naturally from the well-known Cauchy formula for repeated integration:

$$\int_{\mathfrak{a}}^{\mathfrak{z}} d\mathfrak{r}_1 \int_{\mathfrak{a}}^{\mathfrak{r}_1} d\mathfrak{r}_2 ... \int_{\mathfrak{a}}^{\mathfrak{r}_{n-1}} \mathfrak{f}(\mathfrak{r}_n)d\mathfrak{r}_n = \frac{1}{(n-1)!}\int_{\mathfrak{a}}^{\mathfrak{z}} (\mathfrak{z}-r)^{n-1}\mathfrak{f}(r)dr,$$

where $\mathfrak{f}$ is a continuous function on the interval $[\mathfrak{a},\mathfrak{b}]$.

By virtue of the Gamma function $\Gamma(n) = (n-1)!$, Cauchy's repeated integral formula generalizes to fractional orders $\mathsf{v} > 0$ yielding:

Definition 1.2.3. *[54] Let* $\mathfrak{a},\mathfrak{b}$ *be two real numbers and* $\mathfrak{f}\in C([\mathfrak{a},\mathfrak{b}],\mathbb{R})$*. The fractional integral in the Riemann-Liouville sens of* $\mathfrak{f}$ *of order* v *is defined by*

$$\mathcal{I}_{\mathfrak{a}}^{\mathsf{v}}\mathfrak{f}(\mathfrak{z}) = \begin{cases} \dfrac{1}{\Gamma(\mathsf{v})}\int_a^{\mathfrak{z}}(\mathfrak{z}-\mathfrak{r})^{\mathsf{v}-1}f(\mathfrak{r})\,d\mathfrak{r}, & \mathsf{v}>0,\ \mathfrak{a}<\mathfrak{z}\le\mathfrak{b},\\ \mathfrak{f}(\mathfrak{z}), & \mathsf{v}=0, \end{cases}$$

where Γ *is the function defined by the Definition 1.2.1.*

Example 1.2.1. *We compute the Riemann-Liouville fractional integral of order* $\mathsf{v}>0$ *of the function* $\mathfrak{f}$ *such that*

$\mathfrak{f}(\mathfrak{z})=(\mathfrak{z}-\mathfrak{a})^n,\quad \mathfrak{a}\in\mathbb{R},\quad n>-1.$

$$\mathcal{I}_{\mathfrak{a}}^{\mathsf{v}}(\mathfrak{z}-\mathfrak{a})^n=\frac{1}{\Gamma(\mathsf{v})}\int_a^{\mathfrak{z}}(\mathfrak{z}-\mathfrak{r})^{\mathsf{v}-1}(\mathfrak{r}-\mathfrak{a})^n\,d\mathfrak{r}.$$

By performing the following change of variables:

$$\mathfrak{r}=\mathfrak{a}+\xi(\mathfrak{z}-\mathfrak{a}),\qquad with\qquad 0\le\xi\le1,\qquad therefore\ d\,\mathfrak{r}=(\mathfrak{z}-\mathfrak{a})d\,\xi.$$

Hence

$$\begin{aligned}\mathcal{I}_{\mathfrak{a}}^{\mathsf{v}}(\mathfrak{z}-\mathfrak{a})^n &= \frac{1}{\Gamma(\mathsf{v})}(\mathfrak{z}-\mathfrak{a})^{\mathsf{v}+n}\int_0^1(1-\xi)^{\mathsf{v}-1}\xi^n\,d\xi\\ &= \frac{B(\mathsf{v},n+1)}{\Gamma(\mathsf{v})}(\mathfrak{z}-\mathfrak{a})^{\mathsf{v}+n}\\ &= \frac{\Gamma(n+1)}{\Gamma(\mathsf{v}+n+1)}(\mathfrak{z}-\mathfrak{a})^{\mathsf{v}+n}.\end{aligned}$$

Special case: *If* $\mathfrak{a}=2,\mathsf{v}=\frac{3}{2}$ *and* $n=3$*, then*

$$\mathcal{I}_2^{\frac{3}{2}}(\mathfrak{z}-2)^3=\frac{\Gamma(3+1)}{\Gamma(\frac{3}{2}+3+1)}(\mathfrak{z}-2)^{\frac{3}{2}+3}=\frac{\Gamma(4)}{\Gamma(\frac{11}{2})}(\mathfrak{z}-2)^{\frac{9}{2}}=\frac{64}{315}(\mathfrak{z}-2)^{\frac{9}{2}}.$$

Proposition 1.2.2. *[50](**Dirichlet's formula**) Let* $\mathsf{v}>0,\mathfrak{n}>0$*, for any function* $\mathfrak{f}\in L^1([\mathfrak{a},\mathfrak{b}])$*, we have*

$$\mathcal{I}_{\mathfrak{a}}^{\mathsf{v}}[\mathcal{I}_{\mathfrak{a}}^{\mathfrak{n}}\mathfrak{f}(\mathfrak{z})]=\mathcal{I}_{\mathfrak{a}}^{\mathsf{v}+\mathfrak{n}}\mathfrak{f}(\mathfrak{z})=\mathcal{I}_{\mathfrak{a}}^{\mathfrak{n}}[\mathcal{I}_{\mathfrak{a}}^{\mathsf{v}}\mathfrak{f}(\mathfrak{z})],$$

for almost all $\mathfrak{z}\in[\mathfrak{a},\mathfrak{b}]$*. If in addition* $\mathfrak{f}\in C([\mathfrak{a},\mathfrak{b}])$*, then this identity is true* $\forall\mathfrak{z}\in[\mathfrak{a},\mathfrak{b}]$*.*

Proof. From Definition 1.2.3 it follows that

$$\begin{aligned}\mathcal{I}_{\mathfrak{a}}^{\mathsf{v}}[\mathcal{I}_{\mathfrak{a}}^{\mathfrak{n}}\mathfrak{f}(\mathfrak{z})] &= \frac{1}{\Gamma(\mathsf{v})}\int_a^{\mathfrak{z}}(\mathfrak{z}-\mathfrak{r})^{\mathsf{v}-1}\left[\frac{1}{\Gamma(\mathfrak{n})}\int_a^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{s})^{\mathfrak{n}-1}f(\mathfrak{s})\,d\mathfrak{s}\right]d\mathfrak{r}\\ &= \frac{1}{\Gamma(\mathsf{v})\Gamma(\mathfrak{n})}\int_a^{\mathfrak{z}}(\mathfrak{z}-\mathfrak{r})^{\mathsf{v}-1}\left[\int_a^{\mathfrak{r}}(\mathfrak{r}-\mathfrak{s})^{\mathfrak{n}-1}f(\mathfrak{s})\,d\mathfrak{s}\right]d\mathfrak{r}.\end{aligned}$$

Observe that $\mathfrak{a}\leq\mathfrak{s}\leq\mathfrak{r}\leq\mathfrak{z}$. Then, by Fubini's Theorem 1.1.17 we get

$$\mathcal{I}_{\mathfrak{a}}^{\mathsf{v}}[\mathcal{I}_{\mathfrak{a}}^{\mathfrak{n}}\mathfrak{f}(\mathfrak{z})] = \frac{1}{\Gamma(\mathsf{v})\Gamma(\mathfrak{n})}\int_a^{\mathfrak{z}}\left[\int_{\mathfrak{s}}^{\mathfrak{r}}(\mathfrak{z}-\mathfrak{r})^{\mathsf{v}-1}(\mathfrak{r}-\mathfrak{s})^{\mathfrak{n}-1}\,d\mathfrak{r}\right]f(\mathfrak{s})\,d\mathfrak{s}.$$

By making the variable change $\mathfrak{r}-\mathfrak{s}=r(\mathfrak{z}-\mathfrak{s})$,

where $r=0$ when $\mathfrak{r}=\mathfrak{s}$ and $r=1$ when $\mathfrak{r}=\mathfrak{z}$ and $d\mathfrak{r}=(\mathfrak{z}-\mathfrak{s})dr$, we obtain

$$\begin{aligned}\mathcal{I}_{\mathfrak{a}}^{\mathsf{v}}[\mathcal{I}_{\mathfrak{a}}^{\mathfrak{n}}\mathfrak{f}(\mathfrak{z})] &= \frac{1}{\Gamma(\mathsf{v})\Gamma(\mathfrak{n})}\int_a^{\mathfrak{z}}(\mathfrak{z}-\mathfrak{s})^{\mathsf{v}+\mathfrak{n}-1}\left[\int_0^1(1-r)^{\mathsf{v}-1}r^{\mathfrak{n}-1}\,dr\right]f(\mathfrak{s})\,d\mathfrak{s}\\ &= \frac{B(\mathsf{v},\mathfrak{n})}{\Gamma(\mathsf{v})\Gamma(\mathfrak{n})}\int_a^{\mathfrak{z}}(\mathfrak{z}-\mathfrak{s})^{\mathsf{v}+\mathfrak{n}-1}f(\mathfrak{s})\,d\mathfrak{s}.\end{aligned}$$

Using the relation between Beta and Gamma functions (1.2.1), we obtain

$$\begin{aligned}\mathcal{I}_{\mathfrak{a}}^{\mathsf{v}}[\mathcal{I}_{\mathfrak{a}}^{\mathfrak{n}}\mathfrak{f}(\mathfrak{z})] &= \frac{1}{\Gamma(\mathsf{v}+\mathfrak{n})}\int_a^{\mathfrak{z}}(\mathfrak{z}-\mathfrak{s})^{\mathsf{v}+\mathfrak{n}-1}f(\mathfrak{s})\,d\mathfrak{s}\\ &= \mathcal{I}_{\mathfrak{a}}^{\mathsf{v}+\mathfrak{n}}\mathfrak{f}(\mathfrak{z})\\ &= \mathcal{I}_{\mathfrak{a}}^{\mathfrak{n}}[\mathcal{I}_{\mathfrak{a}}^{\mathsf{v}}\mathfrak{f}(\mathfrak{z})].\end{aligned}$$

□

1.2.3 Fractional Derivative in the Sense of Caputo

In this section, we present the Caputo fractional derivative definition along with its fundamental properties.

Definition 1.2.4. *[33] For a function $\mathfrak{f}\in C^n([\mathfrak{a},\mathfrak{b}],\mathbb{R})$, $n\in\mathbb{N}^*$ and $n-1<\mathsf{v}\leq n$, the Caputo fractional derivative of order v is defined by*

$$\begin{aligned}{}^{c}\mathcal{D}_a^{\mathsf{v}}\mathfrak{f}(\mathfrak{z}) &= \begin{cases}\dfrac{1}{\Gamma(n-\mathsf{v})}\int_a^{\mathfrak{z}}(\mathfrak{z}-\mathfrak{r})^{n-\mathsf{v}-1}\mathfrak{f}^{(n)}(\mathfrak{r})\,d\mathfrak{r}, & n-1<\mathsf{v}<n\\ \mathfrak{f}^{(n)}(\mathfrak{z}), & \mathsf{v}=n\end{cases}\\ &= \mathcal{I}_a^{n-\mathsf{v}}\mathfrak{f}^{(n)}(\mathfrak{z}), \qquad \mathfrak{z}\in[\mathfrak{a},\mathfrak{b}],\end{aligned}$$

where $n = [\mathsf{v}] + 1$, *and* $[\mathsf{v}]$ *is the integer part of* v.

For $0 < \mathsf{v} \leq 1$, *we get*

$$
\begin{aligned}
{}^c\mathcal{D}_a^{\mathsf{v}}\mathfrak{f}(\mathfrak{z}) &= \begin{cases} \dfrac{1}{\Gamma(1-\mathsf{v})} \int_a^{\mathfrak{z}} (\mathfrak{z}-\mathfrak{r})^{-\mathsf{v}}\mathfrak{f}'(\mathfrak{r})\, d\mathfrak{r}, & 0 < \mathsf{v} < 1 \\ \mathfrak{f}'(\mathfrak{z}), & \mathsf{v} = 1. \end{cases} \\
&= \mathcal{I}_a^{1-\mathsf{v}}\mathfrak{f}'(\mathfrak{z}), \qquad \mathfrak{z} \in [\mathfrak{a}, \mathfrak{b}].
\end{aligned}
$$

Example 1.2.2. *We consider the function* $\mathfrak{f}$ *defined by*

$\mathfrak{f}(\mathfrak{z}) = (\mathfrak{z}-\mathfrak{a})^{\mathfrak{n}}, \qquad \mathfrak{z} \in [\mathfrak{a}, \mathfrak{b}]$ *where* $\mathfrak{n} \geq 0$.

Let $n \in \mathbb{N}^*$ *and* $\mathsf{v} \geq 0$ *such that* $\mathsf{v} \in [n-1, n[$, *we have*

$$
\begin{aligned}
{}^c\mathcal{D}_a^{\mathsf{v}}(\mathfrak{z}-\mathfrak{a})^{\mathfrak{n}} &= \mathcal{I}_a^{n-\mathsf{v}}[\mathcal{D}^n(\mathfrak{z}-\mathfrak{a})^{\mathfrak{n}}] \\
&= \frac{1}{\Gamma(n-\mathsf{v})} \int_a^{\mathfrak{z}} (\mathfrak{z}-\mathfrak{r})^{n-\mathsf{v}-1}[(\mathfrak{r}-\mathfrak{a})^{\mathfrak{n}}]^{(n)}\, d\mathfrak{r}.
\end{aligned}
$$

It is known that

$$
[(\mathfrak{r}-\mathfrak{a})^{\mathfrak{n}}]^{(n)} = \frac{\Gamma(\mathfrak{n}+1)}{\Gamma(\mathfrak{n}-n+1)}(\mathfrak{r}-\mathfrak{a})^{\mathfrak{n}-n},
$$

hence

$$
{}^c\mathcal{D}_a^{\mathsf{v}}(\mathfrak{z}-\mathfrak{a})^{\mathfrak{n}} = \frac{\Gamma(\mathfrak{n}+1)}{\Gamma(n-\mathsf{v})\Gamma(\mathfrak{n}-n+1)} \int_a^{\mathfrak{z}} (\mathfrak{z}-\mathfrak{r})^{n-\mathsf{v}-1}(\mathfrak{r}-\mathfrak{a})^{\mathfrak{n}-n}\, d\mathfrak{r}.
$$

By making the change of variables $\mathfrak{r} = \mathfrak{a} + \xi(\mathfrak{z}-\mathfrak{a})$, *we obtain:*

$$
\begin{aligned}
{}^c\mathcal{D}_a^{\mathsf{v}}(\mathfrak{z}-\mathfrak{a})^{\mathfrak{n}} &= \frac{\Gamma(\mathfrak{n}+1)}{\Gamma(n-\mathsf{v})\Gamma(\mathfrak{n}-n+1)}(\mathfrak{z}-\mathfrak{a})^{\mathfrak{n}-\mathsf{v}} \int_0^1 (1-\xi)^{n-\mathsf{v}-1}\xi^{\mathfrak{n}-n} d\xi \\
&= \frac{\Gamma(\mathfrak{n}+1)B(n-\mathsf{v}, \mathfrak{n}-n+1)}{\Gamma(n-\mathsf{v})\Gamma(\mathfrak{n}-n+1)}(\mathfrak{z}-\mathfrak{a})^{\mathfrak{n}-\mathsf{v}} \\
&= \frac{\Gamma(\mathfrak{n}+1)\Gamma(n-\mathsf{v})\Gamma(\mathfrak{n}-n+1)}{\Gamma(n-\mathsf{v})\Gamma(\mathfrak{n}-n+1)\Gamma(\mathfrak{n}-\mathsf{v}+1)}(\mathfrak{z}-\mathfrak{a})^{\mathfrak{n}-\mathsf{v}} \\
&= \frac{\Gamma(\mathfrak{n}+1)}{\Gamma(\mathfrak{n}-\mathsf{v}+1)}(\mathfrak{z}-\mathfrak{a})^{\mathfrak{n}-\mathsf{v}}.
\end{aligned}
$$

Special case: *If* $\mathfrak{a} = 1, \mathsf{v} = \frac{9}{2}$ *and* $\mathfrak{n} = 6$, *then*

$$
{}^c\mathcal{D}_1^{\frac{9}{2}}(\mathfrak{z}-1)^6 = \frac{\Gamma(6+1)}{\Gamma(6-\frac{9}{2}+1)}(\mathfrak{z}-1)^{6-\frac{9}{2}} = \frac{\Gamma(7)}{\Gamma(2.5)}(\mathfrak{z}-1)^{\frac{3}{2}} = \frac{960}{\sqrt{\pi}}(\mathfrak{z}-1)^{\frac{3}{2}}.
$$

Remark 1.2.5. *The Caputo derivative of a constant function* $\mathfrak{f}(\mathfrak{z}) = C$ *is zero*

$$ {}^{c}\mathcal{D}^{\mathsf{v}}_{a} C = 0. $$

Lemma 1.2.6. *Let* $\mathsf{v} > 0$ *and* $\mathfrak{f} \in AC^{N}[\mathfrak{a}, \mathfrak{b}]$. *Then the equation*

$$ {}^{c}\mathcal{D}^{\mathsf{v}}_{\mathfrak{a}} \mathfrak{f}(\mathfrak{z}) = 0, $$

has a unique solution

$$ \mathfrak{f}(\mathfrak{z}) = \sum_{i=0}^{N-1} d_i \mathfrak{z}^{i}, $$

and

$$ \mathcal{I}^{\mathsf{v}}_{\mathfrak{a}} {}^{c}\mathcal{D}^{\mathsf{v}}_{\mathfrak{a}} \mathfrak{f}(\mathfrak{z}) = \mathfrak{f}(\mathfrak{z}) + \sum_{i=0}^{N-1} d_i \mathfrak{z}^{i} $$

for some $d_i \in \mathbb{R}$, $i = 0, 1, 2, ..., N-1$, $N = [\mathsf{v}] + 1$.

Lemma 1.2.7. *Let* $\mathsf{v} > \mathfrak{n} > 0$ *and* $\mathfrak{f} \in L^{p}(0,1) \subset L^{1}(0,1), 0 \leq p \leq +\infty$. *Then the next formulas hold.*

(i) $({}^{c}\mathcal{D}^{\mathfrak{n}}_{\mathfrak{a}} \mathcal{I}^{\mathsf{v}}_{\mathfrak{a}} \mathfrak{f})(\mathfrak{z}) = \mathcal{I}^{\mathsf{v}-\mathfrak{n}}_{\mathfrak{a}} \mathfrak{f}(\mathfrak{z})$,

(ii) $({}^{c}\mathcal{D}^{\mathsf{v}}_{\mathfrak{a}} \mathcal{I}^{\mathsf{v}}_{\mathfrak{a}} \mathfrak{f})(\mathfrak{z}) = \mathfrak{f}(\mathfrak{z})$.

Proposition 1.2.3. *[33, 50] Let* $\mathsf{v} \in \mathbb{C}$ *such that* $n - 1 < \mathsf{v} < n$, *where* $n \in \mathbb{N}^{*}$ *and let the two functions* $\mathfrak{f}(.)$, $\mathfrak{g}(.)$ *such that* $\mathcal{D}^{c}_{\mathfrak{a}} \mathfrak{f}(.)$ *and* $\mathcal{D}^{c}_{\mathfrak{a}} \mathfrak{g}(.)$ *exist. Then, the Caputo fractional derivative is a* ***linear*** *operator:*

$$ {}^{c}\mathcal{D}^{\mathsf{v}}_{\mathfrak{a}} (\vartheta \mathfrak{f} + \mathfrak{g})(\mathfrak{z}) = \vartheta {}^{c}\mathcal{D}^{\mathsf{v}}_{\mathfrak{a}} \mathfrak{f}(\mathfrak{z}) +{}^{c} \mathcal{D}^{\mathsf{v}}_{\mathfrak{a}} \mathfrak{g}(\mathfrak{z}), \qquad \vartheta \in \mathbb{R},\ \mathfrak{z} \in [\mathfrak{a}, \mathfrak{b}]. $$

1.3 Multivalued Analysis

Let us recall some basic definitions on multi-valued maps [15, 30].

For a normed space $(\mathfrak{X}, \|.\|)$, let

$$ \mathcal{P}_{cl}(\mathfrak{X}) = \left\{ \mathscr{M} \in \mathcal{P}(\mathfrak{X}) : \mathscr{M} \, is \, closed \right\}, $$

$$\mathcal{P}_b(\mathfrak{X}) = \{\mathscr{M} \in \mathcal{P}(\mathfrak{X}) : \mathscr{M}\, is\, bounded\},$$

$$\mathcal{P}_{cp}(\mathfrak{X}) = \{\mathscr{M} \in \mathcal{P}(\mathfrak{X}) : \mathscr{M}\, is\, compact\},$$

$$\mathcal{P}_{cp,cv}(\mathfrak{X}) = \{\mathscr{M} \in \mathcal{P}(\mathfrak{X}) : \mathscr{M}\, is\, compact\, and\, convex\}$$

and

$$\mathcal{P}_{b,cl,cv}(\mathfrak{X}) = \{\mathscr{M} \in \mathcal{P}(\mathfrak{X}) : \mathscr{M}\, is\, bounded,\ closed\, and\, convex\},$$

where $\mathcal{P}(\mathfrak{X})$ denotes the family of all nonempty subsets of $\mathfrak{X}$.

Definition 1.3.1. *A multifunction $G : \mathfrak{X} \longrightarrow \mathcal{P}(\mathfrak{X})$ is said to be*

- *(i) convex (closed) valued if $G(\mathfrak{u})$ is convex (closed) for all $\mathfrak{u} \in \mathfrak{X}$;*
- *(ii) bounded on bounded sets if $G(\mathbb{B}) = \bigcup_{\mathfrak{u}\in\mathbb{B}} G(\mathfrak{u})$ is bounded in $\mathfrak{X}$ for all $\mathbb{B} \in \mathcal{P}_b(\mathfrak{X})$ (i.e. $\sup_{\mathfrak{u}\in\mathbb{B}} \{\sup\{|v| : v \in G(\mathfrak{u})\}\} < \infty$);*
- *(iii) completely continuous, if G maps the bounded closed set into the relatively compact set.*
- *(iv) upper semicontinuous (u.s.c.), if, for each closed set $Q \subseteq \mathcal{P}(\mathfrak{X})$, the set $G^{-}(Q) = \{\mathfrak{u} \in \mathfrak{X} : G(\mathfrak{u}) \cap Q \neq \emptyset\}$ is closed in $\mathfrak{X}$;*
- *(v) lower semicontinuous (l.s.c.), if, for each closed set $Q \subseteq \mathcal{P}(\mathfrak{X})$, the set $G^{+}(Q) = \{\mathfrak{u} \in \mathfrak{X} : G(\mathfrak{u}) \subseteq Q\}$ is closed in $\mathfrak{X}$.*

Remark 1.3.2. *It is well known that, if the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph,i.e., $\mathfrak{u}_n \longrightarrow \mathfrak{u}, w_n \longrightarrow w, w_n \in G(\mathfrak{u}_n)$ imply $w \in G(\mathfrak{u})$.*

Definition 1.3.3. *A multivalued map $G : [\mathfrak{a}, \mathfrak{b}] \longrightarrow \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$ the function*

$$\mathfrak{z} \longmapsto d(y, G(\mathfrak{z})) = \inf\{\|y - z\| : z \in G(\mathfrak{z})\},$$

is measurable.

Definition 1.3.4. *A multivalued map* $G : [\mathfrak{a}, \mathfrak{b}] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ *is called Carathéodory if*

(i) $\mathfrak{z} \longmapsto G(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2)$ *is measurable for all* $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathbb{R}$*;*

(ii) $(\mathfrak{u}_1, \mathfrak{u}_2) \longmapsto G(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2)$ *is upper semi-continuous for almost all* $\mathfrak{z} \in [\mathfrak{a}, \mathfrak{b}]$*. Further a Carathéodory function* G *is called* L^1*-Carathéodory if*

(iii) for each $\varrho > 0$*, there exists* $\phi_\varrho \in L^1([\mathfrak{a}, \mathfrak{b}], \mathbb{R}^+)$ *such that*

$$\|G(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2)\| = \sup\{|v|,\, v \in G(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2)\} \leq \phi_\varrho(\mathfrak{z})$$

for all $|\mathfrak{u}_1| \leq \varrho$*,* $|\mathfrak{u}_2| \leq \varrho$ *and for a.e.* $\mathfrak{z} \in [\mathfrak{a}, \mathfrak{b}]$*.*

For each $\mathfrak{u} \in C([\mathfrak{a}, \mathfrak{b}], \mathbb{R})$, define the set of selections of G by

$$S_{G,\mathfrak{u}} = \left\{ v \in L^1([\mathfrak{a}, \mathfrak{b}], \mathbb{R}) : v(\mathfrak{z}) \in G(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c D^{\mathfrak{n}}\mathfrak{u}(\mathfrak{z})),\ for\, almost\, all\, \mathfrak{z} \in [\mathfrak{a}, \mathfrak{b}] \right\}.$$

Lemma 1.3.5. *[40] Let* $\mathfrak{X}$ *be a Banach space. Let* $G : [\mathfrak{a}, \mathfrak{b}] \times \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathcal{P}_{cp,cv}(\mathfrak{X})$ *be an* L^1*-Carathéodory multi-valued map and let* Ξ *be a linear continuous mapping from* $L^1([\mathfrak{a}, \mathfrak{b}], \mathfrak{X})$ *to* $C([\mathfrak{a}, \mathfrak{b}], \mathfrak{X})$*. Then the operator*

$$\Xi \circ S_G : C([\mathfrak{a}, \mathfrak{b}], \mathfrak{X}) \longrightarrow P_{cp,cv}(C([\mathfrak{a}, \mathfrak{b}], \mathfrak{X})), \quad \mathfrak{u} \longmapsto (\Xi \circ S_G)(\mathfrak{u}) = \Xi(S_{G,\mathfrak{u}})$$

is a closed graph operator in $C([\mathfrak{a}, \mathfrak{b}], \mathfrak{X}) \times C([\mathfrak{a}, \mathfrak{b}], \mathfrak{X})$*.*

Let $(\mathfrak{X}, d)$ be a metric space induced from the normed space $(\mathfrak{X}, \|.\|)$. Consider $\mathcal{H}_d : \mathcal{P}(\mathfrak{X}) \times \mathcal{P}(\mathfrak{X}) \longrightarrow \mathbb{R} \cup \{\infty\}$ given by

$$\mathcal{H}_d(A, B) = \max\left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\},$$

where $d(A, b) = \inf_{a \in A} d(a, b)$ and $d(a, B) = \inf_{b \in B} d(a, b)$. Then $(\mathcal{P}_{b,cl}(\mathfrak{X}), \mathcal{H}_d)$ is a metric space and $(\mathcal{P}_{cl}(\mathfrak{X}), \mathcal{H}_d)$ is a generalized (complete) metric space (see [35]).

Definition 1.3.6. *[23] A function* $\mathfrak{F} : \mathbb{R}_+ \to \mathbb{R}_+$ *is said to be a strict comparison function if it is continuous strictly increasing and* $\sum_{n=1}^{\infty} \mathfrak{F}^n(\mathfrak{z}) < \infty$*, for each* $\mathfrak{z} > 0$*.*

Definition 1.3.7. *A multivalued operator* $G : \mathfrak{X} \longrightarrow \mathcal{P}_{cl}(\mathfrak{X})$ *is called:*

(a) $\tilde{\rho}$*-Lipschitz if and only if there exists* $\tilde{\rho} > 0$ *such that*

$$\mathcal{H}_d(G(\mathfrak{u}_1), G(\mathfrak{u}_2)) \leq \tilde{\rho} d(\mathfrak{u}_1, \mathfrak{u}_2), \quad for\, each\; \mathfrak{u}_1,\, \mathfrak{u}_2 \in \mathfrak{X};$$

(b) a contraction if and only if it is $\tilde{\rho}$*-Lipschitz with* $\tilde{\rho} < 1$;

(c) a generalized contraction if and only if there is a strict comparison function $\mathfrak{F} : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\mathcal{H}_d(G(\mathfrak{u}_1), G(\mathfrak{u}_2)) \leq \mathfrak{F}(d(\mathfrak{u}_1, \mathfrak{u}_2)), \quad for\, each\; \mathfrak{u}_1,\, \mathfrak{u}_2 \in \mathfrak{X}.$$

1.4 Some Fixed Point Theorems

Fixed-point theorems are mathematical results that guarantee, under certain conditions, the existence of a fixed point for an operator F. These theorems prove to be particularly valuable tools in mathematics, especially in the field of differential and integral equations resolution.

1.4.1 Single-valued case

Definition 1.4.1. *Let* $\mathfrak{G}$ *be an operator defined on a Banach space* $\mathfrak{X}$ *into itself. Then for any* $\mathfrak{u} \in \mathfrak{X}$ *such that* $\mathfrak{u} = \mathfrak{G}(\mathfrak{u})$, $\mathfrak{u}$ *is called a fixed point of the operator* $\mathfrak{G}$.

Theorem 1.4.2. *(**Schauder's fixed point theorem** [23]) Let* U *be a closed, convex, and nonempty subset of Banach space* $\mathfrak{X}$. *Let* $\mathfrak{T} : U \longrightarrow U$ *be a continuous mapping such that* $\mathfrak{T}(U)$ *is a relatively compact subset of* $\mathfrak{X}$. *Then* $\mathfrak{T}$ *has at least one fixed point in* U.

Theorem 1.4.3. *(**Krasnoselskii's fixed point theorem** [37]) Let* M *be a closed, bounded, convex, and nonempty subset of a Banach space* $\mathfrak{X}$. *Let* A, B *be the operators such that* *(i)* $Az_1 + Bz_2 \in M$ *whenever* $z_1, z_2 \in M$*,*

(ii) A *is compact and continuous,*

(iii) B *is a contraction mapping.*

Then there exists $z_3 \in M$ *such that* $z_3 = Az_3 + Bz_3$*.*

Theorem 1.4.4. *(**Banach's fixed point theorem** [33, 59]) Let* (E, d) *be a complete metric space and* $T : E \to E$ *be a contraction mapping. Then* T *has a unique fixed point in* E*.*

Theorem 1.4.5. *(**Leray-Schauder fixed point theorem** [23]) Let* $\mathfrak{X}$ *be a Banach space,* $Y \subseteq \mathfrak{X}$ *be nonempty, bounded and convex,* H *be an open subset of* Y *with* $0 \in H$*. Let map* $\mathfrak{G} : \overline{H} \longrightarrow Y$ *be continuous and compact (that is,* $\mathfrak{G}(\overline{H})$ *is a relatively compact subset of* Y*). Then, one of the following representations is true:*

(i) there exist $z \in \partial H$ *and* $\epsilon \in (0, 1)$ *such that* $z = \epsilon\mathfrak{G}(z)$*;*

(ii) $\mathfrak{G}$ *has a fixed point* $z \in \overline{H}$*.*

1.4.2 Multi-valued case

Definition 1.4.6. *Let* $\mathfrak{X}$ *be a Banach space and* $G : \mathfrak{X} \to \mathcal{P}(\mathfrak{X})$ *a set-valued map. A point* $\mathfrak{u} \in \mathfrak{X}$ *is called a fixed point of* G *if:* $\mathfrak{u} \in G(\mathfrak{u})$*.*

Theorem 1.4.7. *(**Krasnoselskii's fixed point theorem** [49]) Let* $\mathfrak{X}$ *be a Banach space,* $\mathscr{M} \in \mathcal{P}_{b,cl,cv}(\mathfrak{X})$ *and* $\mathsf{A}, \mathsf{B} : \mathscr{M} \to \mathcal{P}_{cp,cv}(\mathfrak{X})$ *are two multivalued operators. Then there exists* $m \in \mathscr{M}$ *such that* $m \in \mathsf{A}m + \mathsf{B}m$ *provided the operators* A *and* B *satisfy the conditions:*

(i) $\mathsf{A}m + \mathsf{B}m \subset \mathscr{M}$ *for all* $m \in \mathscr{M}$*;*

(ii) A *is contraction;*

(iii) B *is u.s.c and compact.*

Theorem 1.4.8. *(**Wegrzyk's fixed point theorem** [64]) Let* $(\mathfrak{X}, d)$ *be a complete metric space. If* $G : \mathfrak{X} \to \mathcal{P}_{cl}(\mathfrak{X})$ *is a generalized contraction, then* $FixG \neq \emptyset$.

Chapter 2

Existence and Uniqueness for Caputo Fractional Differential Equations with Integral Boundary Condition

2.1 Introduction

In recent years, several researchers have investigated fractional differential equations (FDEs) involving nonlocal boundary conditions. For instance, Aljurbua [4] studied the existence of solutions for fractional differential equations of order $1 < \mathsf{v} \leq 2$ with nonlocal two-point boundary conditions involving the Caputo derivative. Similarly, Shivanian [57] considered higher-order FDEs of order $3 < \mathsf{v} \leq 4$ with integral-type nonlocal boundary conditions under the Riemann–Liouville derivative.

Motivated by these contributions, we consider in this chapter a new class of Caputo fractional boundary value problems with a three-point integral condition, in which the nonlinearity depends not only on the unknown function but also on its lower-order Caputo derivative. The problem can be formulated as [26]:

$$\begin{cases} {}^c\mathcal{D}^{\mathsf{v}}\mathfrak{u}(\mathfrak{z}) = \mathfrak{f}(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(\mathfrak{z})), \quad 0 < \mathfrak{n} \leq 1, 2 < \mathsf{v} \leq 3, \mathfrak{z} \in [0, T], \\ \mathfrak{u}(0) = \mathfrak{u}''(0) = 0, \\ \mathfrak{u}(T) = \lambda \int_0^{\eta} \mathfrak{u}(s)ds, \end{cases} \tag{2.1.1}$$

where ${}^cD^{\mathsf{v}}$ and ${}^cD^{\mathfrak{n}}$ denote the Caputo fractional derivatives of orders v and $\mathfrak{n}$, respectively, $\mathfrak{f}$ is a continuous function, and η, λ are parameters satisfying $0 < \eta < T$ and $0 < \lambda < \frac{2T}{\eta^2}$.

2.2 Equivalent Linear System Solutions

In the next Lemma, we characterize the general solution of the linear problem associated with problem (2.1.1).

Lemma 2.2.1. *Let $2T \neq \lambda\eta^2$. Then for $h \in C([0, T], \mathbb{R})$, the problem*

$$\begin{cases} {}^c\mathcal{D}^{\mathsf{v}}\mathfrak{u}(\mathfrak{z}) = h(\mathfrak{z}), \quad 2 < \mathsf{v} \leq 3, \mathfrak{z} \in [0, T], \\ \mathfrak{u}(0) = \mathfrak{u}''(0) = 0, \\ \mathfrak{u}(T) = \lambda \int_0^{\eta} \mathfrak{u}(s)ds, \end{cases} \tag{2.2.1}$$

has a unique solution given by

$$\begin{aligned} \mathfrak{u}(\mathfrak{z}) = &\frac{1}{\Gamma(\mathsf{v})} \int_0^{\mathfrak{z}} (\mathfrak{z} - s)^{\mathsf{v}-1} h(s)ds \\ &+ \frac{2\mathfrak{z}}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \left(\frac{\lambda}{\mathsf{v}} \int_0^{\eta} (\eta - s)^{\mathsf{v}} h(s)ds - \int_0^{T} (T - s)^{\mathsf{v}-1} h(s)ds \right). \end{aligned}$$

Proof. By Definition 1.2.3 and Lemma 1.2.6, the equation (2.2.1) is equivalent to the integral equation,

$$\mathfrak{u}(\mathfrak{z}) = \frac{1}{\Gamma(\mathsf{v})} \int_0^{\mathfrak{z}} (\mathfrak{z} - s)^{\mathsf{v}-1} h(s)ds + a_0 + a_1\mathfrak{z} + a_2\mathfrak{z}^2,$$

where $a_0, a_1, a_2 \in \mathbb{R}$ are arbitrary constants.

We have

$$\mathfrak{u}''(\mathfrak{z}) = \frac{1}{\Gamma(\mathsf{v} - 2)} \int_0^{\mathfrak{z}} (\mathfrak{z} - s)^{\mathsf{v}-3} h(s)ds + 2a_2.$$

By using $\mathfrak{u}(0) = 0$ and $\mathfrak{u}''(0) = 0$, we get

$$\mathfrak{u}(\mathfrak{z}) = \frac{1}{\Gamma(\mathsf{v})} \int_0^{\mathfrak{z}} (\mathfrak{z} - s)^{\mathsf{v}-1} h(s) ds + a_1 \mathfrak{z}. \tag{2.2.2}$$

Integrating the equation (2.2.2) from 0 to η, we get

$$\begin{aligned} \int_0^{\eta} \mathfrak{u}(\mathfrak{z}) d\mathfrak{z} &= \frac{1}{\Gamma(\mathsf{v})} \int_0^{\eta} \left(\int_0^{\mathfrak{z}} (\mathfrak{z} - s)^{\mathsf{v}-1} h(s) ds \right) d\mathfrak{z} + a_1 \int_0^{\eta} \mathfrak{z} d\mathfrak{z} \\ &= \frac{1}{\mathsf{v}\Gamma(\mathsf{v})} \int_0^{\eta} (\eta - s)^{\mathsf{v}} h(s) ds + a_1 \frac{\eta^2}{2}. \end{aligned}$$

Then, by the condition $\mathfrak{u}(T) = \lambda \int_0^{\eta} \mathfrak{u}(s) ds$, we get

$$\frac{1}{\Gamma(\mathsf{v})} \int_0^{T} (T - s)^{\mathsf{v}-1} h(s) ds + a_1 T = \lambda \frac{1}{\mathsf{v}\Gamma(\mathsf{v})} \int_0^{\eta} (\eta - s)^{\mathsf{v}} h(s) ds + a_1 \lambda \frac{\eta^2}{2},$$

which implies:

$$a_1 = \frac{2}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \left(\frac{\lambda}{\mathsf{v}} \int_0^{\eta} (\eta - s)^{\mathsf{v}} h(s) ds - \int_0^{T} (T - s)^{\mathsf{v}-1} h(s) ds \right).$$

Therefore, the solution of the problem (2.2.1)

$$\begin{aligned} \mathfrak{u}(\mathfrak{z}) = &\frac{1}{\Gamma(\mathsf{v})} \int_0^{\mathfrak{z}} (\mathfrak{z} - s)^{\mathsf{v}-1} h(s) ds \\ &+ \frac{2\mathfrak{z}}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \left(\frac{\lambda}{\mathsf{v}} \int_0^{\eta} (\eta - s)^{\mathsf{v}} h(s) ds - \int_0^{T} (T - s)^{\mathsf{v}-1} h(s) ds \right). \end{aligned}$$

□

2.3 Existence Results

This section deals with the existence results for problem (2.1.1).

We define the space

$$\mathfrak{X} = \{\mathfrak{u} \in C([0, T], \mathbb{R}) : {}^{c}\mathcal{D}^{\mathfrak{n}}\mathfrak{u} \in C([0, T], \mathbb{R})\}$$

equipped with the norm

$$\|\mathfrak{u}\|_{\mathfrak{X}} = \sup_{\mathfrak{z}\in[0,T]} |\mathfrak{u}(\mathfrak{z})| + \sup_{\mathfrak{z}\in[0,T]} \left|{}^{c}\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(\mathfrak{z})\right|,$$

where $0 < \mathfrak{n} \leq 1$. $(\mathfrak{X}, \|.\|_{\mathfrak{X}})$ is a Banach space [59].

Now, we define the operator $\mathfrak{G} : \mathfrak{X} \longrightarrow \mathfrak{X}$ as follows:

$$\begin{aligned}
(\mathfrak{G}\mathfrak{u})(\mathfrak{z}) = \frac{1}{\Gamma(\mathsf{v})} \int_0^{\mathfrak{z}} & (\mathfrak{z} - s)^{\mathsf{v}-1} \mathfrak{f}(s, \mathfrak{u}(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s)) ds \\
& + \frac{2\mathfrak{z}}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \left(\frac{\lambda}{\mathsf{v}} \int_0^{\eta} (\eta - s)^{\mathsf{v}} \mathfrak{f}(s, \mathfrak{u}(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s)) ds \right. \\
& \left. - \int_0^T (T - s)^{\mathsf{v}-1} \mathfrak{f}(s, \mathfrak{u}(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s)) ds \right).
\end{aligned}$$

Then, problem (2.1.1) has solutions if and only if the operator $\mathfrak{G}$ has fixed points. So $\mathfrak{G}'\mathfrak{u}$ is given by

$$\begin{aligned}
(\mathfrak{G}'\mathfrak{u})(\mathfrak{z}) = \int_0^{\mathfrak{z}} & \frac{(\mathfrak{z} - s)^{\mathsf{v}-2}}{\Gamma(\mathsf{v} - 1)} \mathfrak{f}(s, \mathfrak{u}(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s)) ds + \frac{2}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \\
& \times \left(\frac{\lambda}{\mathsf{v}} \int_0^{\eta} (\eta - s)^{\mathsf{v}} \mathfrak{f}(s, \mathfrak{u}(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s)) ds \right. \\
& \left. - \int_0^T (T - s)^{\mathsf{v}-1} \mathfrak{f}(s, \mathfrak{u}(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s)) ds \right),
\end{aligned}$$

in Definition 1.2.4, we get

$$\begin{aligned}
& {}^c\mathcal{D}^{\mathfrak{n}}(\mathfrak{G}\mathfrak{u})(\mathfrak{z}) \\
& = \frac{1}{\Gamma(1 - \mathfrak{n})} \int_0^{\mathfrak{z}} (\mathfrak{z} - s)^{-\mathfrak{n}} (\mathfrak{G}\mathfrak{u})'(s) ds \\
& = \frac{1}{\Gamma(\mathsf{v} - \mathfrak{n})} \int_0^{\mathfrak{z}} (\mathfrak{z} - s)^{\mathsf{v}-\mathfrak{n}-1} \mathfrak{f}(s, \mathfrak{u}(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s)) ds + \frac{2\mathfrak{z}^{1-\mathfrak{n}}}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})\Gamma(2 - \mathfrak{n})} \\
& \times \left(\frac{\lambda}{\mathsf{v}} \int_0^{\eta} (\eta - s)^{\mathsf{v}} \mathfrak{f}(s, \mathfrak{u}(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s)) ds - \int_0^T (T - s)^{\mathsf{v}-1} \mathfrak{f}(s, \mathfrak{u}(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s)) ds \right).
\end{aligned}$$

For convenience, we set:

$$\begin{aligned}
\varOmega_1 &= \frac{T^{\mathsf{v}}}{\Gamma(\mathsf{v}+1)} + \frac{2\lambda T^{\mathsf{v}+2}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+2)} + \frac{2T^{\mathsf{v}+1}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+1)}\\
\varOmega_2 &= \frac{T^{\mathsf{v}-\mathfrak{n}}}{\Gamma(\mathsf{v}-\mathfrak{n}+1)} + \frac{2\lambda T^{\mathsf{v}-\mathfrak{n}+2}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+2)\Gamma(2-\mathfrak{n})} + \frac{2T^{\mathsf{v}-\mathfrak{n}+1}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+1)\Gamma(2-\mathfrak{n})}\\
\mathfrak{K}_1 &= \frac{T^{\mathsf{v}-\delta}}{\Gamma(\mathsf{v})}\left(\frac{1-\delta}{\mathsf{v}-\delta}\right)^{1-\delta} + \frac{2\lambda T^{\mathsf{v}-\delta+2}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+1)}\left(\frac{1-\delta}{\mathsf{v}-\delta+1}\right)^{1-\delta}\\
&\quad + \frac{2T^{\mathsf{v}-\delta+1}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v})}\left(\frac{1-\delta}{\mathsf{v}-\delta}\right)^{1-\delta}\\
\mathfrak{K}_2 &= \frac{T^{\mathsf{v}-\mathfrak{n}-\delta}}{\Gamma(\mathsf{v}-\mathfrak{n})}\left(\frac{1-\delta}{\mathsf{v}-\mathfrak{n}-\delta}\right)^{1-\delta} + \frac{2\lambda T^{\mathsf{v}-\mathfrak{n}-\delta+2}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+2)\Gamma(2-\mathfrak{n})}\left(\frac{1-\delta}{\mathsf{v}-\delta+1}\right)^{1-\delta}\\
&\quad + \frac{2T^{\mathsf{v}-\mathfrak{n}-\delta+1}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v})\Gamma(2-\mathfrak{n})}\left(\frac{1-\delta}{\mathsf{v}-\delta}\right)^{1-\delta}.
\end{aligned}\tag{2.3.1}$$

2.3.1 First Result

In the following result, we establish the existence of a solution to Problem (2.1.1) based on the **Schauder fixed point theorem** (Theorem 1.4.2).

Theorem 2.3.1. *Suppose that* $\mathfrak{f} : [0,T]\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ *is a continuous function fulfilling the condition:*

($\mathfrak{B}_1$) $|\mathfrak{f}(\mathfrak{z},\mathfrak{u}_1,\mathfrak{u}_2)| \le w(\mathfrak{z}) + \eta_1 |\mathfrak{u}_1|^{\varrho_1} + \eta_2 |\mathfrak{u}_2|^{\varrho_2}$, $\forall(\mathfrak{z},\mathfrak{u}_1,\mathfrak{u}_2) \in [0,T]\times\mathbb{R}\times\mathbb{R}$, *and* $w \in L^{\frac{1}{\delta}}([0,T],\mathbb{R}^+)$, $\delta\in(0,\mathsf{v}-2)$, $\eta_i \ge 0$, $0\le\varrho_i<1$, $i=1,2$.

Then, the problem (2.1.1) has at least one solution on $[0,T]$.

Proof. Denote $\|w\|_{\frac{1}{\delta}} = \left(\int_0^T |w(s)|^{\frac{1}{\delta}}\,ds\right)^{\delta}$. Let $B_r = \{\mathfrak{u}\in\mathfrak{X} : \|\mathfrak{u}\|_{\mathfrak{X}} \le r\}$ with $r>0$ to be defined later. B_r is a closed, and convex subset of the Banach space $\mathfrak{X}$.

We will show that there exists $r>0$ such that $\mathfrak{G}$ maps B_r into B_r. For $\mathfrak{u}\in B_r$, we have

$$
\begin{aligned}
|(\mathfrak{G}\mathfrak{u})(\mathfrak{z})| \leq{}& \frac{1}{\Gamma(\mathsf{v})}\int_0^{\mathfrak{z}}(\mathfrak{z}-s)^{\mathsf{v}-1}\left[w(s)+\eta_1\left|\mathfrak{u}(s)\right|^{\varrho_1}+\eta_2\left|{}^{c}\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s)\right|^{\varrho_2}\right]ds\\
&+\frac{2T}{(2T-\lambda\eta^2)\Gamma(\mathsf{v})}\left(\frac{\lambda}{\mathsf{v}}\int_0^T(T-s)^{\mathsf{v}}\left[w(s)+\eta_1\left|\mathfrak{u}(s)\right|^{\varrho_1}+\eta_2\left|{}^{c}\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s)\right|^{\varrho_2}\right]ds\right.\\
&\left.+\int_0^T(T-s)^{\mathsf{v}-1}\left[w(s)+\eta_1\left|\mathfrak{u}(s)\right|^{\varrho_1}+\eta_2\left|{}^{c}\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s)\right|^{\varrho_2}\right]ds\right)\\
\leq{}& \frac{\|w\|_{\frac{1}{\delta}}\,T^{\mathsf{v}-\delta}}{\Gamma(\mathsf{v})}\left(\frac{1-\delta}{\mathsf{v}-\delta}\right)^{1-\delta}+\frac{T^{\mathsf{v}}\left(\eta_1r^{\varrho_1}+\eta_2r^{\varrho_2}\right)}{\Gamma(\mathsf{v}+1)}\\
&+\frac{2\lambda T^{\mathsf{v}-\delta+2}\|w\|_{\frac{1}{\delta}}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+1)}\left(\frac{1-\delta}{\mathsf{v}-\delta+1}\right)^{1-\delta}+\frac{2T^{\mathsf{v}-\delta+1}\|w\|_{\frac{1}{\delta}}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v})}\left(\frac{1-\delta}{\mathsf{v}-\delta}\right)^{1-\delta}\\
&+\left(\frac{2\lambda T^{\mathsf{v}+2}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+2)}+\frac{2T^{\mathsf{v}+1}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+1)}\right)\left(\eta_1r^{\varrho_1}+\eta_2r^{\varrho_2}\right)\\
\leq{}& \left(\frac{T^{\mathsf{v}-\delta}}{\Gamma(\mathsf{v})}\left(\frac{1-\delta}{\mathsf{v}-\delta}\right)^{1-\delta}+\frac{2\lambda T^{\mathsf{v}-\delta+2}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+1)}\left(\frac{1-\delta}{\mathsf{v}-\delta+1}\right)^{1-\delta}\right.\\
&\left.+\frac{2T^{\mathsf{v}-\delta+1}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v})}\left(\frac{1-\delta}{\mathsf{v}-\delta}\right)^{1-\delta}\right)\|w\|_{\frac{1}{\delta}}\\
&+\left(\frac{T^{\mathsf{v}}}{\Gamma(\mathsf{v}+1)}+\frac{2\lambda T^{\mathsf{v}+2}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+2)}+\frac{2T^{\mathsf{v}+1}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+1)}\right)\left(\eta_1r^{\varrho_1}+\eta_2r^{\varrho_2}\right),
\end{aligned}
$$

$$
\begin{aligned}
\left|{}^{c}\mathcal{D}^{\mathfrak{n}}(\mathfrak{G}\mathfrak{u})(\mathfrak{z})\right| \leq{}& \frac{1}{\Gamma(\mathsf{v}-\mathfrak{n})}\int_0^{\mathfrak{z}}(\mathfrak{z}-s)^{\mathsf{v}-\mathfrak{n}-1}\left[w(s)+\eta_1\left|\mathfrak{u}(s)\right|^{\varrho_1}+\eta_2\left|{}^{c}\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s)\right|^{\varrho_2}\right]ds\\
&+\frac{2\mathfrak{z}^{1-\mathfrak{n}}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v})\Gamma(2-\mathfrak{n})}\\
&\times\left(\frac{\lambda}{\mathsf{v}}\int_0^{\eta}(\eta-s)^{\mathsf{v}}\left[w(s)+\eta_1\left|\mathfrak{u}(s)\right|^{\varrho_1}+\eta_2\left|{}^{c}\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s)\right|^{\varrho_2}\right]ds\right.\\
&\left.+\int_0^T(T-s)^{\mathsf{v}-1}\left[w(s)+\eta_1\left|\mathfrak{u}(s)\right|^{\varrho_1}+\eta_2\left|{}^{c}\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s)\right|^{\varrho_2}\right]ds\right)\\
\leq{}& \frac{\|w\|_{\frac{1}{\delta}}\,T^{\mathsf{v}-\mathfrak{n}-\delta}}{\Gamma(\mathsf{v}-\mathfrak{n})}\left(\frac{1-\delta}{\mathsf{v}-\mathfrak{n}-\delta}\right)^{1-\delta}+\frac{T^{\mathsf{v}-\mathfrak{n}}}{\Gamma(\mathsf{v}-\mathfrak{n}+1)}\left(\eta_1r^{\varrho_1}+\eta_2r^{\varrho_2}\right)\\
&+\frac{2\lambda T^{\mathsf{v}-\mathfrak{n}-\delta+2}\|w\|_{\frac{1}{\delta}}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+1)\Gamma(2-\mathfrak{n})}\left(\frac{1-\delta}{\mathsf{v}-\delta+1}\right)^{1-\delta}\\
&+\frac{2\lambda T^{\mathsf{v}-\mathfrak{n}+2}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+2)\Gamma(2-\mathfrak{n})}\left(\eta_1r^{\varrho_1}+\eta_2r^{\varrho_2}\right)\\
&+\frac{2T^{\mathsf{v}-\mathfrak{n}-\delta+1}\|w\|_{\frac{1}{\delta}}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v})\Gamma(2-\mathfrak{n})}\left(\frac{1-\delta}{\mathsf{v}-\delta}\right)^{1-\delta}\\
&+\frac{2T^{\mathsf{v}-\mathfrak{n}+1}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+1)\Gamma(2-\mathfrak{n})}\left(\eta_1r^{\varrho_1}+\eta_2r^{\varrho_2}\right).
\end{aligned}
$$

From the above inequalities, we get

$$\|\mathfrak{G}\mathfrak{u}\|_{\mathfrak{X}} \leq H_1 + H_2\left(\eta_1 r^{\varrho_1} + \eta_2 r^{\varrho_2}\right),$$

where

$$\begin{aligned}H_1 = &\left[\frac{T^{\mathsf{v}-\delta}}{\Gamma(\mathsf{v})}\left(\frac{1-\delta}{\mathsf{v}-\delta}\right)^{1-\delta} + \frac{2\lambda T^{\mathsf{v}-\delta+2}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+1)}\left(\frac{1-\delta}{\mathsf{v}-\delta+1}\right)^{1-\delta}\right.\\ &+ \frac{2T^{\mathsf{v}-\delta+1}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v})}\left(\frac{1-\delta}{\mathsf{v}-\delta}\right)^{1-\delta} + \frac{T^{\mathsf{v}-\mathfrak{n}-\delta}}{\Gamma(\mathsf{v}-\mathfrak{n})}\left(\frac{1-\delta}{\mathsf{v}-\mathfrak{n}-\delta}\right)^{1-\delta}\\ &+ \frac{2\lambda T^{\mathsf{v}-\mathfrak{n}-\delta+2}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+2)\Gamma(2-\mathfrak{n})}\left(\frac{1-\delta}{\mathsf{v}-\delta+1}\right)^{1-\delta}\\ &\left.+ \frac{2T^{\mathsf{v}-\mathfrak{n}-\delta+1}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v})\Gamma(2-\mathfrak{n})}\left(\frac{1-\delta}{\mathsf{v}-\delta}\right)^{1-\delta}\right]\|w\|_{\frac{1}{\delta}}\end{aligned} \tag{2.3.2}$$

and

$$\begin{aligned}H_2 = &\frac{T^{\mathsf{v}}}{\Gamma(\mathsf{v}+1)} + \frac{2\lambda T^{\mathsf{v}+2}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+2)} + \frac{2T^{\mathsf{v}+1}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+1)} + \frac{T^{\mathsf{v}-\mathfrak{n}}}{\Gamma(\mathsf{v}-\mathfrak{n}+1)}\\ &+ \frac{2\lambda T^{\mathsf{v}-\mathfrak{n}+2}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+2)\Gamma(2-\mathfrak{n})} + \frac{2T^{\mathsf{v}-\mathfrak{n}+1}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v}+1)\Gamma(2-\mathfrak{n})}.\end{aligned} \tag{2.3.3}$$

Let $r > 0$ with

$$r \geq \max\left\{3H_1, (3H_2\eta_1)^{\frac{1}{1-\varrho_1}}, (3H_2\eta_2)^{\frac{1}{1-\varrho_2}}\right\}.$$

Then, for any $\mathfrak{u} \in B_r$, it follows that

$$\|\mathfrak{G}\mathfrak{u}\|_{\mathfrak{X}} \leq H_1 + H_2\left(\eta_1 r^{\varrho_1} + \eta_2 r^{\varrho_2}\right) \leq \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = r.$$

Since $\mathfrak{f}$ is continuous, we conclude that $\mathfrak{G}$ is continuous.

Next, for every bounded subset $\overline{B} \subset \mathfrak{X}$, we show that the families $\mathfrak{G}(\overline{B})$ and ${}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{G}(\overline{B})$ are equicontinuous. Moreover, the continuity of $\mathfrak{f}$ on a compact set implies that it is bounded. Hence, there exists a constant $Q > 0$ such that $|\mathfrak{f}(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(\mathfrak{z}))| \leq Q$ for all $\mathfrak{u} \in \overline{B}$ and $\mathfrak{z} \in [0, T]$. Now, for $0 \leq \mathfrak{z}_1 < \mathfrak{z}_2 \leq T$, we have

$$\begin{aligned}
&|(\mathfrak{G}\mathfrak{u})(\mathfrak{z}_2) - (\mathfrak{G}\mathfrak{u})(\mathfrak{z}_1)| \\
&\leq \left| \frac{1}{\Gamma(\mathsf{v})} \int_0^{\mathfrak{z}_1} [(\mathfrak{z}_2 - s)^{\mathsf{v}-1} - (\mathfrak{z}_1 - s)^{\mathsf{v}-1}] \mathfrak{f}(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(\mathfrak{z})) ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\mathsf{v})} \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} (\mathfrak{z}_2 - s)^{\mathsf{v}-1} \mathfrak{f}(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(\mathfrak{z})) ds \right| \\
&\quad + \frac{2\,|\mathfrak{z}_2 - \mathfrak{z}_1|}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \left(\frac{\lambda}{\mathsf{v}} \int_0^{\eta} (\eta - s)^{\mathsf{v}} \left| \mathfrak{f}(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(\mathfrak{z})) \right| ds \right. \\
&\quad \left. + \int_0^{T} (T - s)^{\mathsf{v}-1} \left| \mathfrak{f}(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(\mathfrak{z})) \right| ds \right) \\
&\leq \frac{Q}{\mathsf{v}\Gamma(\mathsf{v})} \left| 2(\mathfrak{z}_2 - \mathfrak{z}_1)^{\mathsf{v}} + \mathfrak{z}_1^{\mathsf{v}} - \mathfrak{z}_2^{\mathsf{v}} \right| + \frac{2Q\,|\mathfrak{z}_2 - \mathfrak{z}_1|}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \left(\frac{\lambda\eta^{\mathsf{v}+1}}{\mathsf{v}(\mathsf{v}+1)} + \frac{T^{\mathsf{v}}}{\mathsf{v}} \right) \\
&\leq \frac{Q}{\Gamma(\mathsf{v}+1)} \left| 2(\mathfrak{z}_2 - \mathfrak{z}_1)^{\mathsf{v}} + \mathfrak{z}_1^{\mathsf{v}} - \mathfrak{z}_2^{\mathsf{v}} \right| + \frac{2QT^{\mathsf{v}}\,|\mathfrak{z}_2 - \mathfrak{z}_1|}{(2T - \lambda\eta^2)\Gamma(\mathsf{v}+2)} (\lambda T + \mathsf{v} + 1)
\end{aligned}$$

and

$$\begin{aligned}
&|{}^c\mathcal{D}^{\mathfrak{n}}(\mathfrak{G}\mathfrak{u})(\mathfrak{z}_2) - {}^c\mathcal{D}^{\mathfrak{n}}(\mathfrak{G}\mathfrak{u})(\mathfrak{z}_1)| \\
&\leq \left| \frac{1}{\Gamma(\mathsf{v}-\mathfrak{n})} \int_0^{\mathfrak{z}_1} [(\mathfrak{z}_2 - s)^{\mathsf{v}-\mathfrak{n}-1} - (\mathfrak{z}_1 - s)^{\mathsf{v}-\mathfrak{n}-1}] \mathfrak{f}(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(\mathfrak{z})) ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\mathsf{v}-\mathfrak{n})} \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} (\mathfrak{z}_2 - s)^{\mathsf{v}-\mathfrak{n}-1} \mathfrak{f}(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(\mathfrak{z})) ds \right| \\
&\quad + \frac{2\left| \mathfrak{z}_2^{1-\mathfrak{n}} - \mathfrak{z}_1^{1-\mathfrak{n}} \right|}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})\Gamma(2-\mathfrak{n})} \left(\frac{\lambda}{\mathsf{v}} \int_0^{\eta} (\eta - s)^{\mathsf{v}} \left| \mathfrak{f}(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(\mathfrak{z})) \right| ds \right. \\
&\quad \left. + \int_0^{T} (T - s)^{\mathsf{v}-1} \left| \mathfrak{f}(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(\mathfrak{z})) \right| ds \right) \\
&\leq \frac{Q}{\Gamma(\mathsf{v}-\mathfrak{n}+1)} \left| 2(\mathfrak{z}_2 - \mathfrak{z}_1)^{\mathsf{v}-\mathfrak{n}} + \mathfrak{z}_1^{\mathsf{v}-\mathfrak{n}} - \mathfrak{z}_2^{\mathsf{v}-\mathfrak{n}} \right| \\
&\quad + \frac{2QT^{\mathsf{v}}\left| \mathfrak{z}_2^{1-\mathfrak{n}} - \mathfrak{z}_1^{1-\mathfrak{n}} \right|}{(2T - \lambda\eta^2)\Gamma(\mathsf{v}+2)\Gamma(2-\mathfrak{n})} (\lambda T + \mathsf{v} + 1).
\end{aligned}$$

Consequently, we get

$$\sup_{\mathfrak{z}\in\overline{B}} |(\mathfrak{G}\mathfrak{u})(\mathfrak{z}_2) - (\mathfrak{G}\mathfrak{u})(\mathfrak{z}_1)| + \sup_{\mathfrak{z}\in\overline{B}} |{}^c\mathcal{D}^{\mathfrak{n}}(\mathfrak{G}\mathfrak{u})(\mathfrak{z}_2) - {}^c\mathcal{D}^{\mathfrak{n}}(\mathfrak{G}\mathfrak{u})(\mathfrak{z}_1)| \longrightarrow 0 \quad as\ \mathfrak{z}_2 \longrightarrow \mathfrak{z}_1,$$

independent of $\mathfrak{z} \in \overline{B}$. Therefore, $\mathfrak{G} : B_r \longrightarrow B_r$ is equicontinuous and uniformly bounded. Hence, by the Arzelá-Ascoli theorem 1.1.14, it follows that $\mathfrak{G}(B_r)$ is relatively compact in $\mathfrak{X}$. By applying Theorem 1.4.2, we state that problem (2.1.1) has at least one solution on $[0, T]$. $\square$

2.3.2 Second Result

The following theorem establishes the existence of at least one solution to problem (2.1.1). This result is based on **Krasnoselskii's fixed point theorem** (Theorem 1.4.3) of the sum of two operators.

Theorem 2.3.2. *Let* $\mathfrak{f} : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a continuous function fulfilling the conditions:*

$(\mathfrak{B}_2)$ $|\mathfrak{f}(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2) - \mathfrak{f}(\mathfrak{z}, \bar{\mathfrak{u}}_1, \bar{\mathfrak{u}}_2)| \leq w(\mathfrak{z})\left(|\mathfrak{u}_1 - \bar{\mathfrak{u}}_1| + |\mathfrak{u}_2 - \bar{\mathfrak{u}}_2|\right)$ *for* $\mathfrak{z} \in [0, T]$, $\mathfrak{u}_i, \bar{\mathfrak{u}}_i \in \mathbb{R}, i = 1, 2$ *and* $\delta \in (0, \mathsf{v} - 2)$ *and the function* $w : [0, T] \to \mathbb{R}^+$ *is Lebesgue integrable with power* $\frac{1}{\delta}$, *that is,*
$w \in L^{\frac{1}{\delta}}([0, T], \mathbb{R}^+)$ *with* $\|w\|_{\frac{1}{\delta}} = \left(\int_0^T |w(s)|^{\frac{1}{\delta}}\, ds\right)^{\delta}$.
$(\mathfrak{B}_3)$ $|\mathfrak{f}(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2)| \leq w(\mathfrak{z})$, $\quad \forall(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, *and* $w \in L^{\frac{1}{\delta}}([0, T], \mathbb{R}^+)$, $\delta \in (0, \mathsf{v} - 2)$.

Then problem (2.1.1) has at least one solution on $[0, T]$ *provided that*

$$\|w\|_{\frac{1}{\delta}} \left(\mathfrak{K}_1 - \frac{T^{\mathsf{v}-\delta}}{\Gamma(\mathsf{v})}\left(\frac{1-\delta}{\mathsf{v}-\delta}\right)^{1-\delta} + \mathfrak{K}_2 - \frac{T^{\mathsf{v}-\mathfrak{n}-\delta}}{\Gamma(\mathsf{v}-\mathfrak{n})}\left(\frac{1-\delta}{\mathsf{v}-\mathfrak{n}-\delta}\right)^{1-\delta}\right) < 1, \tag{2.3.4}$$

where $\mathfrak{K}_1, \mathfrak{K}_2$ *are defined in (2.3.1).*

Proof. Selecting $\rho > H_1$, we define $B_\rho = \{\mathfrak{u} \in \mathfrak{X} : \|\mathfrak{u}\|_{\mathfrak{X}} \leq \rho\}$ and define the operators A and B on B_ρ as follows:

$$(A\mathfrak{u})(\mathfrak{z}) = \frac{1}{\Gamma(\mathsf{v})}\int_0^{\mathfrak{z}} (\mathfrak{z} - s)^{\mathsf{v}-1} \mathfrak{f}(s, \mathfrak{u}(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s))ds$$

and

$$\begin{aligned}(B\mathfrak{u})(\mathfrak{z}) = \frac{2\mathfrak{z}}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})}\Bigg(&\frac{\lambda}{\mathsf{v}}\int_0^{\eta} (\eta - s)^{\mathsf{v}} \mathfrak{f}(s, \mathfrak{u}(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s))ds \\ &- \int_0^T (T - s)^{\mathsf{v}-1} \mathfrak{f}(s, \mathfrak{u}(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}(s))ds\Bigg).\end{aligned}$$

For any $z_1, z_2 \in B_\rho$, as in the proof of Theorem 2.3.1, it can be shown that $\|Az_1 + Bz_2\|_{\mathfrak{X}} \leq H_1 < \rho$. This means that $Az_1 + Bz_2 \in B_\rho$. The operator A is completely continuous as in Theorem 2.3.1.

For $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathfrak{X}$, we have

$$\begin{aligned}
&(B\mathfrak{u}_1)(\mathfrak{z}) - (B\mathfrak{u}_2)(\mathfrak{z}) \\
&= \frac{2\mathfrak{z}}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \\
&\quad \times \left(\frac{\lambda}{\mathsf{v}} \int_0^\eta (\eta - s)^{\mathsf{v}} [\mathfrak{f}(s, \mathfrak{u}_1(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s)) - \mathfrak{f}(s, \mathfrak{u}_2(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s))] ds \right. \\
&\quad \left. - \int_0^T (T - s)^{\mathsf{v}-1} [\mathfrak{f}(s, \mathfrak{u}_1(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s)) - \mathfrak{f}(s, \mathfrak{u}_2(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s))] ds \right).
\end{aligned}$$

From condition $(\mathfrak{B}_2)$ together with Hölder's inequality, we obtain

$$\begin{aligned}
&|(B\mathfrak{u}_1)(\mathfrak{z}) - (B\mathfrak{u}_2)(\mathfrak{z})| \\
&\leq \frac{2T}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \\
&\quad \times \left(\frac{\lambda}{\mathsf{v}} \int_0^T (T - s)^{\mathsf{v}} w(s) (|\mathfrak{u}_1(s) - \mathfrak{u}_2(s)| + |{}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s) - {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s)|) \right. \\
&\quad \left. + \int_0^T (T - s)^{\mathsf{v}-1} w(s) (|\mathfrak{u}_1(s) - \mathfrak{u}_2(s)| + |{}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s) - {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s)|) \right) \\
&\leq \frac{2T \|\mathfrak{u}_1 - \mathfrak{u}_2\|_{\mathfrak{X}}}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \left(\frac{\lambda}{\mathsf{v}} \int_0^T (T - s)^{\mathsf{v}} w(s) ds + \int_0^T (T - s)^{\mathsf{v}-1} w(s) ds \right) \\
&\leq \frac{2T \|\mathfrak{u}_1 - \mathfrak{u}_2\|_{\mathfrak{X}} \|w\|_{\frac{1}{\delta}}}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \left[\frac{\lambda}{\mathsf{v}} \left(\int_0^T (T - s)^{\frac{\mathsf{v}}{1-\delta}} ds \right)^{1-\delta} + \left(\int_0^T (T - s)^{\frac{\mathsf{v}-1}{1-\delta}} ds \right)^{1-\delta} \right] \\
&\leq \frac{2T \|w\|_{\frac{1}{\delta}} \|\mathfrak{u}_1 - \mathfrak{u}_2\|_{\mathfrak{X}}}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \left[\frac{\lambda}{\mathsf{v}} \left(\frac{1-\delta}{\mathsf{v} - \delta + 1} \right)^{1-\delta} T^{\mathsf{v}-\delta+1} + \left(\frac{1-\delta}{\mathsf{v} - \delta} \right)^{1-\delta} T^{\mathsf{v}-\delta} \right].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&{}^c\mathcal{D}^{\mathfrak{n}}(B\mathfrak{u}_1)(\mathfrak{z}) - {}^c\mathcal{D}^{\mathfrak{n}}(B\mathfrak{u}_2)(\mathfrak{z}) \\
&= \frac{2\mathfrak{z}^{1-\mathfrak{n}}}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})\Gamma(2 - \mathfrak{n})} \\
&\quad \times \left(\frac{\lambda}{\mathsf{v}} \int_0^\eta (\eta - s)^{\mathsf{v}} [\mathfrak{f}(s, \mathfrak{u}_1(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s)) - \mathfrak{f}(s, \mathfrak{u}_2(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s))] ds \right. \\
&\quad \left. - \int_0^T (T - s)^{\mathsf{v}-1} [\mathfrak{f}(s, \mathfrak{u}_1(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s)) - \mathfrak{f}(s, \mathfrak{u}_2(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s))] ds \right).
\end{aligned}$$

By $(\mathfrak{B}_2)$ and Hölder's inequality, it follows that

$$\begin{aligned}
&|{}^c\mathcal{D}^{\mathfrak{n}}(B\mathfrak{u}_1)(\mathfrak{z}) -{}^c\mathcal{D}^{\mathfrak{n}}(B\mathfrak{u}_2)(\mathfrak{z})| \\
&\leq \frac{2T^{1-\mathfrak{n}}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v})\Gamma(2-\mathfrak{n})} \\
&\quad\times \left(\frac{\lambda}{\mathsf{v}}\int_0^T (T-s)^{\mathsf{v}} w(s)(|\mathfrak{u}_1(s)-\mathfrak{u}_2(s)| + |{}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s) -{}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s)|)ds\right. \\
&\quad\left. + \int_0^T (T-s)^{\mathsf{v}-1} w(s)(|\mathfrak{u}_1(s)-\mathfrak{u}_2(s)| + |{}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s) -{}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s)|)ds\right) \\
&\leq \frac{2T^{1-\mathfrak{n}}\left\|\mathfrak{u}_1-\mathfrak{u}_2\right\|_{\mathfrak{X}}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v})\Gamma(2-\mathfrak{n})}\left(\frac{\lambda}{\mathsf{v}}\int_0^T (T-s)^{\mathsf{v}} w(s)ds + \int_0^T (T-s)^{\mathsf{v}-1} w(s)ds\right) \\
&\leq \frac{2T^{1-\mathfrak{n}}\left\|w\right\|_{\frac{1}{\delta}}\left\|\mathfrak{u}_1-\mathfrak{u}_2\right\|_{\mathfrak{X}}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v})\Gamma(2-\mathfrak{n})}\left[\frac{\lambda}{\mathsf{v}}\left(\frac{1-\delta}{\mathsf{v}-\delta+1}\right)^{1-\delta} T^{\mathsf{v}-\delta+1} + \left(\frac{1-\delta}{\mathsf{v}-\delta}\right)^{1-\delta} T^{\mathsf{v}-\delta}\right].
\end{aligned}$$

Thus, it follows

$$\begin{aligned}
&\left\|B\mathfrak{u}_1 - B\mathfrak{u}_2\right\|_{\mathfrak{X}} \\
&= \sup_{\mathfrak{z}\in[0,T]} |(B\mathfrak{u}_1)(\mathfrak{z}) - (B\mathfrak{u}_2)(\mathfrak{z})| + \sup_{\mathfrak{z}\in[0,T]} |{}^c\mathcal{D}^{\mathfrak{n}}(B\mathfrak{u}_1)(\mathfrak{z}) -{}^c\mathcal{D}^{\mathfrak{n}}(B\mathfrak{u}_2)(\mathfrak{z})| \\
&\leq \left\|w\right\|_{\frac{1}{\delta}}\left(\mathfrak{K}_1 - \frac{T^{\mathsf{v}-\delta}}{\Gamma(\mathsf{v})}\left(\frac{1-\delta}{\mathsf{v}-\delta}\right)^{1-\delta} + \mathfrak{K}_2 - \frac{T^{\mathsf{v}-\mathfrak{n}-\delta}}{\Gamma(\mathsf{v}-\mathfrak{n})}\left(\frac{1-\delta}{\mathsf{v}-\mathfrak{n}-\delta}\right)^{1-\delta}\right) \\
&\quad\times \left\|\mathfrak{u}_1-\mathfrak{u}_2\right\|_{\mathfrak{X}},
\end{aligned}$$

using the condition $(\mathfrak{B}_3)$, we conclude that B is a contraction mapping. Thus all the assumptions of Theorem 1.4.3 are satisfied. Hence the conclusion of Theorem 1.4.3 implies that problem (2.1.1) has at least one solution on $[0, T]$. □

2.3.3 Uniqueness of Solution by Banach Fixed Point Theorem

Our study ends with proving the uniqueness of solution for the problem (2.1.1). The main tool used is Banach fixed point theorem together and also Hölder's inequality.

Theorem 2.3.3. *Let $\mathfrak{f} : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying condition $(\mathfrak{B}_2)$. Then there exists a unique solution for problem (2.1.1) on $[0, T]$ if*

$$\|w\|_{\frac{1}{\delta}} (\mathfrak{K}_1 + \mathfrak{K}_2) < 1,$$

where $\mathfrak{K}_1, \mathfrak{K}_2$ are defined in (2.3.1).

Proof. For $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathfrak{X}$ and for each $\mathfrak{z} \in [0, T]$, by Hölder's inequality, we have

$$\begin{aligned}
&|(\mathfrak{G}\mathfrak{u}_1)(\mathfrak{z}) - (\mathfrak{G}\mathfrak{u}_2)(\mathfrak{z})| \\
&\quad = \Bigg| \frac{1}{\Gamma(\mathsf{v})} \int_0^{\mathfrak{z}} (\mathfrak{z} - s)^{\mathsf{v}-1} [\mathfrak{f}(s, \mathfrak{u}_1(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s)) - \mathfrak{f}(s, \mathfrak{u}_2(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s))] ds \\
&\quad + \frac{2\mathfrak{z}}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \left(\frac{\lambda}{\mathsf{v}} \int_0^{\eta} (\eta - s)^{\mathsf{v}} [\mathfrak{f}(s, \mathfrak{u}_1(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s)) - \mathfrak{f}(s, \mathfrak{u}_2(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s))] ds \right. \\
&\quad \left. - \int_0^T (T - s)^{\mathsf{v}-1} [\mathfrak{f}(s, \mathfrak{u}_1(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s)) - \mathfrak{f}(s, \mathfrak{u}_2(s), {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s))] ds \right) \Bigg| \\
&\quad \le \frac{1}{\Gamma(\mathsf{v})} \int_0^{\mathfrak{z}} (\mathfrak{z} - s)^{\mathsf{v}-1} w(s) \left(|\mathfrak{u}_1(s) - \mathfrak{u}_2(s)| + |{}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s) - {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s)| \right) ds \\
&\quad + \frac{2\mathfrak{z}}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \left(\frac{\lambda}{\mathsf{v}} \int_0^{\eta} (\eta - s)^{\mathsf{v}} w(s) \left(|\mathfrak{u}_1(s) - \mathfrak{u}_2(s)| + |{}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s) - {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s)| \right) ds \right. \\
&\quad \left. + \int_0^T (T - s)^{\mathsf{v}-1} w(s) \left(|\mathfrak{u}_1(s) - \mathfrak{u}_2(s)| + |{}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s) - {}^c\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s)| \right) ds \right) \\
&\quad \le \frac{\|\mathfrak{u}_1 - \mathfrak{u}_2\|_{\mathfrak{X}}}{\Gamma(\mathsf{v})} \left(\left(\int_0^T (T - s)^{\mathsf{v}-1} ds \right)^{\frac{1}{1-\delta}} \right)^{1-\delta} \left(\int_0^T (w(s))^{\frac{1}{\delta}} ds \right)^{\delta} \\
&\qquad + \frac{2T \|\mathfrak{u}_1 - \mathfrak{u}_2\|_{\mathfrak{X}}}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \left[\frac{\lambda}{\mathsf{v}} \left(\left(\int_0^T (T - s)^{\mathsf{v}} ds \right)^{\frac{1}{1-\delta}} \right)^{1-\delta} \left(\int_0^T (w(s))^{\frac{1}{\delta}} ds \right)^{\delta} \right. \\
&\qquad \left. + \left(\left(\int_0^T (T - s)^{\mathsf{v}-1} ds \right)^{\frac{1}{1-\delta}} \right)^{1-\delta} \left(\int_0^T (w(s))^{\frac{1}{\delta}} ds \right)^{\delta} \right] \\
&\le \frac{\|w\|_{\frac{1}{\delta}} \|\mathfrak{u}_1 - \mathfrak{u}_2\|_{\mathfrak{X}} T^{\mathsf{v}-\delta}}{\Gamma(\mathsf{v})} \left(\frac{1 - \delta}{\mathsf{v} - \delta} \right)^{1-\delta} + \frac{2T \|w\|_{\frac{1}{\delta}} \|\mathfrak{u}_1 - \mathfrak{u}_2\|_{\mathfrak{X}}}{(2T - \lambda\eta^2)\Gamma(\mathsf{v})} \\
&\quad \times \left[\frac{\lambda}{\mathsf{v}} \left(\frac{1 - \delta}{\mathsf{v} - \delta + 1} \right)^{1-\delta} T^{\mathsf{v}-\delta+1} + \left(\frac{1 - \delta}{\mathsf{v} - \delta} \right)^{1-\delta} T^{\mathsf{v}-\delta} \right]
\end{aligned}$$

$$= \mathfrak{K}_1 \left\| w \right\|_{\frac{1}{\delta}} \left\| \mathfrak{u}_1 - \mathfrak{u}_2 \right\|_{\mathfrak{X}} .$$

Analogously, we have

$$\begin{aligned}
&\left| {}^{c}\mathcal{D}^{\mathfrak{n}}(\mathfrak{G}\mathfrak{u}_1)(\mathfrak{z}) -{}^{c}\mathcal{D}^{\mathfrak{n}}(\mathfrak{G}\mathfrak{u}_2)(\mathfrak{z}) \right| \\
&\le \frac{1}{\Gamma(\mathsf{v}-\mathfrak{n})} \int_0^{\mathfrak{z}} (\mathfrak{z}-s)^{\mathsf{v}-\mathfrak{n}-1} w(\mathfrak{z}) \left(|\mathfrak{u}_1(s) - \mathfrak{u}_2(s)| + |{}^{c}\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s) -{}^{c}\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s)| \right) ds \\
&\quad + \frac{2\mathfrak{z}^{1-\mathfrak{n}}}{(2T+\lambda\eta^2)\Gamma(\mathsf{v})\Gamma(2-\mathfrak{n})} \left(\frac{\lambda}{\mathsf{v}} \int_0^{\eta} (\eta-s)^{\mathsf{v}} w(\mathfrak{z}) \left(|\mathfrak{u}_1(s) - \mathfrak{u}_2(s)| + |{}^{c}\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s) -{}^{c}\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s)| \right) ds \right. \\
&\quad \left. + \int_0^{T} (T-s)^{\mathsf{v}-1} w(\mathfrak{z}) \left(|\mathfrak{u}_1(s) - \mathfrak{u}_2(s)| + |{}^{c}\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_1(s) -{}^{c}\mathcal{D}^{\mathfrak{n}}\mathfrak{u}_2(s)| \right) ds \right) \\
&\le \frac{\left\| w \right\|_{\frac{1}{\delta}} \left\| \mathfrak{u}_1 - \mathfrak{u}_2 \right\|_{\mathfrak{X}} T^{\mathsf{v}-\mathfrak{n}-\delta}}{\Gamma(\mathsf{v}-\mathfrak{n})} \left(\frac{1-\delta}{\mathsf{v}-\mathfrak{n}-\delta} \right)^{1-\delta} + \frac{2T^{1-\mathfrak{n}} \left\| w \right\|_{\frac{1}{\delta}} \left\| \mathfrak{u}_1 - \mathfrak{u}_2 \right\|_{\mathfrak{X}}}{(2T-\lambda\eta^2)\Gamma(\mathsf{v})\Gamma(2-\mathfrak{n})} \\
&\quad \times \left[\frac{\lambda}{\mathsf{v}} \left(\frac{1-\delta}{\mathsf{v}-\delta+1} \right)^{1-\delta} T^{\mathsf{v}-\delta+1} + \left(\frac{1-\delta}{\mathsf{v}-\delta} \right)^{1-\delta} T^{\mathsf{v}-\delta} \right] \\
&= \mathfrak{K}_2 \left\| w \right\|_{\frac{1}{\delta}} \left\| \mathfrak{u}_1 - \mathfrak{u}_2 \right\|_{\mathfrak{X}} .
\end{aligned}$$

From the above inequalities, we obtain

$$\left\| \mathfrak{G}\mathfrak{u}_1 - \mathfrak{G}\mathfrak{u}_2 \right\|_{\mathfrak{X}} \le (\mathfrak{K}_1 + \mathfrak{K}_2) \left\| w \right\|_{\frac{1}{\delta}} \left\| \mathfrak{u}_1 - \mathfrak{u}_2 \right\|_{\mathfrak{X}} .$$

Given the condition $\left\| w \right\|_{\frac{1}{\delta}} (\mathfrak{K}_1 + \mathfrak{K}_2) < 1$, the operator $\mathfrak{G}$ constitutes a contraction. By Banach fixed point theorem, $\mathfrak{G}$ admits a unique fixed point, ensuring a unique solution to problem (2.1.1). □

Corollary 2.3.4. *Let the continuous function* $\mathfrak{f} : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be such that it satisfies the following condition:*

$(\mathfrak{B}_4)$ $|\mathfrak{f}(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2) - \mathfrak{f}(\mathfrak{z}, \bar{\mathfrak{u}_1}, \bar{\mathfrak{u}_2})| \le H_1 \left(|\mathfrak{u}_1 - \bar{\mathfrak{u}_1}| + |\mathfrak{u}_2 - \bar{\mathfrak{u}_2}| \right)$ *for all* $\mathfrak{z} \in [0,T]$, $\mathfrak{u}_i, \bar{\mathfrak{u}_i} \in \mathbb{R}\, (i = 1, 2)$, *where* $H_1 > 0$ *is a constant. If the inequality*

$$H_1(\Omega_1 + \Omega_2) < 1,$$

holds, with Ω_1, Ω_2 *defined in (2.3.1) and* H_1 *as in (2.3.2). Consequently, problem (2.1.1) has a unique solution on* $[0,T]$.

2.4 Examples

Les us consider problem (2.1.1) with specific data:

$$T=1.5, \qquad \eta=1.25, \qquad \lambda=1, \qquad \mathsf{v}=2.5, \qquad \mathfrak{n}=0.5, \qquad \delta=0.5. \tag{2.4.1}$$

Using the given values of the parameters in (2.3.1) , we find that

$$\begin{aligned} \mathfrak{K}_1 &\simeq 3.477680593, \qquad & \mathfrak{K}_2 &\simeq 3.485012168. \\ \Omega_1 &\simeq 3.301293165, \qquad & \Omega_2 &\simeq 3.4025962053. \end{aligned} \tag{2.4.2}$$

For the illustration of Theorem 2.3.2, let us take

$$\begin{aligned} &\mathfrak{f}\left(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^{c}\mathcal{D}^{1/2}\mathfrak{u}(\mathfrak{z})\right) \\ &= \frac{1}{10(\mathfrak{z}+2)^{1/3}}\left(\frac{1}{5}+\frac{2\left|\mathfrak{u}(\mathfrak{z})\right|}{2+(\mathfrak{u}(\mathfrak{z}))^2}+\arctan\left({}^{c}\mathcal{D}^{1/2}\mathfrak{u}(\mathfrak{z})+1\right)\right) \end{aligned} \tag{2.4.3}$$

in (2.1.1) and note that

$$\begin{aligned} &\left|\mathfrak{f}\left(\mathfrak{z}, \mathfrak{u}_1(\mathfrak{z}), {}^{c}\mathcal{D}^{1/2}\mathfrak{u}_1(\mathfrak{z})\right)-\mathfrak{f}\left(\mathfrak{z}, \mathfrak{u}_2(\mathfrak{z}), {}^{c}\mathcal{D}^{1/2}\mathfrak{u}_2(\mathfrak{z})\right)\right| \\ &\leq \frac{1}{10(\mathfrak{z}+2)^{1/3}}\left(\left|\mathfrak{u}_1-\mathfrak{u}_2\right|+\left|{}^{c}\mathcal{D}^{1/2}\mathfrak{u}_1-{}^{c}\mathcal{D}^{1/2}\mathfrak{u}_2\right|\right). \end{aligned}$$

Here $w(\mathfrak{z})=\frac{1}{10(\mathfrak{z}+2)^{1/3}}$ with $\|w\|_2 \simeq 0.088040917$. Using the values of $\mathfrak{K}_1$ and $\mathfrak{K}_2$ given by (2.4.2), we find that

$$\|w\|_2\left(\mathfrak{K}_1-\frac{T^{\mathsf{v}-\delta}}{\Gamma(\mathsf{v})}\left(\frac{1-\delta}{\mathsf{v}-\delta}\right)^{1-\delta}+\mathfrak{K}_2-\frac{T^{\mathsf{v}-\mathfrak{n}-\delta}}{\Gamma(\mathsf{v}-\mathfrak{n})}\left(\frac{1-\delta}{\mathsf{v}-\mathfrak{n}-\delta}\right)^{1-\delta}\right) \simeq 0.445112708<1.$$

Since the conditions of Theorem 2.3.2 are satisfied, therefore there exists a one solution of problem (2.1.1) with the data (2.4.1) and (2.4.3) on $[0, T]$.

For the illustration of Theorem 2.3.3, we take

$$\begin{aligned} &\mathfrak{f}\left(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^{c}\mathcal{D}^{1/2}\mathfrak{u}(\mathfrak{z})\right) \\ &= \frac{1}{50(\mathfrak{z}+2)^{1/3}}\left(\frac{3}{7}+\ln(|\mathfrak{u}(\mathfrak{z})|+1)+\frac{2\left|{}^{c}\mathcal{D}^{1/2}\mathfrak{u}(\mathfrak{z})\right|}{2+\left|{}^{c}\mathcal{D}^{1/2}\mathfrak{u}(\mathfrak{z})\right|}\right) \end{aligned} \tag{2.4.4}$$

in (2.1.1) and note that

$$\begin{aligned}&\left|\mathfrak{f}\left(\mathfrak{z},\mathfrak{u}_1(\mathfrak{z}),{}^c\mathcal{D}^{1/2}\mathfrak{u}_1(\mathfrak{z})\right)-\mathfrak{f}\left(\mathfrak{z},\mathfrak{u}_2(\mathfrak{z}),{}^c\mathcal{D}^{1/2}\mathfrak{u}_2(\mathfrak{z})\right)\right|\\ &\leq\frac{1}{50(\mathfrak{z}+2)^{1/3}}\left(|\mathfrak{u}_1-\mathfrak{u}_2|+\left|{}^c\mathcal{D}^{1/2}\mathfrak{u}_1-{}^c\mathcal{D}^{1/2}\mathfrak{u}_2\right|\right).\end{aligned}$$

Here $w(\mathfrak{z})=\frac{1}{50(\mathfrak{z}+2)^{1/3}}$ with $\|w\|_2\simeq 0.017608183$. Using the values of $\mathfrak{K}_1$ and $\mathfrak{K}_2$ given by (2.4.2), we find that

$$\|w\|_2\left(\mathfrak{K}_1+\mathfrak{K}_2\right)\simeq 0.122600371<1.$$

As all the conditions of Theorem 2.3.3 are satisfied, therefore there exists a unique solution of problem (2.1.1) with (2.4.1) and (2.4.4) on $[0,T]$.

Next we demonstrate the application of Corollary 2.3.4 and considering

$$\mathfrak{f}(\mathfrak{z},\mathfrak{u}(\mathfrak{z}),{}^c\mathcal{D}^{1/2}\mathfrak{u}(\mathfrak{z}))=H_1\left(\frac{3}{11}+\ln|\mathfrak{u}(\mathfrak{z})|+\left|{}^c\mathcal{D}^{1/2}\mathfrak{u}(\mathfrak{z})\right|\right),\tag{2.4.5}$$

which is a Lipschitz function with Lipschitz constant H_1. Notice that condition $H_1(\Omega_1+\Omega_2)<1$ is verified for $0<H_1<0.149167139$ (Ω_1 and Ω_2 are given by (2.4.2)). Thus, by Corollary 2.3.4, problem (2.1.1) with (2.4.1) and (2.4.5) has a unique solution on $[0,T]$.

Chapter 3

A Study of Caputo Sequential Fractional Differential Equations with Mixed Boundary Conditions

3.1 Introduction

Inspired by the fondational work of [8, 43, 65], this chapter investigated the existence results for sequential FDEs presented in [27] :

$$\begin{cases} ({}^c\mathcal{D}^{\mathsf{v}+1} + \mathtt{k}\,{}^c\mathcal{D}^{\mathsf{v}})\mathfrak{u}(\mathfrak{z}) = \mathfrak{f}(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z})), \quad 1 < \mathsf{v} \leq 2, \mathtt{k} > 0, \mathfrak{z} \in [0,1], \\ \mathfrak{u}(0) + \mathfrak{p}\mathfrak{u}(1) = \mathcal{I}^{\mathsf{v}-1}\mathfrak{u}(\eta) + \mathcal{I}^{\mathsf{v}}\mathfrak{u}(\eta), \quad 0 < \eta < 1, \\ \mathfrak{u}^{'}(0) + \mathfrak{q}\mathfrak{u}^{'}(1) = {}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\eta) + {}^c\mathcal{D}^{\mathsf{v}}\mathfrak{u}(\eta), \quad \mathfrak{p}, \mathfrak{q} \in \mathbb{R}, \\ \mathfrak{u}^{''}(0) = 0, \end{cases} \tag{3.1.1}$$

here ${}^c\mathcal{D}^{\mathsf{v}+1}$, ${}^c\mathcal{D}^{\mathsf{v}}$, and ${}^c\mathcal{D}^{\mathsf{v}-1}$ denote the Caputo fractional derivatives of orders $\mathsf{v}+1$, v, and $\mathsf{v}-1$, respectively, $\mathfrak{f} \in C([0,1]\times\mathbb{R}^2, \mathbb{R})$, and $1+\mathfrak{q}-\frac{\eta^{2-\mathsf{v}}}{\Gamma(3-\mathsf{v})} \neq 0$, $1+\mathfrak{p}-\frac{\mathsf{v}\eta^{\mathsf{v}-1}+\eta^{\mathsf{v}}}{\Gamma(\mathsf{v}+1)} \neq 0$. Examples are also given to illustrate the main results.

Definition 3.1.1. *[46] The sequential fractional derivative of a function $\mathfrak{u}$ can be expressed as*

$$\mathcal{D}^{\mathsf{v}}\mathfrak{u}(\mathfrak{z}) = \mathcal{D}^{\mathsf{v}_1}\mathcal{D}^{\mathsf{v}_2}...\mathcal{D}^{\mathsf{v}_m}\mathfrak{u}(\mathfrak{z}),$$

where $\mathsf{v} = (\mathsf{v}_1, \mathsf{v}_2, ..., \mathsf{v}_m)$ is a multi-index.

3.2 Equivalent Linear System Solutions

In Lemma next, we characterize the general solution of the linear problem associated the problem (3.1.1).

Lemma 3.2.1. *Assume that $\xi \in C([0,1],\mathbb{R})$. The boundary value problem*

$$\begin{cases} ({}^{c}\mathcal{D}^{\mathsf{v}+1} + \mathtt{k}\,{}^{c}\mathcal{D}^{\mathsf{v}})\mathfrak{u}(\mathfrak{z}) = \xi(\mathfrak{z}), \quad \mathfrak{z} \in [0,1], \\ \mathfrak{u}(0) + \mathfrak{p}\mathfrak{u}(1) = \mathcal{I}^{\mathsf{v}-1}\mathfrak{u}(\eta) + \mathcal{I}^{\mathsf{v}}\mathfrak{u}(\eta), \\ \mathfrak{u}'(0) + \mathfrak{q}\mathfrak{u}'(1) =^{c} \mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\eta) +^{c} \mathcal{D}^{\mathsf{v}}\mathfrak{u}(\eta), \\ \mathfrak{u}''(0) = 0, \end{cases} \tag{3.2.1}$$

possesses a unique solution, which can be explicitly expressed as

$$\mathfrak{u}(\mathfrak{z}) = \int_0^{\mathfrak{z}} e^{-\mathtt{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})d\mathfrak{n} + \frac{1}{\Theta_1}\left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2}\right)\Lambda_\xi(\eta) + \frac{1}{\Theta_2}\Delta_\xi(\eta) \tag{3.2.2}$$

where

$$\begin{aligned} \Theta_1 &= 1 + \mathfrak{q} - \frac{\eta^{2-\mathsf{v}}}{\Gamma(3-\mathsf{v})}, \quad \Theta_2 = 1 + \mathfrak{p} - \frac{\mathsf{v}\eta^{\mathsf{v}-1} + \eta^{\mathsf{v}}}{\Gamma(\mathsf{v}+1)}, \quad \Theta_3 = \frac{\eta^{\mathsf{v}}(\eta + \mathsf{v} + 1)}{\Gamma(\mathsf{v}+2)} - \mathfrak{p}, \\ \Lambda_\xi(\eta) &= (\mathtt{k}^2 - \mathtt{k})\int_0^\eta \frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}\left(\int_0^{\mathfrak{n}} e^{-\mathtt{k}(\mathfrak{n}-\mathfrak{r})}\mathcal{I}^{\mathsf{v}}\xi(\vartheta)d\vartheta\right)d\mathfrak{n} \\ &\quad + \int_0^\eta \frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}[(1-\mathtt{k})\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n}) + \mathcal{I}^{\mathsf{v}-1}\xi(\mathfrak{n})]d\mathfrak{n} \\ &\quad + \mathfrak{q}\mathtt{k}\int_0^1 e^{-\mathtt{k}(1-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})d\mathfrak{n} - \mathfrak{q}\mathcal{I}^{\mathsf{v}}\xi(1), \\ \Delta_\xi(\eta) &= \int_0^\eta \left[\frac{(\eta-\mathfrak{n})^{\mathsf{v}-2}}{\Gamma(\mathsf{v}-1)} + \frac{(\eta-\mathfrak{n})^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}\right]\left(\int_0^{\mathfrak{n}} e^{-\mathtt{k}(\mathfrak{n}-\vartheta)}\mathcal{I}^{\mathsf{v}}\xi(\vartheta)d\vartheta\right)d\mathfrak{n} \\ &\quad - \mathfrak{p}\int_0^1 e^{-\mathtt{k}(1-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})d\mathfrak{n}. \end{aligned} \tag{3.2.3}$$

Proof. By Lemma 1.2.6, we find

$$\left(\frac{d}{d\mathfrak{z}}+\mathtt{k}\right)\mathfrak{u}(\mathfrak{z})=\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{z})+a_0+a_1\mathfrak{z}, \tag{3.2.4}$$

where $a_0, a_1 \in \mathbb{R}$. Then (3.2.4) is equivalent to

$$\frac{d}{d\mathfrak{z}}(e^{\mathtt{k}\mathfrak{z}}\mathfrak{u}(\mathfrak{z}))=e^{\mathtt{k}\mathfrak{z}}(\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{z})+a_0+a_1\mathfrak{z}),$$

integrating this equation from 0 to $\mathfrak{z}$, we get

$$e^{\mathtt{k}\mathfrak{z}}\mathfrak{u}(\mathfrak{z})=\int_0^{\mathfrak{z}}e^{\mathtt{k}\mathfrak{n}}\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})d\mathfrak{n}+\left(\frac{a_0}{\mathtt{k}}-\frac{a_1}{\mathtt{k}^2}\right)e^{\mathtt{k}\mathfrak{z}}+\frac{a_1\mathfrak{z}}{\mathtt{k}}e^{\mathtt{k}\mathfrak{z}}+\left(\frac{a_1}{\mathtt{k}^2}-\frac{a_0}{\mathtt{k}}+\mathfrak{u}(0)\right).$$

Hence, we obtain that

$$\mathfrak{u}(\mathfrak{z})=\mathcal{A}+\mathcal{B}\mathfrak{z}+\mathcal{C}e^{-\mathtt{k}\mathfrak{z}}+\int_0^{\mathfrak{z}}e^{-\mathtt{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})d\mathfrak{n}, \tag{3.2.5}$$

where $\mathcal{A}=\frac{a_0}{\mathtt{k}}-\frac{a_1}{\mathtt{k}^2}$, $\mathcal{B}=\frac{a_1}{\mathtt{k}}$ and $\mathcal{C}=\frac{a_1}{\mathtt{k}^2}-\frac{a_0}{\mathtt{k}}+\mathfrak{u}(0)$.

Thus, the second derivative of $\mathfrak{u}$ with respect to $\mathfrak{z}$ can be written as

$$\mathfrak{u}''(\mathfrak{z})=\mathcal{C}\mathtt{k}^2e^{-\mathtt{k}\mathfrak{z}}+\mathtt{k}^2\int_0^{\mathfrak{z}}e^{-\mathtt{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})d\mathfrak{n}-\mathtt{k}\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{z})+\mathcal{I}^{\mathsf{v}-1}\xi(\mathfrak{z}).$$

Using the condition $\mathfrak{u}''(0)=0$, it follows that $\mathcal{C}=0$.

From (3.2.5), we obtain

$$\begin{aligned}{}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\eta)+{}^c\mathcal{D}^{\mathsf{v}}\mathfrak{u}(\eta)=\int_0^{\eta}\frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}\Big((\mathtt{k}^2-\mathtt{k})\int_0^{\mathfrak{n}}e^{-\mathtt{k}(\mathfrak{n}-\vartheta)}\mathcal{I}^{\mathsf{v}}\xi(\vartheta)d\vartheta\\ +(1-\mathtt{k})\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})+\mathcal{I}^{\mathsf{v}-1}\xi(\mathfrak{n})+\mathcal{B}\Big)\,d\mathfrak{n},\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}^{\mathsf{v}-1}\mathfrak{u}(\eta)+\mathcal{I}^{\mathsf{v}}\mathfrak{u}(\eta)=\mathcal{A}\int_0^{\eta}\left(\frac{(\eta-\mathfrak{n})^{\mathsf{v}-2}}{\Gamma(\mathsf{v}-1)}+\frac{(\eta-\mathfrak{n})^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}\right)d\mathfrak{n}\\ +\mathcal{B}\int_0^{\eta}\left(\frac{(\eta-\mathfrak{n})^{\mathsf{v}-2}}{\Gamma(\mathsf{v}-1)}+\frac{(\eta-\mathfrak{n})^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}\right)\mathfrak{n}d\mathfrak{n}\\ +\int_0^{\eta}\left(\frac{(\eta-\mathfrak{n})^{\mathsf{v}-2}}{\Gamma(\mathsf{v}-1)}+\frac{(\eta-\mathfrak{n})^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}\right)\left(\int_0^{\mathfrak{n}}e^{-\mathtt{k}(\mathfrak{n}-\vartheta)}\mathcal{I}^{\mathsf{v}}\xi(\vartheta)d\vartheta\right)d\mathfrak{n}.\end{aligned}$$

The condition $\mathfrak{u}(0)+\mathfrak{p}\mathfrak{u}(1)=\mathcal{I}^{\mathsf{v}-1}\mathfrak{u}(\eta)+\mathcal{I}^{\mathsf{v}}\mathfrak{u}(\eta)$ gives

$$\begin{aligned}
&\mathcal{A}\left(1+\mathfrak{p}-\int_0^{\eta}\left(\frac{(\eta-\mathfrak{n})^{\mathsf{v}-2}}{\Gamma(\mathsf{v}-1)}+\frac{(\eta-\mathfrak{n})^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}\right)d\mathfrak{n}\right)\\
&+\mathcal{B}\left(\mathfrak{p}-\int_0^{\eta}\left(\frac{(\eta-\mathfrak{n})^{\mathsf{v}-2}}{\Gamma(\mathsf{v}-1)}+\frac{(\eta-\mathfrak{n})^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}\right)\mathfrak{n}d\mathfrak{n}\right)\\
&=\int_0^{\eta}\left(\frac{(\eta-\mathfrak{n})^{\mathsf{v}-2}}{\Gamma(\mathsf{v}-1)}+\frac{(\eta-\mathfrak{n})^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}\right)\left(\int_0^{\mathfrak{n}}e^{-\mathsf{k}(\mathfrak{n}-\vartheta)}\mathcal{I}^{\mathsf{v}}\xi(\vartheta)d\vartheta\right)d\mathfrak{n}\\
&\quad-\mathfrak{p}\int_0^1 e^{-\mathsf{k}(1-\mathfrak{n})}I^{\mathsf{v}}\xi(\mathfrak{n})d\mathfrak{n},
\end{aligned}\tag{3.2.6}$$

while the condition $\mathfrak{u}'(0)+\mathfrak{q}\mathfrak{u}'(1)={}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\eta)+{}^{c}\mathcal{D}^{\mathsf{v}}\mathfrak{u}(\eta)$ implies

$$\begin{aligned}
\mathcal{B}\left(1+\mathfrak{q}-\frac{\eta^{2-\mathsf{v}}}{\Gamma(3-\mathsf{v})}\right)&=(\mathsf{k}^2-\mathsf{k})\int_0^{\eta}\frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}\left(\int_0^{\mathfrak{n}}e^{-\mathsf{k}(\mathfrak{n}-\vartheta)}\mathcal{I}^{\mathsf{v}}\xi(\vartheta)d\vartheta\right)d\mathfrak{n}\\
&\quad+\int_0^{\eta}\frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}[(1-\mathsf{k})\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})+\mathcal{I}^{\mathsf{v}-1}\xi(\mathfrak{n})]d\mathfrak{n}\\
&\quad+\mathfrak{q}\mathsf{k}\int_0^1 e^{-\mathsf{k}(1-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})d\mathfrak{n}-\mathfrak{q}\mathcal{I}^{\mathsf{v}}\xi(1).
\end{aligned}\tag{3.2.7}$$

A simultaneous solution of (3.2.6) and (3.2.7) yields to

$$\begin{aligned}
\mathcal{A}=&\frac{1}{\Theta_2}\int_0^{\eta}\left[\frac{(\eta-\mathfrak{n})^{\mathsf{v}-2}}{\Gamma(\mathsf{v}-1)}+\frac{(\eta-\mathfrak{n})^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}\right]\left(\int_0^{\mathfrak{n}}e^{-\mathsf{k}(\mathfrak{n}-\vartheta)}\mathcal{I}^{\mathsf{v}}\xi(\vartheta)d\vartheta\right)d\mathfrak{n}\\
&-\frac{\mathfrak{p}}{\Theta_2}\int_0^1 e^{-\mathsf{k}(1-\mathfrak{n})}I^{\mathsf{v}}\xi(\mathfrak{n})d\mathfrak{n}\\
&+\frac{\Theta_3}{\Theta_1\Theta_2}\left[(\mathsf{k}^2-\mathsf{k})\int_0^{\eta}\frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}\left(\int_0^{\mathfrak{n}}e^{-\mathsf{k}(\mathfrak{n}-\vartheta)}\mathcal{I}^{\mathsf{v}}\xi(\vartheta)d\vartheta\right)d\mathfrak{n}\right.\\
&+\int_0^{\eta}\frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}[(1-\mathsf{k})\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})+\mathcal{I}^{\mathsf{v}-1}\xi(\mathfrak{n})]d\mathfrak{n}\\
&\left.+\mathfrak{q}\mathsf{k}\int_0^1 e^{-\mathsf{k}(1-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})d\mathfrak{n}-\mathfrak{q}\mathcal{I}^{\mathsf{v}}\xi(1)\right],
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}=&\frac{1}{\Theta_1}\left[(\mathsf{k}^2-\mathsf{k})\int_0^{\eta}\frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}\left(\int_0^{\mathfrak{n}}e^{-\mathsf{k}(\mathfrak{n}-\vartheta)}\mathcal{I}^{\mathsf{v}}\xi(\vartheta)d\vartheta\right)d\mathfrak{n}\right.\\
&+\int_0^{\eta}\frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}[(1-\mathsf{k})\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})+\mathcal{I}^{\mathsf{v}-1}\xi(\mathfrak{n})]d\mathfrak{n}\\
&\left.+\mathfrak{q}\mathsf{k}\int_0^1 e^{-\mathsf{k}(1-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})d\mathfrak{n}-\mathfrak{q}\mathcal{I}^{\mathsf{v}}\xi(1)\right].
\end{aligned}$$

Inserting the values of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ into (3.2.5), we get (3.2.2). ☐

Lemma 3.2.2. *Let* $\xi\in C([0,1],\mathbb{R})$ *and define* $\|\xi\|=\sup_{\mathfrak{z}\in[0,1]}|\xi(\mathfrak{z})|$. *Then, we have* *(i)* $\left|\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{z})\right| \leq \frac{1}{\Gamma(\mathsf{v}+1)}\left\|\xi\right\|,$

(ii) $\left|\int_0^{\mathfrak{z}} e^{-\mathsf{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})d\mathfrak{n}\right| \leq \frac{1-e^{-\mathsf{k}}}{\mathsf{k}\Gamma(\mathsf{v}+1)}\left\|\xi\right\|,$

(iii) $\left|\int_0^{\eta} \frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}\left(\int_0^{\mathfrak{n}} e^{-\mathsf{k}(\mathfrak{n}-\vartheta)}\mathcal{I}^{\mathsf{v}}\xi(\vartheta)d\vartheta\right)d\mathfrak{n}\right| \leq \frac{\eta(\mathsf{k}\eta+e^{-\mathsf{k}\eta}-1)}{\mathsf{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)}\left\|\xi\right\|,$

(iv) $\left|\int_0^{\eta} \frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}[(1-\mathsf{k})\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})+\mathcal{I}^{\mathsf{v}-1}\xi(\mathfrak{n})]d\mathfrak{n}\right| \leq \frac{|1-\mathsf{k}|\eta^2+\mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})}\left\|\xi\right\|,$

(v) $\left|\int_0^{\eta}\left[\frac{(\eta-\mathfrak{n})^{\mathsf{v}-2}}{\Gamma(\mathsf{v}-1)}+\frac{(\eta-\mathfrak{n})^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}\right]\left(\int_0^{\mathfrak{n}} e^{-\mathsf{k}(\mathfrak{n}-\vartheta)}\mathcal{I}^{\mathsf{v}}\xi(\vartheta)d\vartheta\right)d\mathfrak{n}\right| \leq \frac{(1-e^{-\mathsf{k}\eta})(\mathsf{v}\eta^{2\mathsf{v}-1}+\eta^{2\mathsf{v}})}{\mathsf{k}[\Gamma(\mathsf{v}+1)]^2}\left\|\xi\right\|,$

where

$\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{z}) = \int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-\eta)^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}\xi(\eta)d\eta, \quad \mathcal{I}^{\mathsf{v}-1}\xi(\mathfrak{z}) = \int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-\eta)^{\mathsf{v}-2}}{\Gamma(\mathsf{v}-1)}\xi(\eta)d\eta.$

Proof. Let $\xi \in C([0,1],\mathbb{R})$ and define $\|\xi\| = \sup_{\mathfrak{z}\in[0,1]}|\xi(\mathfrak{z})|$. Then, we have

(i)

$$\begin{aligned}
\left|\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{z})\right| &= \left|\int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-\mathfrak{n})^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}\xi(\mathfrak{n})d\mathfrak{n}\right| \\
&\leq \int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-\mathfrak{n})^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}\left|\xi(\mathfrak{n})\right|d\mathfrak{n} \\
&\leq \frac{\mathfrak{z}^{\mathsf{v}}}{\Gamma(\mathsf{v}+1)}\left\|\xi\right\| \\
&\leq \frac{1}{\Gamma(\mathsf{v}+1)}\left\|\xi\right\|.
\end{aligned}$$

(ii)

$$\begin{aligned}
\left|\int_0^{\mathfrak{z}} e^{-\mathsf{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})d\mathfrak{n}\right| &\leq \int_0^{\mathfrak{z}} e^{-\mathsf{k}(\mathfrak{z}-\mathfrak{n})}\left|\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n})\right|d\mathfrak{n} \\
&\leq \frac{\|\xi\|}{\Gamma(\mathsf{v}+1)}\int_0^{\mathfrak{z}} e^{-\mathsf{k}(\mathfrak{z}-\mathfrak{n})}d\mathfrak{n} \\
&\leq \frac{1-e^{-\mathsf{k}}}{\mathsf{k}\Gamma(\mathsf{v}+1)}\left\|\xi\right\|.
\end{aligned}$$

(iii)

$$\int_0^{\vartheta} \frac{(\vartheta-\eta)^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}d\eta = \frac{\vartheta^{\mathsf{v}}}{\Gamma(\mathsf{v}+1)}$$

and

$$\int_0^{\mathfrak{n}} e^{-\mathsf{k}(\mathfrak{n}-\vartheta)} \frac{\vartheta^{\mathsf{v}}}{\Gamma(\mathsf{v}+1)} d\vartheta \leq \frac{\mathfrak{n}^{\mathsf{v}}}{\Gamma(\mathsf{v}+1)} \int_0^{\mathfrak{n}} e^{-\mathsf{k}(\mathfrak{n}-\vartheta)} d\vartheta = \frac{\mathfrak{n}^{\mathsf{v}}(1-e^{-\mathsf{k}\mathfrak{n}})}{\mathsf{k}\Gamma(\mathsf{v}+1)}.$$

Therefore

$$\begin{aligned}
&\left| \int_0^{\eta} \frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} \left(\int_0^{\mathfrak{n}} e^{-\mathsf{k}(\mathfrak{n}-\vartheta)} \mathcal{I}^{\mathsf{v}} \xi(\vartheta) d\vartheta \right) d\mathfrak{n} \right| \\
&= \left| \int_0^{\eta} \frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} \left(\int_0^{\mathfrak{n}} e^{-\mathsf{k}(\mathfrak{n}-\vartheta)} \left(\int_0^{\vartheta} \frac{(\vartheta-\eta)^{\mathsf{v}-1}}{\Gamma(\mathsf{v})} \xi(\eta) d\eta \right) d\vartheta \right) d\mathfrak{n} \right| \\
&\leq \|\xi\| \int_0^{\eta} \frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} \frac{\mathfrak{n}^{\mathsf{v}}(1-e^{-\mathsf{k}\mathfrak{n}})}{\mathsf{k}\Gamma(\mathsf{v}+1)} d\mathfrak{n} \\
&\leq \|\xi\| \frac{\eta^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} \frac{\eta^{\mathsf{v}}}{\mathsf{k}\Gamma(\mathsf{v}+1)} \int_0^{\eta} (1-e^{-\mathsf{k}\mathfrak{n}}) d\mathfrak{n} \\
&\leq \frac{\eta(\mathsf{k}\eta + e^{-\mathsf{k}\eta} - 1)}{\mathsf{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} \|\xi\| .
\end{aligned}$$

(iv)

$$\int_0^{\eta-1} \frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} d\mathfrak{n} \leq \int_0^{\eta} \frac{\eta^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} d\mathfrak{n},$$

and

$$\left| \mathcal{I}^{\mathsf{v}-1} \xi(\mathfrak{n}) \right| = \left| \int_0^{\mathfrak{n}} \frac{(\mathfrak{n}-\eta)^{\mathsf{v}-2}}{\Gamma(\mathsf{v}-1)} \xi(\eta d\eta \right| \leq \frac{\mathfrak{n}^{\mathsf{v}-1}}{\Gamma(\mathsf{v})} \|\xi\| .$$

Hence

$$\begin{aligned}
&\left| \int_0^{\eta} \frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} [(1-\mathsf{k})\mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n}) + \mathcal{I}^{\mathsf{v}-1}\xi(\mathfrak{n})] d\mathfrak{n} \right| \\
&\leq \int_0^{\eta} \frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} \left[|1-\mathsf{k}| \left| \mathcal{I}^{\mathsf{v}}\xi(\mathfrak{n}) \right| + \left| \mathcal{I}^{\mathsf{v}-1}\xi(\mathfrak{n}) \right| \right] d\mathfrak{n} \\
&\leq \int_0^{\eta} \frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} \left[\frac{|1-\mathsf{k}| \mathfrak{n}^{\mathsf{v}} + \mathsf{v}\mathfrak{n}^{\mathsf{v}-1}}{\Gamma(\mathsf{v}+1)} \right] \|\xi\| d\mathfrak{n} \\
&\leq \frac{|1-\mathsf{k}| \eta^{\mathsf{v}} + \mathsf{v}\eta^{\mathsf{v}-1}}{\Gamma(\mathsf{v}+1)} \|\xi\| \int_0^{\eta} \frac{(\eta-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} d\mathfrak{n} \\
&\leq \frac{|1-\mathsf{k}| \eta^2 + \mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} \|\xi\| .
\end{aligned}$$

(v)

$$\int_0^{\eta} \frac{(\eta-\mathfrak{n})^{\mathsf{v}-2}}{\Gamma(\mathsf{v}-1)} d\mathfrak{n} = \frac{\eta^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}, \quad \int_0^{\eta} \frac{(\eta-\mathfrak{n})^{\mathsf{v}-1}}{\Gamma(\mathsf{v})} d\mathfrak{n} = \frac{\eta^{\mathsf{v}}}{\Gamma(\mathsf{v}+1)}$$

and

$$\left|\int_0^{\mathfrak{n}} e^{-\mathtt{k}(\mathfrak{n}-\vartheta)}\mathcal{I}^{\mathsf{v}}\xi(\vartheta)d\vartheta\right| \leq \frac{\mathfrak{n}^{\mathsf{v}}(1-e^{-\mathtt{k}\mathfrak{n}})}{\mathtt{k}\Gamma(\mathsf{v}+1)}\left\|\xi\right\|.$$

Hence

$$\begin{aligned}
&\left|\int_0^{\eta}\left[\frac{(\eta-\mathfrak{n})^{\mathsf{v}-2}}{\Gamma(\mathsf{v}-1)}+\frac{(\eta-\mathfrak{n})^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}\right]\left(\int_0^{\mathfrak{n}} e^{-\mathtt{k}(\mathfrak{n}-\vartheta)}\mathcal{I}^{\mathsf{v}}\xi(\vartheta)d\vartheta\right)d\mathfrak{n}\right| \\
&\leq \int_0^{\eta}\left[\frac{(\eta-\mathfrak{n})^{\mathsf{v}-2}}{\Gamma(\mathsf{v}-1)}+\frac{(\eta-\mathfrak{n})^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}\right]\left|\int_0^{\mathfrak{n}} e^{-\mathtt{k}(\mathfrak{n}-\vartheta)}\mathcal{I}^{\mathsf{v}}\xi(\vartheta)d\vartheta\right|d\mathfrak{n} \\
&\leq \frac{\eta^{\mathsf{v}}(1-e^{-\mathtt{k}})}{\mathtt{k}\Gamma(\mathsf{v}+1)}\left\|\xi\right\|\int_0^{\eta}\left[\frac{(\eta-\mathfrak{n})^{\mathsf{v}-2}}{\Gamma(\mathsf{v}-1)}+\frac{(\eta-\mathfrak{n})^{\mathsf{v}-1}}{\Gamma(\mathsf{v})}\right]d\mathfrak{n} \\
&\leq \frac{(1-e^{-\mathtt{k}\eta})(\mathsf{v}\eta^{2\mathsf{v}-1}+\eta^{2\mathsf{v}})}{\mathtt{k}[\Gamma(\mathsf{v}+1)]^2}\left\|\xi\right\|.
\end{aligned}$$

□

From Lemma 3.2.2, we deduce the following result:

Corollary 3.2.3. *Let* $\xi \in C([0,1],\mathbb{R})$ *and define* $\|\xi\| = \sup_{\mathfrak{z}\in[0,1]}|\xi(\mathfrak{z})|$. *Then, we get*

$$\begin{aligned}
|\Lambda_{\xi}(\eta)| &\leq \left(\frac{|\mathtt{k}^2-\mathtt{k}|\,\eta(\mathtt{k}\eta+e^{-\mathtt{k}\eta}-1)}{\mathtt{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)}+\frac{|1-\mathtt{k}|\,\eta^2+\mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})}+\frac{|\mathfrak{q}|\,(2-e^{-\mathtt{k}})}{\Gamma(\mathsf{v}+1)}\right)\left\|\xi\right\|; \\
|\Delta_{\xi}(\eta)| &\leq \left(\frac{(1-e^{-\mathtt{k}\eta})(\mathsf{v}\eta^{2\mathsf{v}-1}+\eta^{2\mathsf{v}})}{\mathtt{k}[\Gamma(\mathsf{v}+1)]^2}+\frac{|\mathfrak{p}|\,(1-e^{-\mathtt{k}})}{\mathtt{k}\Gamma(\mathsf{v}+1)}\right)\left\|\xi\right\|.
\end{aligned}$$

3.3 Existence Results

This section presents the existence results for problem (3.1.1).

Let $\mathfrak{X} = \{\mathfrak{u} \in C([0,1],\mathbb{R}) : {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u} \in C([0,1],\mathbb{R})\}$ be the Banach space of continuous functions on $[0,1]$, endowed with the norm

$$\|\mathfrak{u}\|_{\mathfrak{X}} = \|\mathfrak{u}\| + \|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}\| = \sup_{\mathfrak{z}\in[0,1]}|\mathfrak{u}(\mathfrak{z})| + \sup_{\mathfrak{z}\in[0,1]}|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z})|,$$

where $1 < \mathsf{v} \leq 2$.

According to Lemma 3.2.1, problem (3.1.1) can be rewritten as an equivalent fixed point problem of the form

$$\mathfrak{u} = \mathfrak{G}\mathfrak{u}.$$

$$\begin{aligned}(\mathfrak{G}\mathfrak{u})(\mathfrak{z}) = {} & \int_0^{\mathfrak{z}} e^{-\mathfrak{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathfrak{v}}\mathfrak{f}(\mathfrak{n},\mathfrak{u}(\mathfrak{n}),{}^{c}\mathcal{D}^{\mathfrak{v}-1}\mathfrak{u}(\mathfrak{n}))d\mathfrak{n} + \frac{1}{\varTheta_1}\left(\mathfrak{z}+\frac{\varTheta_3}{\varTheta_2}\right)\Lambda_{\mathfrak{f}(.,\mathfrak{u}(.),{}^{c}\mathcal{D}^{\mathfrak{v}-1}\mathfrak{u}(.))}(\eta) \\ & + \frac{1}{\varTheta_2}\Delta_{\mathfrak{f}(.,\mathfrak{u}(.),{}^{c}\mathcal{D}^{\mathfrak{v}-1}\mathfrak{u}(.))}(\eta).\end{aligned} \tag{3.3.1}$$

For convenience, we let

$$\begin{aligned}\varPi_1 = {} & \frac{(|\Theta_2|+|\mathfrak{p}|)(1-e^{-\mathfrak{k}})}{|\Theta_2|\,\mathfrak{k}\Gamma(\mathfrak{v}+1)} + \frac{|\Theta_2|+|\Theta_3|}{|\Theta_1|\,|\Theta_2|}\left(\frac{\eta(e^{-\mathfrak{k}\eta}+\mathfrak{k}\eta-1)\,|\mathfrak{k}^2-\mathfrak{k}|}{\mathfrak{k}^2\Gamma(2-\mathfrak{v})\Gamma(\mathfrak{v}+1)} + \frac{|1-\mathfrak{k}|\,\eta^2+\mathfrak{v}\eta}{\Gamma(\mathfrak{v}+1)\Gamma(3-\mathfrak{v})}\right. \\ & \left. + \frac{|\mathfrak{q}|\,(2-e^{-\mathfrak{k}})}{\Gamma(\mathfrak{v}+1)}\right) + \frac{(1-e^{-\mathfrak{k}\eta})(\mathfrak{v}\eta^{2\mathfrak{v}-1}+\eta^{2\mathfrak{v}})}{\mathfrak{k}\,|\Theta_2|\,[\Gamma(\mathfrak{v}+1)]^2}, \\ \varPi_2 = {} & \frac{2-e^{-\mathfrak{k}}}{\Gamma(\mathfrak{v}+1)} + \frac{1}{|\Theta_1|}\left(\frac{|\mathfrak{k}^2-\mathfrak{k}|\,\eta(\mathfrak{k}\eta+e^{-\mathfrak{k}\eta}-1)}{\mathfrak{k}^2\Gamma(2-\mathfrak{v})\Gamma(\mathfrak{v}+1)} + \frac{|1-\mathfrak{k}|\,\eta^2+\mathfrak{v}\eta}{\Gamma(\mathfrak{v}+1)\Gamma(3-\mathfrak{v})} + \frac{|\mathfrak{q}|\,(2-e^{-\mathfrak{k}})}{\Gamma(\mathfrak{v}+1)}\right).\end{aligned} \tag{3.3.2}$$

$$\widetilde{\varPi}_1 = \varPi_1 - \frac{1-e^{-\mathfrak{k}}}{\mathfrak{k}\Gamma(\mathfrak{v}+1)}, \qquad \widetilde{\varPi}_2 = \varPi_2 - \frac{2-e^{-\mathfrak{k}}}{\Gamma(\mathfrak{v}+1)}. \tag{3.3.3}$$

3.3.1 Existence Results via Krasnoselskii's Fixed Point Theorem

To apply Krasnoselskii's fixed point theorem (Theorem 1.4.3), we decompose the operator $\mathfrak{G}$ as follows: $\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2$, where,

$$\begin{aligned}(\mathfrak{G}_1\mathfrak{u})(\mathfrak{z}) &= \int_0^{\mathfrak{z}} e^{-\mathfrak{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathfrak{v}}\mathfrak{f}(\mathfrak{n},\mathfrak{u}(\mathfrak{n}),{}^{c}\mathcal{D}^{\mathfrak{v}-1}\mathfrak{u}(\mathfrak{n}))d\mathfrak{n}, \\ (\mathfrak{G}_2\mathfrak{u})(\mathfrak{z}) &= \frac{1}{\varTheta_1}\left(\mathfrak{z}+\frac{\varTheta_3}{\varTheta_2}\right)\Lambda_{\mathfrak{f}(.,\mathfrak{u}(.),{}^{c}\mathcal{D}^{\mathfrak{v}-1}\mathfrak{u}(.))}(\eta) + \frac{1}{\varTheta_2}\Delta_{\mathfrak{f}(.,\mathfrak{u}(.),{}^{c}\mathcal{D}^{\mathfrak{v}-1}\mathfrak{u}(.))}(\eta).\end{aligned}$$

Theorem 3.3.1. *Suppose that* $\mathfrak{f} : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ *is a continuous function satisfying the following condition:*

$(\mathfrak{B}_5)$ *There exists a positive constant* q *such that*

$$|\mathfrak{f}(\mathfrak{z},\mathfrak{u}_1,\mathfrak{u}_2) - \mathfrak{f}(\mathfrak{z},\tilde{\mathfrak{u}}_1,\tilde{\mathfrak{u}}_2)| \leq q\left(\|\mathfrak{u}_1-\tilde{\mathfrak{u}}_1\| + \|\mathfrak{u}_2-\tilde{\mathfrak{u}}_2\|\right),$$

$\forall \mathfrak{z} \in [0,1]$, $\mathfrak{u}_i, \widetilde{\mathfrak{u}}_i \in \mathbb{R}, i = 1, 2$.

$(\mathfrak{B}_6)$ $\forall \mathfrak{z} \in [0,1]$, $\forall\, \mathfrak{u}_1, \mathfrak{u}_2 \in \mathbb{R}, \exists \theta \in C([0,1], \mathbb{R}^+) :\ |\mathfrak{f}(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2)| \leq \theta(\mathfrak{z})$.

Then the problem (3.1.1) *has at least one solution on* $[0,1]$ *if*

$$q\left(\widetilde{\varPi_1} + \frac{\widetilde{\varPi_2}}{\Gamma(3-\mathsf{v})}\right) < 1,$$

where $\widetilde{\varPi_1}, \widetilde{\varPi_2}$ *are given by* (3.3.3).

Proof. Set $\sup_{\mathfrak{z}\in[0,1]} |\mathfrak{u}(\mathfrak{z})| = \|\mathfrak{u}\|$, we fix $\varrho \geq \left(\varPi_1 + \frac{\varPi_2}{\Gamma(3-\mathsf{v})}\right)\|\mathfrak{u}\|$, where $\varPi_1, \varPi_2$ given by (3.3.2) and define the ball $\mathcal{S}_\varrho = \{\mathfrak{u} \in \mathfrak{X} : \|\mathfrak{u}\|_{\mathfrak{X}} \leq \varrho\}$.

We divide the proof of the theorem into three main steps.

Step 1. $\forall \mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{S}_\varrho$, $(\mathfrak{G}_1\mathfrak{u}_1)(\mathfrak{z}) + (\mathfrak{G}_2\mathfrak{u}_2)(\mathfrak{z}) \in \mathcal{S}_\varrho$.

Using Lemma 3.2.2 and Corollary 3.2.3, for every $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{S}_\varrho$, we have

$$\begin{aligned}
&|(\mathfrak{G}_1\mathfrak{u}_1)(\mathfrak{z}) + (\mathfrak{G}_2\mathfrak{u}_2)(\mathfrak{z})| \\
&\leq \left|\int_0^{\mathfrak{z}} e^{-\mathtt{k}(\mathfrak{z}-\mathfrak{n})} \mathcal{I}^{\mathsf{v}} \mathfrak{f}(\mathfrak{n}, \mathfrak{u}_1(\mathfrak{n}), {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1(\mathfrak{n}))d\mathfrak{n}\right| \\
&+ \frac{1}{|\varTheta_1|}\left|\mathfrak{z} + \frac{\varTheta_3}{\varTheta_2}\right| \left|\Lambda_{\mathfrak{f}(.,\mathfrak{u}_2(.),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2(.))}(\eta)\right| + \frac{1}{|\varTheta_2|}\left|\Delta_{\mathfrak{f}(.,\mathfrak{u}_2(.),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2(.))}(\eta)\right| \\
&\leq \frac{1-e^{-\mathtt{k}}}{\mathtt{k}\Gamma(\mathsf{v}+1)}\|\theta\| + \frac{1}{|\varTheta_1|}\left(1 + \frac{|\varTheta_3|}{|\varTheta_2|}\right)\left(\frac{|\mathtt{k}^2 - \mathtt{k}|\,\eta(\mathtt{k}\eta + e^{-\mathtt{k}\eta} - 1)}{\mathtt{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{|1-\mathtt{k}|\,\eta^2 + \mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})}\right. \\
&\quad \left. + \frac{|\mathfrak{q}|\,(2-e^{-\mathtt{k}})}{\Gamma(\mathsf{v}+1)}\right)\|\theta\| + \frac{1}{|\varTheta_2|}\left(\frac{(1-e^{-\mathtt{k}\eta})(\mathsf{v}\eta^{2\mathsf{v}-1} + \eta^{2\mathsf{v}})}{\mathtt{k}[\Gamma(\mathsf{v}+1)]^2} + \frac{|\mathfrak{p}|\,(1-e^{-\mathtt{k}})}{\mathtt{k}\Gamma(\mathsf{v}+1)}\right)\|\theta\| \\
&\leq \left[\frac{(|\Theta_2| + |\mathfrak{p}|)(1-e^{-\mathtt{k}})}{|\Theta_2|\,\mathtt{k}\Gamma(\mathsf{v}+1)} + \frac{|\Theta_2| + |\Theta_3|}{|\Theta_1|\,|\Theta_2|}\left(\frac{\eta(e^{-\mathtt{k}\eta} + \mathtt{k}\eta - 1)\,|\mathtt{k}^2 - \mathtt{k}|}{\mathtt{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{|1-\mathtt{k}|\,\eta^2 + \mathsf{v}\eta)}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})}\right.\right. \\
&\quad \left.\left. + \frac{|\mathfrak{q}|\,(2-e^{-\mathtt{k}})}{\Gamma(\mathsf{v}+1)}\right) + \frac{(1-e^{-\mathtt{k}\eta})(\mathsf{v}\eta^{2\mathsf{v}-1} + \eta^{2\mathsf{v}})}{\mathtt{k}\,|\Theta_2|\,[\Gamma(\mathsf{v}+1)]^2}\right]\|\theta\| \\
&\leq \varPi_1\|\theta\|.
\end{aligned}$$

Thus

$$\|(\mathfrak{G}_1\mathfrak{u}_1) + (\mathfrak{G}_2\mathfrak{u}_2)\| \leq \varPi_1\|\theta\|.$$

Also we have

$$\begin{aligned}
&(\mathfrak{G}'{}_1\mathfrak{u}_1)(\mathfrak{z}) = -\mathtt{k}\int_0^{\mathfrak{z}} e^{-\mathtt{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{n}, \mathfrak{u}_1(\mathfrak{n}), {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1(\mathfrak{n}))d\mathfrak{n} + \mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{z}, \mathfrak{u}_1(\mathfrak{z}), {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1(\mathfrak{z})), \\
&(\mathfrak{G}'{}_2\mathfrak{u}_2)(\mathfrak{z}) = \frac{1}{\varTheta_1}\Lambda_{\mathfrak{f}(.,\mathfrak{u}(.),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(.))}(\eta).
\end{aligned}$$

Hence

$$\begin{aligned}
&\left|(\mathfrak{G}'_1\mathfrak{u}_1)(\mathfrak{z}) + (\mathfrak{G}'_2\mathfrak{u}_2)(\mathfrak{z})\right| \\
&\leq \mathsf{k}\left|\int_0^{\mathfrak{z}} e^{-\mathsf{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{n},\mathfrak{u}_1(\mathfrak{n}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1(\mathfrak{n}))d\mathfrak{n}\right| + \left|\mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{z},\mathfrak{u}_1(\mathfrak{z}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1(\mathfrak{z}))\right| \\
&\quad + \frac{1}{|\varTheta_1|}\left|\Lambda_{\mathfrak{f}(.,\mathfrak{u}(.),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(.))}(\eta)\right| \\
&\leq \frac{1-e^{-\mathsf{k}}}{\Gamma(\mathsf{v}+1)}\|\theta\| + \frac{1}{\Gamma(\mathsf{v}+1)}\|\theta\| + \frac{1}{|\varTheta_1|}\left(\frac{|\mathsf{k}^2-\mathsf{k}|\,\eta(\mathsf{k}\eta+e^{-\mathsf{k}\eta}-1)}{\mathsf{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{|1-\mathsf{k}|\,\eta^2+\mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})}\right. \\
&\quad \left. + \frac{|\mathfrak{q}|\,(2-e^{-\mathsf{k}})}{\Gamma(\mathsf{v}+1)}\right)\|\theta\| \\
&\leq \left[\frac{2-e^{-\mathsf{k}}}{\Gamma(\mathsf{v}+1)} + \frac{1}{|\varTheta_1|}\left(\frac{|\mathsf{k}^2-\mathsf{k}|\,\eta(\mathsf{k}\eta+e^{-\mathsf{k}\eta}-1)}{\mathsf{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{|1-\mathsf{k}|\,\eta^2+\mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} + \frac{|\mathfrak{q}|\,(2-e^{-\mathsf{k}})}{\Gamma(\mathsf{v}+1)}\right)\right]\|\theta\| \\
&\leq \varPi_2\,\|\theta\|.
\end{aligned}$$

By the definition of the Caputo fractional derivative with $1 < \mathsf{v} \leq 2$, we get

$$\begin{aligned}
\left|{}^{c}\mathcal{D}^{\mathsf{v}-1}(\mathfrak{G}_1\mathfrak{u}_1 + \mathfrak{G}_2\mathfrak{u}_2)(\mathfrak{z})\right| &\leq \int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}\left|(\mathfrak{G}'_1\mathfrak{u}_1 + \mathfrak{G}'_2\mathfrak{u}_2)(\mathfrak{n})\right| d\mathfrak{n} \\
&\leq \varPi_2\,\|\theta\|\int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}d\mathfrak{n} \\
&\leq \varPi_2\,\|\theta\|\,\frac{\mathfrak{z}^{2-\mathsf{v}}}{\Gamma(3-\mathsf{v})} \\
&\leq \frac{\varPi_2}{\Gamma(3-\mathsf{v})}\|\theta\|.
\end{aligned}$$

From the above inequalities, we get

$$\begin{aligned}
\|\mathfrak{G}_1\mathfrak{u}_1 + \mathfrak{G}_2\mathfrak{u}_2\|_{\mathfrak{X}} &= \|\mathfrak{G}_1\mathfrak{u}_1 + \mathfrak{G}_2\mathfrak{u}_2\| + \left\|{}^{c}\mathcal{D}^{\mathsf{v}-1}(\mathfrak{G}_1\mathfrak{u}_1 + \mathfrak{G}_2\mathfrak{u}_2)\right\| \\
&\leq \left(\varPi_1 + \frac{\varPi_2}{\Gamma(3-\mathsf{v})}\right)\|\theta\| \\
&\leq \varrho.
\end{aligned}$$

Thus, $\mathfrak{G}_1\mathfrak{u}_1 + \mathfrak{G}_2\mathfrak{u}_2 \in \mathcal{S}_{\varrho}$.

Step 2. The operator $\mathfrak{G}_1 : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is both continuous and compact.

Let $\mathfrak{z}_1, \mathfrak{z}_2 \in [0,1]$ with $\mathfrak{z}_1 < \mathfrak{z}_2$ and $\mathfrak{u} \in \mathcal{S}_{\varrho}$. By using Lemma 3.2.2, one can find

$$\begin{aligned}
|(\mathfrak{G}_1\mathfrak{u})(\mathfrak{z}_2)-(\mathfrak{G}_1\mathfrak{u})(\mathfrak{z}_1)| &= \bigg|\int_0^{\mathfrak{z}_2} e^{-\mathsf{k}(\mathfrak{z}_2-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{n},\mathfrak{u}(\mathfrak{n}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{n}))d\mathfrak{n}\\
&\quad -\int_0^{\mathfrak{z}_1} e^{-\mathsf{k}(\mathfrak{z}_1-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{n},\mathfrak{u}(\mathfrak{n}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{n}))d\mathfrak{n}\bigg|\\
&= \bigg|\int_0^{\mathfrak{z}_1} e^{\mathsf{k}\mathfrak{n}}(e^{-\mathsf{k}\mathfrak{z}_2}-e^{-\mathsf{k}\mathfrak{z}_1})\mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{n},\mathfrak{u}(\mathfrak{n}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{n}))d\mathfrak{n}\\
&\quad +\int_{\mathfrak{z}_1}^{\mathfrak{z}_2} e^{-\mathsf{k}(\mathfrak{z}_2-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{n},\mathfrak{u}(\mathfrak{n}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{n}))d\mathfrak{n}\bigg|\\
&\le \int_0^{\mathfrak{z}_1} e^{\mathsf{k}\mathfrak{n}}\left|e^{-\mathsf{k}\mathfrak{z}_2}-e^{-\mathsf{k}\mathfrak{z}_1}\right|\left|\mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{n},\mathfrak{u}(\mathfrak{n}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{n}))\right|d\mathfrak{n}\\
&\quad +\int_{\mathfrak{z}_1}^{\mathfrak{z}_2} e^{-\mathsf{k}(\mathfrak{z}_2-\mathfrak{n})}\left|\mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{n},\mathfrak{u}(\mathfrak{n}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{n}))\right|d\mathfrak{n}\\
&\le \frac{1}{\Gamma(\mathsf{v}+1)}\bigg(\int_0^{\mathfrak{z}_1} e^{\mathsf{k}\mathfrak{n}}\left|e^{-\mathsf{k}\mathfrak{z}_2}-e^{-\mathsf{k}\mathfrak{z}_1}\right|d\mathfrak{n}\\
&\quad +\int_{\mathfrak{z}_1}^{\mathfrak{z}_2} e^{-\mathsf{k}(\mathfrak{z}_2-\mathfrak{n})}d\mathfrak{n}\bigg)\|\theta\|,
\end{aligned}$$

and

$$\begin{aligned}
&\left|{}^{c}\mathcal{D}^{\mathsf{v}-1}(\mathfrak{G}_1\mathfrak{u})(\mathfrak{z}_2)-{}^{c}\mathcal{D}^{\mathsf{v}-1}(\mathfrak{G}_1\mathfrak{u})(\mathfrak{z}_1)\right|\\
&=\left|\int_0^{\mathfrak{z}_2}\frac{(\mathfrak{z}_2-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}({\mathfrak{G}'}_1\mathfrak{u})(\mathfrak{n})d\mathfrak{n}-\int_0^{\mathfrak{z}_1}\frac{(\mathfrak{z}_1-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}({\mathfrak{G}'}_1\mathfrak{u})(\mathfrak{n})d\mathfrak{n}\right|\\
&=\bigg|\int_0^{\mathfrak{z}_1}\frac{(\mathfrak{z}_2-\mathfrak{n})^{1-\mathsf{v}}-(\mathfrak{z}_1-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}({\mathfrak{G}'}_1\mathfrak{u})(\mathfrak{n})d\mathfrak{n}\\
&\quad+\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\frac{(\mathfrak{z}_2-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}({\mathfrak{G}'}_1\mathfrak{u})(\mathfrak{n})d\mathfrak{n}\bigg|\\
&\le\frac{1}{\Gamma(2-\mathsf{v})}\bigg(\int_0^{\mathfrak{z}_1}\left|(\mathfrak{z}_2-\mathfrak{n})^{1-\mathsf{v}}-(\mathfrak{z}_1-\mathfrak{n})^{1-\mathsf{v}}\right|\left|({\mathfrak{G}'}_1\mathfrak{u})(\mathfrak{n})\right|d\mathfrak{n}\\
&\quad+\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\left|(\mathfrak{z}_2-\mathfrak{n})^{1-\mathsf{v}}\right|\left|({\mathfrak{G}'}_1\mathfrak{u})(\mathfrak{n})\right|d\mathfrak{n}\bigg)\\
&\le\frac{2-e^{-\mathsf{k}}}{\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)}\bigg(\int_0^{\mathfrak{z}_1}\left|(\mathfrak{z}_2-\mathfrak{n})^{1-\mathsf{v}}-(\mathfrak{z}_1-\mathfrak{n})^{1-\mathsf{v}}\right|d\mathfrak{n}\\
&\quad+\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\left|(\mathfrak{z}_2-\mathfrak{n})^{1-\mathsf{v}}\right|d\mathfrak{n}\bigg)\|\theta\|.
\end{aligned}$$

Clearly,

$$|(\mathfrak{G}_1\mathfrak{u})(\mathfrak{z}_2)-(\mathfrak{G}_1\mathfrak{u})(\mathfrak{z}_1)|\to 0\quad\text{and}\quad\left|{}^{c}\mathcal{D}^{\mathsf{v}-1}(\mathfrak{G}_1\mathfrak{u})(\mathfrak{z}_2)-{}^{c}\mathcal{D}^{\mathsf{v}-1}(\mathfrak{G}_1\mathfrak{u})(\mathfrak{z}_1)\right|\to 0$$

independent of $\mathfrak{u}$ as $\mathfrak{z}_1\to\mathfrak{z}_2$. Thus, $\mathfrak{G}_1$ is relatively compact on $\mathcal{S}_\varrho$, and by the Arzelà-Ascoli theorem, it is compact on $\mathcal{S}_\varrho$.

Step 3. The operator $\mathfrak{G}_2 : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is a contraction.

For $\mathfrak{z} \in [0,1]$, $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{S}_\varrho$, we can derive

$$
\begin{aligned}
&|(\mathfrak{G}_2\mathfrak{u}_1)(\mathfrak{z}) - (\mathfrak{G}_2\mathfrak{u}_2)(\mathfrak{z})| \\
&\leq \frac{1}{|\Theta_1|}\left|\mathfrak{z} + \frac{\Theta_3}{\Theta_2}\right| \left|\Lambda_{\mathfrak{f}(.,\mathfrak{u}_1(.),{}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1(.))-\mathfrak{f}(.,\mathfrak{u}_2(.),{}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2(.))}(\eta)\right| \\
&\quad + \frac{1}{|\Theta_2|}\left|\Delta_{\mathfrak{f}(.,\mathfrak{u}_1(.),{}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1(.))-\mathfrak{f}(.,\mathfrak{u}_2(.),{}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2(.))}(\eta)\right| \\
&\leq \frac{1}{|\Theta_1|}\left(1 + \frac{|\Theta_3|}{|\Theta_2|}\right)\left(\frac{|\mathtt{k}^2 - \mathtt{k}|\,\eta(\mathtt{k}\eta + e^{-\mathtt{k}\eta} - 1)}{\mathtt{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{|1-\mathtt{k}|\,\eta^2 + \mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} + \frac{|\mathfrak{q}|\,(2-e^{-\mathtt{k}})}{\Gamma(\mathsf{v}+1)}\right) \\
&\qquad \times q(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \left\|{}^cD^{\mathsf{v}-1}\mathfrak{u}_1 -{}^c D^{\mathsf{v}-1}\mathfrak{u}_2\right\|) \\
&\quad + \frac{1}{|\Theta_2|}\left(\frac{(1-e^{-\mathtt{k}\eta})(\mathsf{v}\eta^{2\mathsf{v}-1} + \eta^{2\mathsf{v}})}{\mathtt{k}[\Gamma(\mathsf{v}+1)]^2} + \frac{|\mathfrak{p}|\,(1-e^{-\mathtt{k}})}{\mathtt{k}\Gamma(\mathsf{v}+1)}\right) q(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \left\|{}^cD^{\mathsf{v}-1}\mathfrak{u}_1 -{}^c D^{\mathsf{v}-1}\mathfrak{u}_2\right\|) \\
&\leq \left[\frac{|\Theta_2| + |\Theta_3|}{|\Theta_1|\,|\Theta_2|}\left(\frac{\eta(e^{-\mathtt{k}\eta} + \mathtt{k}\eta - 1)\,|\mathtt{k}^2 - \mathtt{k}|}{\mathtt{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{|1-\mathtt{k}|\,\eta^2 + \mathsf{v}\eta)}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})}\frac{|\mathfrak{q}|\,(2-e^{-\mathtt{k}})}{\Gamma(\mathsf{v}+1)}\right)\right. \\
&\quad \left.\frac{|\mathfrak{p}|\,(1-e^{-\mathtt{k}})}{|\Theta_2|\,\mathtt{k}\Gamma(\mathsf{v}+1)} + \frac{(1-e^{-\mathtt{k}\eta})(\mathsf{v}\eta^{2\mathsf{v}-1} + \eta^{2\mathsf{v}})}{\mathtt{k}\,|\Theta_2|\,[\Gamma(\mathsf{v}+1)]^2}\right] q(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \left\|{}^cD^{\mathsf{v}-1}\mathfrak{u}_1 -{}^c D^{\mathsf{v}-1}\mathfrak{u}_2\right\|) \\
&\leq q\widetilde{\mathit{\Pi}_1}(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \left\|{}^cD^{\mathsf{v}-1}\mathfrak{u}_1 -{}^c D^{\mathsf{v}-1}\mathfrak{u}_2\right\|).
\end{aligned}
$$

Also

$$
\left|(\mathfrak{G}'_2\mathfrak{u}_1)(\mathfrak{z}) - (\mathfrak{G}'_2\mathfrak{u}_2)(\mathfrak{z})\right| \leq q\widetilde{\mathit{\Pi}_2}(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \left\|{}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1 -{}^c \mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2\right\|).
$$

Which implies that

$$
\begin{aligned}
&\left|{}^c\mathcal{D}^{\mathsf{v}-1}(\mathfrak{G}_2\mathfrak{u}_1)(\mathfrak{z}) -{}^c \mathcal{D}^{\mathsf{v}-1}(\mathfrak{G}_2\mathfrak{u}_2)(\mathfrak{z})\right| \\
&\leq \int_0^{\mathfrak{z}} \frac{(\mathfrak{z} - \mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}\left|(\mathfrak{G}'_2\mathfrak{u}_1)(\mathfrak{n}) - (\mathfrak{G}'_2\mathfrak{u}_2)(\mathfrak{n})\right| d\mathfrak{n} \\
&\leq q\widetilde{\mathit{\Pi}_2}(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \left\|{}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1 -{}^c \mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2\right\|)\int_0^{\mathfrak{z}} \frac{(\mathfrak{z} - \mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} d\mathfrak{n} \\
&\leq q\widetilde{\mathit{\Pi}_2}\left(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \left\|{}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1 -{}^c \mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2\right\|\right)\frac{\mathfrak{z}^{2-\mathsf{v}}}{\Gamma(3-\mathsf{v})} \\
&\leq \frac{q\widetilde{\mathit{\Pi}_2}}{\Gamma(3-\mathsf{v})}\left(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \left\|{}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1 -{}^c \mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2\right\|\right).
\end{aligned}
$$

From the above inequalities, we have

$$
\begin{aligned}
\|\mathfrak{G}_2\mathfrak{u}_1-\mathfrak{G}_2\mathfrak{u}_2\|_{\mathfrak{X}} &= \|\mathfrak{G}_2\mathfrak{u}_1-\mathfrak{G}_2\mathfrak{u}_2\| + \left\|{}^c\mathcal{D}^{\mathsf{v}-1}(\mathfrak{G}_2\mathfrak{u}_1) -{}^c\mathcal{D}^{\mathsf{v}-1}(\mathfrak{G}_2\mathfrak{u}_2)\right\| \\
&\leq q\left(\widetilde{\mathit{\Pi}_1}+\frac{\widetilde{\mathit{\Pi}_2}}{\Gamma(3-\mathsf{v})}\right)\left(\|\mathfrak{u}_1-\mathfrak{u}_2\|+\left\|{}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1-{}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2\right\|\right) \\
&\leq q\left(\widetilde{\mathit{\Pi}_1}+\frac{\widetilde{\mathit{\Pi}_2}}{\Gamma(3-\mathsf{v})}\right)\|\mathfrak{u}_1-\mathfrak{u}_2\|_{\mathfrak{X}}\,.
\end{aligned}
$$

As $q\left(\widetilde{\mathit{\Pi}_1}+\frac{\widetilde{\mathit{\Pi}_2}}{\Gamma(3-\mathsf{v})}\right)<1$, $\mathfrak{G}_2$ is contraction.
By Theorem 1.4.3, there exists $\mathfrak{u}\in\mathcal{S}_{\varrho}$ satisfying

$$
\mathfrak{u}(\mathfrak{z})=(\mathfrak{G}_1\mathfrak{u})(\mathfrak{z})+(\mathfrak{G}_2\mathfrak{u})(\mathfrak{z})=(\mathfrak{G}\mathfrak{u})(\mathfrak{z}),
$$

implying that $\mathfrak{u}$ is a solution of problem (3.1.1). ☐

3.3.2 Existence Results via Leray-Schauder Fixed Point Theorem

The next main result applies Leray-Schauder fixed point theorem (Theorem 1.4.5) to show that there is at least one solution to the problem (3.1.1). This is the objective of the following theorem.

Theorem 3.3.2. *Consider* $\mathfrak{f}\in C([0,1]\times\mathbb{R}^2,\mathbb{R})$ *and assume that*

($\mathfrak{B}_7$) $\forall\,(\mathfrak{z},\mathfrak{u}_1,\mathfrak{u}_2)\in[0,1]\times\mathbb{R}^2$, *there exist* $\mathcal{J}\in C([0,1],\mathbb{R}^+)$ *and a nondecreasing continuous function* $\mathcal{R}:[0,\infty)\to[0,\infty)$ *such that*

$$
|\mathfrak{f}(\mathfrak{z},\mathfrak{u}_1,\mathfrak{u}_2)|\leq\mathcal{J}(\mathfrak{z})\mathcal{R}(\|\mathfrak{u}_1\|+\|\mathfrak{u}_2\|);
$$

($\mathfrak{B}_8$) *there exists a constant* $N>0$ *such that*

$$
\frac{N}{\|\mathcal{J}\|\,\mathcal{R}(N)}>\mathit{\Pi}_1+\frac{\mathit{\Pi}_2}{\Gamma(3-\mathsf{v})},
$$

where $\mathit{\Pi}_1,\mathit{\Pi}_2$ *are given by (3.3.2).*

Then problem (3.1.1) has at least one solution on $[0,1]$.

Proof. Consider the operator $\mathfrak{G} : \mathfrak{X} \to \mathfrak{X}$ defined by (3.3.1). At first, we show that $\mathfrak{G}$ maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For $\rho > 0$, let $\mathfrak{D}_\rho = \{\mathfrak{u} \in C([0,1],\mathbb{R}) : \|\mathfrak{u}\|_{\mathfrak{X}} \leq \rho\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}
&|(\mathfrak{G}\mathfrak{u})(\mathfrak{z})| \\
&\leq \left|\int_0^{\mathfrak{z}} e^{-\mathsf{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{n},\mathfrak{u}(\mathfrak{n}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{n}))d\mathfrak{n}\right| + \left|\frac{1}{\Theta_1}\left(\mathfrak{z}+\frac{\Theta_3}{\Theta_2}\right)\right|\left|\Lambda_{\mathfrak{f}(.,\mathfrak{u}(.),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(.))}(\eta)\right| \\
&\quad + \frac{1}{|\Theta_2|}\left|\Delta_{\mathfrak{f}(.,\mathfrak{u}(.),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(.))}(\eta)\right| \\
&\leq \frac{1-e^{-\mathsf{k}}}{\mathsf{k}\Gamma(\mathsf{v}+1)}\|\mathcal{J}\|\,\mathcal{R}\left(\|\mathfrak{u}\| + \left\|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}\right\|\right) + \frac{1}{|\Theta_1|}\left(1+\frac{|\Theta_3|}{|\Theta_2|}\right)\left(\frac{|\mathsf{k}^2-\mathsf{k}|\,\eta(\mathsf{k}\eta+e^{-\mathsf{k}\eta}-1)}{\mathsf{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)}\right. \\
&\quad \left. + \frac{|1-\mathsf{k}|\,\eta^2+\mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} + \frac{|\mathfrak{q}|\,(2-e^{-\mathsf{k}})}{\Gamma(\mathsf{v}+1)}\right)\|\mathcal{J}\|\,\mathcal{R}\left(\|\mathfrak{u}\| + \left\|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}\right\|\right) \\
&\quad + \frac{1}{|\Theta_2|}\left(\frac{(1-e^{-\mathsf{k}\eta})(\mathsf{v}\eta^{2\mathsf{v}-1}+\eta^{2\mathsf{v}})}{\mathsf{k}[\Gamma(\mathsf{v}+1)]^2} + \frac{|\mathfrak{p}|\,(1-e^{-\mathsf{k}})}{\mathsf{k}\Gamma(\mathsf{v}+1)}\right)\|\mathcal{J}\|\,\mathcal{R}\left(\|\mathfrak{u}\| + \left\|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}\right\|\right) \\
&\leq \left[\frac{(|\Theta_2|+|\mathfrak{p}|)(1-e^{-\mathsf{k}})}{|\Theta_2|\,\mathsf{k}\Gamma(\mathsf{v}+1)} + \frac{|\Theta_2|+|\Theta_3|}{|\Theta_1|\,|\Theta_2|}\left(\frac{\eta(e^{-\mathsf{k}\eta}+\mathsf{k}\eta-1)\,|\mathsf{k}^2-\mathsf{k}|}{\mathsf{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{|1-\mathsf{k}|\,\eta^2+\mathsf{v}\eta)}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})}\right.\right. \\
&\quad \left.\left. + \frac{|\mathfrak{q}|\,(2-e^{-\mathsf{k}})}{\Gamma(\mathsf{v}+1)}\right) + \frac{(1-e^{-\mathsf{k}\eta})(\mathsf{v}\eta^{2\mathsf{v}-1}+\eta^{2\mathsf{v}})}{\mathsf{k}\,|\Theta_2|\,[\Gamma(\mathsf{v}+1)]^2}\right]\|\mathcal{J}\|\,\mathcal{R}\left(\|\mathfrak{u}\| + \left\|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}\right\|\right) \\
&\leq \mathit{\Pi}_1\,\|\mathcal{J}\|\,\mathcal{R}\left(\|\mathfrak{u}\| + \left\|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}\right\|\right) = \mathit{\Pi}_1\,\|\mathcal{J}\|\,\mathcal{R}\left(\|\mathfrak{u}\|_{\mathfrak{X}}\right),
\end{aligned}$$

where $\mathit{\Pi}_1$ are given in (3.3.2).

Hence

$$\|\mathfrak{G}\mathfrak{u}\| \leq \mathit{\Pi}_1\,\|\mathcal{J}\|\,\mathcal{R}\left(\|\mathfrak{u}\|_{\mathfrak{X}}\right).$$

Also we have

$$\begin{aligned}
&\left|(\mathfrak{G}'\mathfrak{u})(\mathfrak{z})\right| \\
&\leq \mathsf{k}\left|\int_0^{\mathfrak{z}} e^{-\mathsf{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{n},\mathfrak{u}(\mathfrak{n}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{n}))d\mathfrak{n}\right| + \left|\mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{z},\mathfrak{u}(\mathfrak{z}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z}))\right| \\
&\quad + \frac{1}{|\Theta_1|}\left|\Lambda_{\mathfrak{f}(.,\mathfrak{u}(.),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(.))}(\eta)\right| \\
&\leq \left[\frac{2-e^{-\mathsf{k}}}{\Gamma(\mathsf{v}+1)} + \frac{1}{|\Theta_1|}\left(\frac{|\mathsf{k}^2-\mathsf{k}|\,\eta(\mathsf{k}\eta+e^{-\mathsf{k}\eta}-1)}{\mathsf{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{|1-\mathsf{k}|\,\eta^2+\mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} + \frac{|\mathfrak{q}|\,(2-e^{-\mathsf{k}})}{\Gamma(\mathsf{v}+1)}\right)\right] \\
&\qquad \times \|\mathcal{J}\|\,\mathcal{R}\left(\|\mathfrak{u}\| + \left\|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}\right\|\right) \\
&\leq \mathit{\Pi}_2\,\|\mathcal{J}\|\,\mathcal{R}\left(\|\mathfrak{u}\| + \left\|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}\right\|\right) = \mathit{\Pi}_2\,\|\mathcal{J}\|\,\mathcal{R}\left(\|\mathfrak{u}\|_{\mathfrak{X}}\right),
\end{aligned}$$

where $\mathit{\Pi}_2$ given by (3.3.2).

By Definition 1.2.4 for $\mathsf{v} \in (1,2]$, we get

$$\begin{aligned}
\left|{}^{c}\mathcal{D}^{\mathsf{v}-1}(\mathfrak{G}\mathfrak{u})(\mathfrak{z})\right| &\leq \int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} \left|(\mathfrak{G}'\mathfrak{u})(\mathfrak{n})\right| d\mathfrak{n} \\
&\leq \mathit{\Pi}_2 \left\|\mathcal{J}\right\| \mathcal{R}\left(\|\mathfrak{u}\|_{\mathfrak{x}}\right) \int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} d\mathfrak{n} \\
&\leq \mathit{\Pi}_2 \left\|\mathcal{J}\right\| \mathcal{R}\left(\|\mathfrak{u}\|_{\mathfrak{x}}\right) \frac{\mathfrak{z}^{2-\mathsf{v}}}{\Gamma(3-\mathsf{v})} \\
&\leq \frac{\mathit{\Pi}_2}{\Gamma(3-\mathsf{v})} \left\|\mathcal{J}\right\| \mathcal{R}\left(\|\mathfrak{u}\|_{\mathfrak{x}}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\|\mathfrak{G}\mathfrak{u}\|_{\mathfrak{x}} = \|\mathfrak{G}\mathfrak{u}\| + \left\|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{G}\mathfrak{u}\right\| &\leq \left(\mathit{\Pi}_1 + \frac{\mathit{\Pi}_2}{\Gamma(3-\mathsf{v})}\right) \left\|\mathcal{J}\right\| \mathcal{R}\left(\|\mathfrak{u}\|_{\mathfrak{x}}\right) \\
&\leq \left(\mathit{\Pi}_1 + \frac{\mathit{\Pi}_2}{\Gamma(3-\mathsf{v})}\right) \left\|\mathcal{J}\right\| \mathcal{R}(\rho).
\end{aligned}$$

Next we show that $\mathfrak{G}$ maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $\mathfrak{z}_1, \mathfrak{z}_2 \in [0,1]$ with $\mathfrak{z}_1 < \mathfrak{z}_2$ and $\mathfrak{u} \in \mathfrak{D}_\rho$, where $\mathfrak{D}_\rho$ is a bounded set of $C([0,1],\mathbb{R})$. Then we obtain

$$\begin{aligned}
&\left|(\mathfrak{G}\mathfrak{u})(\mathfrak{z}_2) - (\mathfrak{G}\mathfrak{u})(\mathfrak{z}_1)\right| \\
&\leq \left|\int_0^{\mathfrak{z}_2} e^{-\mathsf{k}(\mathfrak{z}_2-\mathfrak{n})} \mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{n},\mathfrak{u}(\mathfrak{n}), {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{n}))d\mathfrak{n} - \int_0^{\mathfrak{z}_1} e^{-\mathsf{k}(\mathfrak{z}_1-\mathfrak{n})} \mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{n},\mathfrak{u}(\mathfrak{n}), {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{n}))d\mathfrak{n}\right| \\
&\quad + \left|\frac{\mathfrak{z}_2-\mathfrak{z}_1}{\mathit{\Theta}_1}\right| \left|\Lambda_{\mathfrak{f}(.,\mathfrak{u}(.),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(.))}(\eta)\right| \\
&\leq \int_0^{\mathfrak{z}_1} e^{\mathsf{k}\mathfrak{n}} \left|e^{-\mathsf{k}\mathfrak{z}_2} - e^{-\mathsf{k}\mathfrak{z}_1}\right| \left|\mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{n},\mathfrak{u}(\mathfrak{n}), {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{n}))\right| d\mathfrak{n} \\
&\quad + \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} e^{-\mathsf{k}(\mathfrak{z}_2-\mathfrak{n})} \left|\mathcal{I}^{\mathsf{v}}\mathfrak{f}(\mathfrak{n},\mathfrak{u}(\mathfrak{n}), {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{n}))\right| d\mathfrak{n} \\
&\quad + \left|\frac{\mathfrak{z}_2-\mathfrak{z}_1}{\mathit{\Theta}_1}\right| \left(\frac{\left|\mathsf{k}^2-\mathsf{k}\right| \eta(\mathsf{k}\eta + e^{-\mathsf{k}\eta} - 1)}{\mathsf{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{\left|1-\mathsf{k}\right| \eta^2 + \mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} + \frac{\left|\mathsf{q}\right| (2-e^{-\mathsf{k}})}{\Gamma(\mathsf{v}+1)}\right) \\
&\qquad \times \left\|\mathcal{J}\right\| \mathcal{R}(\|\mathfrak{u}\|_{\mathfrak{x}}) \\
&\leq \left(\int_0^{\mathfrak{z}_1} e^{\mathsf{k}\mathfrak{n}} \left|e^{-\mathsf{k}\mathfrak{z}_2} - e^{-\mathsf{k}\mathfrak{z}_1}\right| d\mathfrak{n} + \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} e^{-\mathsf{k}(\mathfrak{z}_2-\mathfrak{n})} d\mathfrak{n}\right) \frac{\left\|\mathcal{J}\right\| \mathcal{R}(\|\mathfrak{u}\|_{\mathfrak{x}})}{\Gamma(\mathsf{v}+1)} \\
&\quad + \left|\frac{\mathfrak{z}_2-\mathfrak{z}_1}{\mathit{\Theta}_1}\right| \left(\frac{\left|\mathsf{k}^2-\mathsf{k}\right| \eta(\mathsf{k}\eta + e^{-\mathsf{k}\eta} - 1)}{\mathsf{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{\left|1-\mathsf{k}\right| \eta^2 + \mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} + \frac{\left|\mathsf{q}\right| (2-e^{-\mathsf{k}})}{\Gamma(\mathsf{v}+1)}\right) \\
&\qquad \times \left\|\mathcal{J}\right\| \mathcal{R}(\|\mathfrak{u}\|_{\mathfrak{x}}).
\end{aligned}$$

Also

$$\begin{aligned}
&\left|{}^{c}\mathcal{D}^{\mathsf{v}-1}(\mathfrak{G}\mathfrak{u})(\mathfrak{z}_2)-{}^{c}\,\mathcal{D}^{\mathsf{v}-1}(\mathfrak{G}\mathfrak{u})(\mathfrak{z}_1)\right|\\
&=\left|\int_0^{\mathfrak{z}_2}\frac{(\mathfrak{z}_2-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}(\mathfrak{G}'\mathfrak{u})(\mathfrak{n})d\mathfrak{n}-\int_0^{\mathfrak{z}_1}\frac{(\mathfrak{z}_1-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}(\mathfrak{G}'\mathfrak{u})(\mathfrak{n})d\mathfrak{n}\right|\\
&\leq\int_0^{\mathfrak{z}_1}\frac{|(\mathfrak{z}_2-\mathfrak{n})^{1-\mathsf{v}}-(\mathfrak{z}_1-\mathfrak{n})^{1-\mathsf{v}}|}{\Gamma(2-\mathsf{v})}\left|(\mathfrak{G}'\mathfrak{u})(\mathfrak{n})\right|d\mathfrak{n}+\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\frac{|(\mathfrak{z}_2-\mathfrak{n})^{1-\mathsf{v}}|}{\Gamma(2-\mathsf{v})}\left|(\mathfrak{G}'\mathfrak{u})(\mathfrak{n})\right|d\mathfrak{n}\\
&\leq\left(\int_0^{\mathfrak{z}_1}\frac{|(\mathfrak{z}_2-\mathfrak{n})^{1-\mathsf{v}}-(\mathfrak{z}_1-\mathfrak{n})^{1-\mathsf{v}}|}{\Gamma(2-\mathsf{v})}d\mathfrak{n}+\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\frac{|(\mathfrak{z}_2-\mathfrak{n})^{1-\mathsf{v}}|}{\Gamma(2-\mathsf{v})}d\mathfrak{n}\right)\mathit{\Pi}_2\left\|\mathcal{J}\right\|\mathcal{R}(\|\mathfrak{u}\|_{\mathfrak{X}}).
\end{aligned}$$

Clearly, the right-hand side of the above inequality tends to zero independently of $\mathfrak{u}\in\mathfrak{D}_\rho$ as $\mathfrak{z}_2-\mathfrak{z}_1\to 0$. Since $\mathfrak{G}$ satisfies the above conditions, it follows from the Arzelà-Ascoli theorem that $\mathfrak{G}: C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ is completely continuous. To verify the hypotheses of the Leray-Schauder nonlinear alternative theorem, it is sufficient to show that the set of all solutions to the equation

$$\mathfrak{u}=\lambda\mathfrak{G}\mathfrak{u},\quad \lambda\in(0,1),$$

is bounded. Suppose that $\mathfrak{u}$ is a solution; then, by reasoning similar to that used to establish the boundedness of $\mathfrak{G}$, we can obtain

$$|\mathfrak{u}(\mathfrak{z})|=|\lambda(\mathfrak{G}\mathfrak{u})(\mathfrak{z})|\leq\mathit{\Pi}_1\left\|\mathcal{J}\right\|\mathcal{R}(\|\mathfrak{u}\|_{\mathfrak{X}}).$$

Moreover, we have

$$\left|\mathfrak{u}'(\mathfrak{z})\right|=\left|\lambda(\mathfrak{G}'\mathfrak{u})(\mathfrak{z})\right|\leq\mathit{\Pi}_2\left\|\mathcal{J}\right\|\mathcal{R}(\|\mathfrak{u}\|_{\mathfrak{X}}).$$

According to Definition 1.2.4, for $1<\mathsf{v}\leq 2$, we obtain

$$\left|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z})\right|=\left|\lambda{}^{c}\mathcal{D}^{\mathsf{v}-1}(\mathfrak{G}\mathfrak{u})(\mathfrak{z})\right|\leq\frac{\mathit{\Pi}_2}{\Gamma(3-\mathsf{v})}\left\|\mathcal{J}\right\|\mathcal{R}(\|\mathfrak{u}\|_{\mathfrak{X}}).$$

Therefore

$$\|\mathfrak{u}\|_{\mathfrak{X}}=\|\mathfrak{u}\|+\left\|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}\right\|\leq\left(\mathit{\Pi}_1+\frac{\mathit{\Pi}_2}{\Gamma(3-\mathsf{v})}\right)\left\|\mathcal{J}\right\|\mathcal{R}(\|\mathfrak{u}\|_{\mathfrak{X}}).$$

Thus,

$$\frac{\|\mathfrak{u}\|_{\mathfrak{X}}}{\|\mathcal{J}\|\,\mathcal{R}(\|\mathfrak{u}\|_{\mathfrak{X}})} \leq \varPi_1 + \frac{\varPi_2}{\Gamma(3-\mathsf{v})}.$$

In light of $(\mathfrak{B}_8)$, there exists a constant N such that $\|\mathfrak{u}\|_{\mathfrak{X}} \neq N$.

We define

$$\mathcal{V} = \left\{\mathfrak{u} \in C([0,1],\mathbb{R}) : \|\mathfrak{u}\|_{\mathfrak{X}} < N\right\}.$$

Observe that the operator $\mathfrak{G} : \overline{\mathcal{V}} \to C([0,1],\mathbb{R})$ is continuous and completely continuous. Based on the choice of $\mathcal{V}$, there exists no $\mathfrak{u} \in \partial\mathcal{V}$ such that $\mathfrak{u} = \lambda\mathfrak{G}\mathfrak{u}$ for some $\lambda \in (0,1)$. Thus, by Theorem 1.4.5, we conclude that $\mathfrak{G}$ has a fixed point $\mathfrak{u} \in \overline{\mathcal{V}}$ which is a solution of the problem (3.1.1). $\square$

3.4 Examples

Example 3.4.1. *We examine the following sequential fractional boundary value problem involving the Caputo-type derivative:*

$$\begin{cases} ({}^{c}\mathcal{D}^{\frac{5}{2}} + 2\,{}^{c}\mathcal{D}^{\frac{3}{2}})\mathfrak{u}(\mathfrak{z}) = \mathfrak{f}\Big(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}(\mathfrak{z})\Big), \quad \mathfrak{z} \in [0,1], \\ \mathfrak{u}(0) + \mathfrak{u}(1) = \mathcal{I}^{\frac{1}{2}}\mathfrak{u}\Big(\frac{1}{2}\Big) + \mathcal{I}^{\frac{3}{2}}\mathfrak{u}\Big(\frac{1}{2}\Big), \\ \mathfrak{u}'(0) + \mathfrak{u}'(1) = {}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}\Big(\frac{1}{2}\Big) + {}^{c}\mathcal{D}^{\frac{3}{2}}\mathfrak{u}\Big(\frac{1}{2}\Big), \\ \mathfrak{u}''(0) = 0, \end{cases} \tag{3.4.1}$$

here $\mathsf{v} = \frac{3}{2}$, $\mathtt{k} = 2$, $\mathfrak{p} = \mathfrak{q} = 1$, $\eta = \frac{1}{2}$.

Using the specified values, we obtain $\varTheta_1 \simeq 1.202115439$, $\varTheta_2 \simeq 0.936153918$, $\varTheta_3 \simeq -0.202115439$, *where* $\varTheta_1, \varTheta_2$ *and* $\varTheta_3$ *defined by (3.2.3).*

We take

$\mathfrak{f}(\mathfrak{z},\mathfrak{u}(\mathfrak{z}),{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}(\mathfrak{z}))=\frac{2}{5(\mathfrak{z}^2+42)}\left(\cos(\mathfrak{u}(\mathfrak{z})+1)+\frac{3\left|{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}(\mathfrak{z})\right|}{3+{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}(\mathfrak{z})}+e^{-\mathfrak{z}}\sin\mathfrak{z}\right)$ *in (3.4.1). Then*

$$\begin{aligned}
&\left|\mathfrak{f}(\mathfrak{z},\mathfrak{u}_1(\mathfrak{z}),{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}_1(\mathfrak{z}))-\mathfrak{f}(\mathfrak{z},\mathfrak{u}_2(\mathfrak{z}),{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}_2(\mathfrak{z}))\right|\\
&\leq\frac{2}{5(\mathfrak{z}^2+42)}\left(\left|\cos(\mathfrak{u}_1(\mathfrak{z})+1)-\cos(\mathfrak{u}_2(\mathfrak{z})+1)\right|+\left|\frac{3\left|{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}_1(\mathfrak{z})\right|}{3+{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}_1(\mathfrak{z})}-\frac{3\left|{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}_2(\mathfrak{z})\right|}{3+{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}_2(\mathfrak{z})}\right|\right)\\
&\leq\frac{2}{5(\mathfrak{z}^2+42)}\left(\left|\mathfrak{u}_1(\mathfrak{z})-\mathfrak{u}_2(\mathfrak{z})\right|+\left|{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}_1(\mathfrak{z})-{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}_2(\mathfrak{z})\right|\right)\\
&\leq q\left(\|\mathfrak{u}_1-\mathfrak{u}_2\|+\left\|{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}_1-{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}_2\right\|\right),
\end{aligned}$$

with $q=\frac{1}{105}$,

and

$$\begin{aligned}
\left|\mathfrak{f}(\mathfrak{z},\mathfrak{u}(\mathfrak{z}),{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}(\mathfrak{z}))\right|&=\left|\frac{2}{5(\mathfrak{z}^2+42)}\right|\left|\cos(\mathfrak{u}(\mathfrak{z})+1)+\frac{3\left|{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}(\mathfrak{z})\right|}{3+{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}(\mathfrak{z})}+e^{-\mathfrak{z}}\sin\mathfrak{z}\right|\\
&\leq\frac{2(4+e^{-\mathfrak{z}}\sin\mathfrak{z})}{5(\mathfrak{z}^2+42)}=\theta(\mathfrak{z}).
\end{aligned}$$

We found $\widetilde{\Pi}_1\simeq 2.7597533428$ *and* $\widetilde{\Pi}_2\simeq 1.905439962$ *(*$\widetilde{\Pi}_1$, $\widetilde{\Pi}_2$ *defined by (3.3.3)). Further* $q\left(\widetilde{\Pi}_1+\frac{\widetilde{\Pi}_2}{\Gamma(3-\mathrm{v})}\right)\simeq 0.0467601152<1$. *Thus, all the conditions of Theorem 3.3.1 are fulfilled. Hence, the problem (3.4.1) has a solution on* $[0,1]$.

Example 3.4.2. *Consider the problem (3.4.1) and*

$$\mathfrak{f}(\mathfrak{z},\mathfrak{u}(\mathfrak{z}),{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}(\mathfrak{z}))=\frac{1}{2\sqrt{\mathfrak{z}+729}}\left(\tan^{-1}(\mathfrak{u}(\mathfrak{z})+1)+\ln(\left|{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}(\mathfrak{z})\right|+2)\right).$$

Clearly, we get

$$\begin{aligned}
\left|\mathfrak{f}(\mathfrak{z},\mathfrak{u}(\mathfrak{z}),{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}(\mathfrak{z}))\right|&\leq\left|\frac{1}{2\sqrt{\mathfrak{z}+729}}\right|\left|\tan^{-1}(\mathfrak{u}(\mathfrak{z})+1)+\ln(\left|{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}(\mathfrak{z})\right|+2)\right|\\
&\leq\frac{1}{2\sqrt{\mathfrak{z}+729}}\left(|\mathfrak{u}(\mathfrak{z})|+\left|{}^{c}\mathcal{D}^{\frac{1}{2}}\mathfrak{u}(\mathfrak{z})\right|+3\right)\\
&\leq\mathcal{J}(\mathfrak{z})\mathcal{R}(\|\mathfrak{u}\|_{\mathfrak{X}}),
\end{aligned}$$

where $\mathcal{J}(\mathfrak{z})=\frac{1}{2\sqrt{\mathfrak{z}+729}}$, $\mathcal{R}(\|\mathfrak{u}\|_{\mathfrak{X}})=\|\mathfrak{u}\|_{\mathfrak{X}}+3$.

With the above assumption, we can obtain

$\mathit{\Pi}_1 \simeq 3.0849765598$, $\mathit{\Pi}_2 \simeq 3.308139177$ *($\mathit{\Pi}_1$, $\mathit{\Pi}_2$ defined by (3.3.2)),* $\|\mathcal{J}\| = \frac{1}{54}$. *Applying condition* $(\mathfrak{B}_8)$, *we obtain* $N > 0.4334990912$. *Thus, by Theorem 3.3.2, problem* (3.4.1) *admits a solution on the interval* $[0, 1]$.

Chapter 4

Mixed Boundary Value Problems for Caputo Sequential Fractional Differential Inclusions

4.1 Introduction

In [3], B. Ahmad et al. investigated the following sequential fractional differential inclusions (SFDI) of Caputo type, subject to nonlocal boundary conditions involving Riemann–Liouville fractional integrals:

$$\begin{cases} ({}^{c}\mathcal{D}^{\mathsf{v}} + \mathtt{k}\,{}^{c}\mathcal{D}^{\mathsf{v}-1})\mathfrak{u}(\mathfrak{z}) \in G\left(\mathfrak{z}, \mathfrak{u}(\mathfrak{z})\right), \quad \mathfrak{z} \in [0,1],\ 2 < \mathsf{v} \le 3,\ \mathtt{k} > 0, \\ \mathfrak{u}(0) = 0,\ \mathfrak{u}'(0) = 0,\ \mathfrak{u}\left(\varsigma\right) = \upsilon \int_0^{\eta} \frac{(\eta - s)^{\varpi - 1}}{\Gamma(\varpi)} \mathfrak{u}\left(s\right) ds,\ 0 < \eta < \varsigma < 1,\ \varpi, \\ \upsilon > 0. \end{cases}$$

In [21], S. Gao et al. investigates an affine-periodic boundary value problem with Caputo Sequential fractional derivatives, proving the existence of solutions for the associated differential inclusion via fixed-point theorems and set-valued analysis given by

$$\begin{cases} {}^{c}\mathcal{D}^{\mathsf{v}+1}\mathfrak{u}(\mathfrak{z}) + \mathtt{k}{}^{c}\mathcal{D}^{\mathsf{v}}\mathfrak{u}(\mathfrak{z}) \in G(\mathfrak{z}, \mathfrak{u}(\mathfrak{z})), \quad 0 < \mathsf{v} < 1, \mathfrak{z} \in [0, T], \\ \mathfrak{u}(T) = \tilde{\lambda}\mathfrak{u}(0), \quad \mathfrak{u}^{'}(T) = \tilde{\lambda}\mathfrak{u}^{'}(0), \end{cases}$$

where $\mathtt{k} \in \mathbb{R}$ and $\tilde{\lambda}$ are constants that satisfy $\tilde{\lambda} \neq 1$, $\tilde{\lambda} \neq e^{-\mathtt{k}T}$. $G : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multifunction.

In this chapter, we study the multivalued variant of problem (3.1.1) introduced in Chapter 3 [28]:

$$\begin{cases} ({}^{c}\mathcal{D}^{\mathsf{v}+1} + \mathtt{k}{}^{c}\mathcal{D}^{\mathsf{v}})\mathfrak{u}(\mathfrak{z}) \in G(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z})), \quad 1 < \mathsf{v} \leq 2, \mathtt{k} > 0, \mathfrak{z} \in [0, 1], \\ \mathfrak{u}(0) + \mathfrak{p}\mathfrak{u}(1) = \mathcal{I}^{\mathsf{v}-1}\mathfrak{u}(\eta) + \mathcal{I}^{\mathsf{v}}\mathfrak{u}(\eta), \quad 0 < \eta < 1, \\ \mathfrak{u}^{'}(0) + \mathfrak{q}\mathfrak{u}^{'}(1) = {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\eta) + {}^{c}\mathcal{D}^{\mathsf{v}}\mathfrak{u}(\eta), \quad \mathfrak{p}, \mathfrak{q} \in \mathbb{R}, \\ \mathfrak{u}^{''}(0) = 0, \end{cases} \tag{4.1.1}$$

here ${}^{c}\mathcal{D}^{\mathsf{v}+1}$, ${}^{c}\mathcal{D}^{\mathsf{v}}, {}^{c}\mathcal{D}^{\mathsf{v}-1}$ are the Caputo fractional derivatives of order $\mathsf{v}+1$, v, and $\mathsf{v}-1$ respectively, $G : [0, 1] \times \mathbb{R}^2 \longrightarrow \mathcal{P}(\mathbb{R})$ be a multivalued map, where $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and $1 + \mathfrak{q} - \frac{\eta^{2-\mathsf{v}}}{\Gamma(3-\mathsf{v})} \neq 0$, $1 + \mathfrak{p} - \frac{\mathsf{v}\eta^{\mathsf{v}-1}+\eta^{\mathsf{v}}}{\Gamma(\mathsf{v}+1)} \neq 0$. We discuss the existence of solutions for problem (4.1.1) when the multivalued map $G(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^{c}D^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z}))$ is convex or nonconvex set-valued by using Krasnoselskii's fixed-point theorem for multivalued maps, and Wegrzyk's fixed-point theorem for generalized contractions, we establish new existence criteria under mild assumptions. The applicability of our results is illustrated through numerical examples.

Remark 4.1.1. *This chapter is based on the general solution of the linear problem associated with problem (3.1.1), as stated in Lemma 3.2.1.*

Definition 4.1.2. *A function $\mathfrak{u} \in AC^1([0, 1], \mathbb{R})$ is solution of the problem (4.1.1) if there exists a function $\mathscr{V} \in L^1([0, 1], \mathbb{R})$ with $\mathscr{V}(\mathfrak{z}) \in G(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z}))$ a.e. on $[0, 1]$ such that*

$\mathfrak{u}(0) + \mathfrak{p}\mathfrak{u}(1) = \mathcal{I}^{\mathsf{v}-1}\mathfrak{u}(\eta) + \mathcal{I}^{\mathsf{v}}\mathfrak{u}(\eta)$, $\mathfrak{u}^{'}(0) + \mathfrak{q}\mathfrak{u}^{'}(1) = {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\eta) + {}^{c}\mathcal{D}^{\mathsf{v}}\mathfrak{u}(\eta)$, $\mathfrak{u}^{''}(0) = 0$, *and*

$$\mathfrak{u}(\mathfrak{z}) = \int_0^{\mathfrak{z}} e^{-\mathtt{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathcal{V}(\mathfrak{n})d\mathfrak{n} + \frac{1}{\Theta_1}\left(\mathfrak{z}+\frac{\Theta_3}{\Theta_2}\right)\Lambda_{\mathcal{V}}(\eta) + \frac{1}{\Theta_2}\Delta_{\mathcal{V}}(\eta).$$

4.2 Existence Results

Let us develop the existence results of the problem (4.1.1). The vector normed space $(\mathfrak{X}, \|.\|_{\mathfrak{X}})$ is a Banach space, where $\mathfrak{X} = \{\mathfrak{u} \in C([0,1],\mathbb{R}) : {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u} \in C([0,1],\mathbb{R})\}$, and

$\|\mathfrak{u}\|_{\mathfrak{X}} = \sup_{\mathfrak{z}\in[0,1]} |\mathfrak{u}(\mathfrak{z})| + \sup_{\mathfrak{z}\in[0,1]} |{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z})|$.

4.2.1 Convex set-valued case

We apply **Krasnoselskii's fixed point theorem** (Theorem 1.4.7) to prove the existence of solutions for inclusion problem (4.1.1) with a convex set-valued mapping. For $\mathcal{V} \in \mathcal{S}_{G,\mathfrak{u}}$, we define the operator $\mathsf{G} : \mathfrak{X} \to \mathcal{P}(\mathfrak{X})$ as follows:

$$\mathsf{G}(\mathfrak{u}) = \left\{ \begin{array}{l} \hbar \in \mathfrak{X},\ \textit{there exists}\ \mathcal{V} \in \mathcal{S}_{G,\mathfrak{u}}\ \textit{such that} \\ \hbar(\mathfrak{z}) = \left\{ \begin{array}{l} \int_0^{\mathfrak{z}} e^{-\mathtt{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathcal{V}(\mathfrak{n})d\mathfrak{n} + \frac{1}{\Theta_1}\left(\mathfrak{z}+\frac{\Theta_3}{\Theta_2}\right)\Lambda_{\mathcal{V}}(\eta) + \frac{1}{\Theta_2}\Delta_{\mathcal{V}}(\eta), \\ \mathfrak{z} \in [0,1]. \end{array} \right. \end{array} \right\} \tag{4.2.1}$$

Theorem 4.2.1. *Assume that*

($\mathfrak{B}_9$) $G : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{P}_{cp,cv}(\mathbb{R})$ *is Carathéodory multivalued map;*

($\mathfrak{B}_{10}$) *there exists a continuous function* $\mathscr{E} \in C([0,1],\mathbb{R}^+)$ *such that*

$$\|G(\mathfrak{z},\mathfrak{u}_1,\mathfrak{u}_2)\|_{\mathcal{P}} = \sup\{|\mathcal{V}| : \mathcal{V} \in G(\mathfrak{z},\mathfrak{u}_1,\mathfrak{u}_2)\} \le \mathscr{E}(\mathfrak{z}),$$

for each $(\mathfrak{z},\mathfrak{u}_1,\mathfrak{u}_2) \in [0,1] \times \mathbb{R} \times \mathbb{R}$;

($\mathfrak{B}_{11}$) *there exists a function* $\mathscr{S} \in C([0,1],\mathbb{R})$ *satisfying*

$$\mathcal{H}_d(G(\mathfrak{z},\mathfrak{u}_1,\mathfrak{u}_2), G(\mathfrak{z},\bar{\mathfrak{u}}_1,\bar{\mathfrak{u}}_2)) \le \|\mathscr{S}\|\left(\|\mathfrak{u}_1-\bar{\mathfrak{u}}_1\| + \|\mathfrak{u}_2-\bar{\mathfrak{u}}_2\|\right),$$

for a.e $\mathfrak{z} \in [0,1]$ *and all* $\mathfrak{u}_i, \bar{\mathfrak{u}}_i \in C([0,1],\mathbb{R}), i = 1, 2,$ *with*

$$\|\mathscr{S}\| \left(\frac{1-e^{-\mathsf{k}}}{\mathsf{k}\Gamma(\mathsf{v}+1)} + \frac{2-e^{-\mathsf{k}}}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} \right) < 1.$$

Then the boundary value problem (4.1.1) has at least one solution on $[0,1]$.

Proof. For each $\mathfrak{u} \in \mathfrak{X}$, since $\mathcal{S}_{G,\mathfrak{u}}$ is nonempty [40], there exists $\mathscr{V} \in \mathcal{S}_{G,\mathfrak{u}}$. We define the operators $\mathsf{A} : \mathfrak{X} \longrightarrow \mathcal{P}(\mathfrak{X})$ and $\mathsf{B} : \mathfrak{X} \longrightarrow \mathcal{P}(\mathfrak{X})$ by

$$\mathsf{A}\,(\mathfrak{u}) = \left\{ \hbar \in \mathfrak{X} : \hbar\,(\mathfrak{z}) = \int_0^{\mathfrak{z}} e^{-\mathsf{k}(\mathfrak{z}-\mathfrak{n})} \mathcal{I}^{\mathsf{v}} \mathscr{V}(\mathfrak{n}) d\mathfrak{n}, \quad \mathfrak{z} \in [0,1] \right\},$$

$$\mathsf{B}\,(\mathfrak{u}) = \left\{ \hbar \in \mathfrak{X} : \hbar\,(\mathfrak{z}) = \frac{1}{\Theta_1} \left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2} \right) \Lambda_{\mathscr{V}}(\eta) + \frac{1}{\Theta_2} \Delta_{\mathscr{V}}(\eta), \quad \mathfrak{z} \in [0,1] \right\}.$$

Then $\mathsf{G} = \mathsf{A} + \mathsf{B}$, where G is given by (4.2.1).

Now, we will show in several steps that the conditions of Theorem 1.4.7 are fulfilled by the operators A and B.

Let $\mathsf{a} > 0$ and $\mathbf{B}_{\mathsf{a}} = \{\mathfrak{u} \in \mathfrak{X} : \|\mathfrak{u}\| \le \mathsf{a}\} \subset \mathfrak{X}$. We define the operators $\mathsf{A}, \mathsf{B} : \mathbf{B}_{\mathsf{a}} \to \mathcal{P}_{cp,cv}(\mathfrak{X})$. Note that B is equivalent to the composition $\mathscr{L} \circ \mathcal{S}_G$, where $\mathscr{L} : L^1([0,1],\mathbb{R}) \longrightarrow \mathfrak{X}$ is the continuous linear operator defined by :

$$\mathscr{L}(\mathscr{V})(\mathfrak{z}) = \frac{1}{\Theta_1} \left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2} \right) \Lambda_{\mathscr{V}}(\eta) + \frac{1}{\Theta_2} \Delta_{\mathscr{V}}(\eta).$$

Let $\mathfrak{u} \in \mathbf{B}_{\mathsf{a}}$ is arbitrary and let $(\mathscr{V}_n)$ be a sequence in $\mathcal{S}_{G,\mathfrak{u}}$. Then $\mathscr{V}_n(\mathfrak{z}) \in G(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z}))$ a.e in $[0,1]$. As $G(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z}))$ is compact for all $\mathfrak{z} \in [0,1]$, then $(\mathscr{V}_n)$ has a subsequence (still denoted $(\mathscr{V}_n)$) that converges in measure to a limit $\mathscr{V}(\mathfrak{z}) \in \mathcal{S}_{G,\mathfrak{u}}$ for almost every $\mathfrak{z} \in [0,1]$. Since, $\mathscr{L}$ is continuous, so $\mathscr{L}(\mathscr{V}_n)(\mathfrak{z}) \to \mathscr{L}(\mathscr{V})(\mathfrak{z})$ pointwise on $[0,1]$.

To establish uniform convergence, It is enough to prove that $\mathscr{L}(\mathscr{V}_n)$ is equicontinuous.

Let $\mathfrak{z}_1, \mathfrak{z}_2 \in [0,1]$ with $\mathfrak{z}_1 < \mathfrak{z}_2$. By using Corollary 3.2.3 and condition $(\mathfrak{B}_{10})$, we have

$$\begin{aligned}
&|\mathscr{L}(\mathscr{V}_n)(\mathfrak{z}_2) - \mathscr{L}(\mathscr{V}_n)(\mathfrak{z}_1)| \\
&= \left| \frac{\mathfrak{z}_2 - \mathfrak{z}_1}{\Theta_1} \Lambda_{\mathscr{V}_n}(\eta) \right| \\
&\leq \frac{|\mathfrak{z}_2 - \mathfrak{z}_1|}{|\Theta_1|} \left| \Lambda_{\mathscr{V}_n}(\eta) \right| \\
&\leq |\mathfrak{z}_2 - \mathfrak{z}_1| \times \frac{1}{|\Theta_1|} \left(\frac{|\mathsf{k}^2 - \mathsf{k}|\, \eta(e^{-\mathsf{k}\eta} + \mathsf{k}\eta - 1)}{\mathsf{k}^2 \Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{|1-\mathsf{k}|\, \eta^2 + \mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} + \frac{|\mathsf{q}|\,(2 - e^{-\mathsf{k}})}{\Gamma(\mathsf{v}+1)} \right) \|\mathscr{E}\| \\
&\leq |\mathfrak{z}_2 - \mathfrak{z}_1|\, \widetilde{\Pi}_2 \, \|\mathscr{E}\|,
\end{aligned}$$

and

$$\begin{aligned}
&\left| {}^c\mathcal{D}^{\mathsf{v}-1} \mathscr{L}(\mathscr{V}_n)(\mathfrak{z}_2) - {}^c\mathcal{D}^{\mathsf{v}-1} \mathscr{L}(\mathscr{V}_n)(\mathfrak{z}_1) \right| \\
&= \left| \frac{\mathfrak{z}_2^{2-\mathsf{v}} - \mathfrak{z}_1^{2-\mathsf{v}}}{\Theta_1 \Gamma(3-\mathsf{v})} \Lambda_{\mathscr{V}_n}(\eta) \right| \\
&\leq \frac{\left| \mathfrak{z}_2^{2-\mathsf{v}} - \mathfrak{z}_1^{2-\mathsf{v}} \right|}{|\Theta_1|\, \Gamma(3-\mathsf{v})} \left| \Lambda_{\mathscr{V}_n}(\eta) \right| \\
&\leq \frac{\left| \mathfrak{z}_2^{2-\mathsf{v}} - \mathfrak{z}_1^{2-\mathsf{v}} \right|}{\Gamma(3-\mathsf{v})} \times \frac{1}{|\Theta_1|} \left(\frac{|\mathsf{k}^2 - \mathsf{k}|\, \eta(e^{-\mathsf{k}\eta} + \mathsf{k}\eta - 1)}{\mathsf{k}^2 \Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} \right. \\
&\quad \left. + \frac{|1-\mathsf{k}|\, \eta^2 + \mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} + \frac{|\mathsf{q}|\,(2 - e^{-\mathsf{k}})}{\Gamma(\mathsf{v}+1)} \right) \|\mathscr{E}\| \\
&\leq \frac{\left| \mathfrak{z}_2^{2-\mathsf{v}} - \mathfrak{z}_1^{2-\mathsf{v}} \right|}{\Gamma(3-\mathsf{v})} \widetilde{\Pi}_2 \, \|\mathscr{E}\|.
\end{aligned}$$

Hence, $\lim_{\mathfrak{z}_2 \to \mathfrak{z}_1} |\mathscr{L}(\mathscr{V}_n)(\mathfrak{z}_2) - \mathscr{L}(\mathscr{V}_n)(\mathfrak{z}_1)| = \lim_{\mathfrak{z}_2 \to \mathfrak{z}_1} |{}^c\mathcal{D}^{\mathsf{v}-1} \mathscr{L}(\mathscr{V}_n)(\mathfrak{z}_2) - {}^c\mathcal{D}^{\mathsf{v}-1} \mathscr{L}(\mathscr{V}_n)(\mathfrak{z}_1)| = 0$. Thus, the sequence $\{\mathscr{L}(\mathscr{V}_n)\}$ is equicontinuous and by using the Arzelá-Ascoli theorem, It follows that there is a uniformly convergent subsequence. So, there is a subsequence of $\{\mathscr{V}_n\}$ (still denoted $\{\mathscr{V}_n\}$) such that $\mathscr{L}(\mathscr{V}_n) \to \mathscr{L}(\mathscr{V})$. Note that $\mathscr{L}(\mathscr{V}) \in \mathscr{L}(\mathcal{S}_{G,\mathfrak{u}})$. Hence $\mathsf{B}(\mathfrak{u}) = \mathscr{L}(\mathcal{S}_{G,\mathfrak{u}})$ is compact for all $\mathfrak{u} \in \mathbf{B}_{\mathsf{a}}$. So $\mathsf{B}(\mathfrak{u})$ is compact.

Now, we show that $\mathsf{B}(\mathfrak{u})$ is convex for all $\mathfrak{u} \in \mathfrak{X}$, Let $\varkappa_1, \varkappa_2 \in \mathsf{B}(\mathfrak{u})$. Taking $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathcal{S}_{G,\mathfrak{u}}$ where

$$\varkappa_i(\mathfrak{z}) = \frac{1}{\Theta_1} \left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2} \right) \Lambda_{\mathfrak{g}_i}(\eta) + \frac{1}{\Theta_2} \Delta_{\mathfrak{g}_i}(\eta), \qquad i = 1, 2,$$

for almost all $\mathfrak{z} \in [0,1]$. Let $0 \leq \mathfrak{b} \leq 1$. Then, we get

$$\begin{aligned}&[\mathfrak{b}\varkappa_1 + (1-\mathfrak{b})\varkappa_2](\mathfrak{z})\\ &= \frac{1}{\Theta_1}\left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2}\right)\Lambda_{\mathfrak{b}\mathfrak{g}_1+(1-\mathfrak{b})\mathfrak{g}_2}(\eta) + \frac{1}{\Theta_2}\Delta_{\mathfrak{b}\mathfrak{g}_1+(1-\mathfrak{b})\mathfrak{g}_2}(\eta).\end{aligned}$$

Owing to $G(\mathfrak{z},\mathfrak{u}(\mathfrak{z}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z}))$ is convex, $\mathcal{S}_{G,\mathfrak{u}}$ is convex, then

$$\mathfrak{b}\varkappa_1 + (1-\mathfrak{b})\varkappa_2 \in \mathsf{B}(\mathfrak{u}),$$

which implies that B is convex-valued. Similarly, A is compact and convex-valued.

We split the remaining proof in several steps and claims.

Step 1: Let us prove that A is a multi-valued contraction on $\mathfrak{X}$. Take $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathfrak{X}$ and $\hbar_1 \in \mathsf{A}(\mathfrak{u}_1)$. Then, for each $\mathfrak{z} \in [0,1]$, there exists $\mathscr{V}_1(\mathfrak{z}) \in G(\mathfrak{z},\mathfrak{u}_1(\mathfrak{z}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1(\mathfrak{z}))$ such that

$$\hbar_1(\mathfrak{z}) = \int_0^{\mathfrak{z}} e^{-\mathtt{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathscr{V}_1(\mathfrak{n})d\mathfrak{n}.$$

Since

$\mathcal{H}_d\left(G(\mathfrak{z},\mathfrak{u}_1(\mathfrak{z}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1(\mathfrak{z})), G(\mathfrak{z},\mathfrak{u}_2(\mathfrak{z}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2(\mathfrak{z}))\right) \leq \mathscr{S}(\mathfrak{z})\left(\|\mathfrak{u}_1-\mathfrak{u}_2\| + \|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1 - {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2\|\right)$, there exists $\mathscr{W} \in G(\mathfrak{z},\mathfrak{u}_2(\mathfrak{z}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2(\mathfrak{z}))$ such that

$$|\mathscr{V}_1(\mathfrak{z}) - \mathscr{W}| \leq \mathscr{S}(\mathfrak{z})\left(\|\mathfrak{u}_1-\mathfrak{u}_2\| + \left\|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1 - {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2\right\|\right).$$

Then the multifunction $\mathsf{U}(\mathfrak{z}) = \mathcal{S}_{G,\mathfrak{u}_2} \bigcap \mathsf{W}(\mathfrak{z})$, where

$$\mathsf{W}(\mathfrak{z}) = \left\{\mathscr{W} \in \mathbb{R} : |\mathscr{V}_1(\mathfrak{z}) - \mathscr{W}| \leq \mathscr{S}(\mathfrak{z})\left(\|\mathfrak{u}_1-\mathfrak{u}_2\| + \left\|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1 - {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2\right\|\right)\right\},$$

is measurable and nonempty. Let $\mathscr{V}_2$ be a measurable selection for U (see [30], Theorem 2.1). Thus, $\mathscr{V}_2(\mathfrak{z}) \in G(\mathfrak{z},\mathfrak{u}_2(\mathfrak{z}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2(\mathfrak{z}))$ and for each $\mathfrak{z} \in [0,1]$, we have $|\mathscr{V}_1(\mathfrak{z}) - \mathscr{V}_2(\mathfrak{z})| \leq \mathscr{S}(\mathfrak{z})\left(\|\mathfrak{u}_1-\mathfrak{u}_2\| + \|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1 - {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2\|\right)$ a.e. on $[0,1]$.

For each $\mathfrak{z} \in [0,1]$, let us define

$$\hbar_2(\mathfrak{z}) = \int_0^{\mathfrak{z}} e^{-\mathtt{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathscr{V}_2(\mathfrak{n})d\mathfrak{n}.$$

Then, $\hbar_2 \in \mathsf{A}(\mathfrak{u}_2)$, and by applying Lemma 3.2.2, we obtain

$$\begin{aligned}
|\hbar_1(\mathfrak{z}) - \hbar_2(\mathfrak{z})| &= \left| \int_0^{\mathfrak{z}} e^{-\mathsf{k}(\mathfrak{z}-\mathfrak{n})} \mathcal{I}^{\mathsf{v}} (\mathscr{V}_1 - \mathscr{V}_2)(\mathfrak{n}) d\mathfrak{n} \right| \\
&\leq \frac{1 - e^{-\mathsf{k}}}{\mathsf{k}\Gamma(\mathsf{v}+1)} \left| \mathscr{V}_1(\mathfrak{z}) - \mathscr{V}_2(\mathfrak{z}) \right| \\
&\leq \frac{1 - e^{-\mathsf{k}}}{\mathsf{k}\Gamma(\mathsf{v}+1)} \|\mathscr{S}\| \left(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \left\| {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1 -^{c} \mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2 \right\| \right).
\end{aligned}$$

Also we have

$$ {}^{c}\mathcal{D}^{\mathsf{v}-1}\hbar(\mathfrak{z}) = \int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} \hbar'(\mathfrak{n}) d\mathfrak{n},$$

and

$$\hbar'(\mathfrak{z}) = -\mathsf{k} \int_0^{\mathfrak{z}} e^{-\mathsf{k}(\mathfrak{z}-\mathfrak{n})} \mathcal{I}^{\mathsf{v}} \mathscr{V}(\mathfrak{n}) d\mathfrak{n} + \mathcal{I}^{\mathsf{v}} \mathscr{V}(\mathfrak{z}).$$

Hence

$$\begin{aligned}
\left| \hbar_1'(\mathfrak{z}) - \hbar_2'(\mathfrak{z}) \right| &= \left| -\mathsf{k} \int_0^{\mathfrak{z}} e^{-\mathsf{k}(\mathfrak{z}-\mathfrak{n})} \mathcal{I}^{\mathsf{v}} (\mathscr{V}_1 - \mathscr{V}_2)(\mathfrak{n}) d\mathfrak{n} + \mathcal{I}^{\mathsf{v}} (\mathscr{V}_1 - \mathscr{V}_2)(\mathfrak{z}) \right| \\
&\leq \frac{1 - e^{-\mathsf{k}}}{\Gamma(\mathsf{v}+1)} \left| \mathscr{V}_1(\mathfrak{z}) - \mathscr{V}_2(\mathfrak{z}) \right| + \frac{1}{\Gamma(\mathsf{v}+1)} \left| \mathscr{V}_1(\mathfrak{z}) - \mathscr{V}_2(\mathfrak{z}) \right| \\
&\leq \frac{2 - e^{-\mathsf{k}}}{\Gamma(\mathsf{v}+1)} \left| \mathscr{V}_1(\mathfrak{z}) - \mathscr{V}_2(\mathfrak{z}) \right| \\
&\leq \frac{2 - e^{-\mathsf{k}}}{\Gamma(\mathsf{v}+1)} \|\mathscr{S}\| \left(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \left\| {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1 -^{c} \mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2 \right\| \right).
\end{aligned}$$

From Definition 1.2.4 with $1 < \mathsf{v} \leq 2$, we get

$$\begin{aligned}
\left| {}^{c}\mathcal{D}^{\mathsf{v}-1}(\hbar_1 - \hbar_2)(\mathfrak{z}) \right| &= \left| \int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} (\hbar_1' - \hbar_2')(\mathfrak{n}) d\mathfrak{n} \right| \\
&\leq \int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} \left| (\hbar_1' - \hbar_2')(\mathfrak{n}) \right| d\mathfrak{n} \\
&\leq \frac{2 - e^{-\mathsf{k}}}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} \|\mathscr{S}\| \left(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \left\| {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1 -^{c} \mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2 \right\| \right).
\end{aligned}$$

From the above inequalities, we get

$$\begin{aligned}
\|\hbar_1 - \hbar_2\|_{\mathfrak{X}} &= \|\hbar_1 - \hbar_2\| + \left\| {}^{c}\mathcal{D}^{\mathsf{v}-1}(\hbar_1 - \hbar_2) \right\| \\
&\leq \left(\frac{1 - e^{-\mathsf{k}}}{\mathsf{k}\Gamma(\mathsf{v}+1)} + \frac{2 - e^{-\mathsf{k}}}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} \right) \|\mathscr{S}\| \left(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \left\| {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1 -^{c} \mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2 \right\| \right).
\end{aligned}$$

By swapping beetwen $\mathfrak{u}_1$ and $\mathfrak{u}_2$, we derive a similar inequality:

$$\mathcal{H}_d\left(\mathsf{A}\mathfrak{u}_1, \mathsf{B}\mathfrak{u}_2\right) \leq \left(\frac{1-e^{-\mathsf{k}}}{\mathsf{k}\Gamma(\mathsf{v}+1)} + \frac{2-e^{-\mathsf{k}}}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})}\right) \|\mathscr{S}\| \left(\|\mathfrak{u}_1 - \mathfrak{u}_2\| + \left\|{}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_1 -{}^c \mathcal{D}^{\mathsf{v}-1}\mathfrak{u}_2\right\|\right),$$

for each $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathfrak{X}$. This shows that A is a multi-valued contraction, since

$$\left(\frac{1-e^{-\mathsf{k}}}{\mathsf{k}\Gamma(\mathsf{v}+1)} + \frac{2-e^{-\mathsf{k}}}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})}\right) \|\mathscr{S}\| < 1.$$

Step 2: B is compact and upper semicontinuous. This will be established in several claims.

Claim I: B maps bounded sets into bounded sets in $\mathfrak{X}$.

For a positive number r, let $\mathbf{B}_\mathsf{r} = \{\mathfrak{u} \in \mathfrak{X} : \|\mathfrak{u}\| \leq \mathsf{r}\}$ be a bounded ball in $\mathfrak{X}$. Then, for each $\hbar \in \mathsf{B}(\mathfrak{u})$, $\mathfrak{u} \in \mathbf{B}_\mathsf{r}$, there exists $\mathscr{V} \in \mathcal{S}_{G,\mathfrak{u}}$ such that

$$\hbar(\mathfrak{z}) = \frac{1}{\Theta_1}\left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2}\right)\Lambda_{\mathscr{V}}(\eta) + \frac{1}{\Theta_2}\Delta_{\mathscr{V}}(\eta).$$

By Lemma 3.2.2, Corollary 3.2.3 and condition $(\mathfrak{B}_{10})$, we find

$$\begin{aligned} |\hbar(\mathfrak{z})| \leq & \left[\frac{|\Theta_2| + |\Theta_3|}{|\Theta_1|\,|\Theta_2|}\left(\frac{\eta(e^{-\mathsf{k}\eta} + \mathsf{k}\eta - 1)\,|\mathsf{k}^2 - \mathsf{k}|}{\mathsf{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{|1-\mathsf{k}|\,\eta^2 + \mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})}\right.\right. \\ & \left.\left. + \frac{|\mathfrak{q}|\,(2-e^{-\mathsf{k}})}{\Gamma(\mathsf{v}+1)}\right) + \frac{|\mathfrak{p}|\,(1-e^{-\mathsf{k}})}{|\Theta_2|\,\mathsf{k}\Gamma(\mathsf{v}+1)} + \frac{(1-e^{-\mathsf{k}\eta})(\mathsf{v}\eta^{2\mathsf{v}-1} + \eta^{2\mathsf{v}})}{\mathsf{k}\,|\Theta_2|\,[\Gamma(\mathsf{v}+1)]^2}\right] \|\mathscr{E}\| \\ \leq & \widetilde{\Pi}_1 \|\mathscr{E}\|. \end{aligned}$$

Also we have

$$\begin{aligned} \left|\hbar^{'}(\mathfrak{z})\right| = & \left|\frac{1}{\Theta_1}\Lambda_{\mathscr{V}}(\eta)\right| \\ \leq & \frac{1}{|\Theta_1|}\left(\frac{|\mathsf{k}^2 - \mathsf{k}|\,\eta(\mathsf{k}\eta + e^{-\mathsf{k}\eta} - 1)}{\mathsf{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{|1-\mathsf{k}|\,\eta^2 + \mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} + \frac{|\mathfrak{q}|\,(2-e^{-\mathsf{k}})}{\Gamma(\mathsf{v}+1)}\right)\|\mathscr{E}\| \\ \leq & \widetilde{\Pi}_2 \|\mathscr{E}\|. \end{aligned}$$

By Definition 1.2.4 for $\mathsf{v} \in (1,2]$, we get

$$\begin{aligned} \left|{}^c\mathcal{D}^{\mathsf{v}-1}\hbar(\mathfrak{z})\right| \leq & \int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}\left|\hbar^{'}(\mathfrak{n})\right| d\mathfrak{n} \\ \leq & \widetilde{\Pi}_2 \|\mathscr{E}\| \int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})} d\mathfrak{n} \\ \leq & \frac{\widetilde{\Pi}_2}{\Gamma(3-\mathsf{v})}\|\mathscr{E}\|. \end{aligned}$$

Consequently, we get

$$\|\hbar\|_{\mathfrak{X}} = \|\hbar\| + \left\|{}^{c}\mathcal{D}^{\mathsf{v}-1}\hbar\right\| \leq \left(\widetilde{\Pi}_1 + \frac{\widetilde{\Pi}_2}{\Gamma(3-\mathsf{v})}\right)\|\mathscr{E}\|.$$

Claim II: B maps bounded sets into equi-continuous sets.

Let $\mathfrak{z}_1, \mathfrak{z}_2 \in [0,1]$ with $\mathfrak{z}_1 < \mathfrak{z}_2$ and $\mathfrak{u} \in \mathbf{B}_{\mathsf{r}}$. For each $\hbar \in \mathsf{B}(\mathfrak{u})$, we obtain

$$\begin{aligned}
&|\hbar(\mathfrak{z}_2) - \hbar(\mathfrak{z}_1)| \\
&= \left|\frac{\mathfrak{z}_2 - \mathfrak{z}_1}{\Theta_1}\Lambda_{\mathscr{V}}(\eta)\right| \\
&\leq \left|\frac{\mathfrak{z}_2 - \mathfrak{z}_1}{\Theta_1}\right| |\Lambda_{\mathscr{V}}(\eta)| \\
&\leq \frac{|\mathfrak{z}_2 - \mathfrak{z}_1|}{|\Theta_1|}\left(\frac{|\mathsf{k}^2 - \mathsf{k}|\,\eta(\mathsf{k}\eta + e^{-\mathsf{k}\eta} - 1)}{\mathsf{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{|1-\mathsf{k}|\,\eta^2 + \mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} + \frac{|\mathsf{q}|\,(2-e^{-\mathsf{k}})}{\Gamma(\mathsf{v}+1)}\right)\|\mathscr{E}\| \\
&\leq |\mathfrak{z}_2 - \mathfrak{z}_1|\,\widetilde{\Pi}_2\,\|\mathscr{E}\|,
\end{aligned}$$

and

$$\begin{aligned}
&\left|{}^{c}\mathcal{D}^{\mathsf{v}-1}\hbar(\mathfrak{z}_2) -{}^{c}\mathcal{D}^{\mathsf{v}-1}\hbar(\mathfrak{z}_1)\right| \\
&\leq \left|\frac{\mathfrak{z}_2^{2-\mathsf{v}} - \mathfrak{z}_1^{2-\mathsf{v}}}{\Theta_1\Gamma(3-\mathsf{v})}\right| |\Lambda_{\mathscr{V}}(\eta)| \\
&\leq \frac{\left|\mathfrak{z}_2^{2-\mathsf{v}} - \mathfrak{z}_1^{2-\mathsf{v}}\right|}{\Gamma(3-\mathsf{v})} \times \frac{1}{|\Theta_1|}\left(\frac{|\mathsf{k}^2 - \mathsf{k}|\,\eta(\mathsf{k}\eta + e^{-\mathsf{k}\eta} - 1)}{\mathsf{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)}\right. \\
&\quad \left. + \frac{|1-\mathsf{k}|\,\eta^2 + \mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} + \frac{|\mathsf{q}|\,(2-e^{-\mathsf{k}})}{\Gamma(\mathsf{v}+1)}\right)\|\mathscr{E}\| \\
&\leq \frac{\left|\mathfrak{z}_2^{2-\mathsf{v}} - \mathfrak{z}_1^{2-\mathsf{v}}\right|}{\Gamma(3-\mathsf{v})}\widetilde{\Pi}_2\,\|\mathscr{E}\|.
\end{aligned}$$

Obviously $|\hbar(\mathfrak{z}_2) - \hbar(\mathfrak{z}_1)|$, $|{}^{c}\mathcal{D}^{\mathsf{v}-1}\hbar(\mathfrak{z}_2) -{}^{c}\mathcal{D}^{\mathsf{v}-1}\hbar(\mathfrak{z}_1)|$ tends to zero independently of $\mathfrak{u} \in \mathbf{B}_{\mathsf{r}}$ as $\mathfrak{z}_2 - \mathfrak{z}_1 \to 0$. Therefore we deduce by the Ascoli-Arzelá theorem that $\mathsf{B} : \mathfrak{X} \to \mathcal{P}(\mathfrak{X})$ is completely continuous.

By Remark 1.3.2, B is upper semicontinuous provided it has a closed graph, as shown in the next claim.

Claim III: B has a closed graph.

Let $\mathfrak{u}_n \to \mathfrak{u}_*$, $\hbar_n \in \mathsf{B}(\mathfrak{u}_n)$ and $\hbar_n \to \hbar_*$, we show that $\hbar_* \in \mathsf{B}(\mathfrak{u}_*)$. Associated with $\hbar_n \in \mathsf{B}(\mathfrak{u}_n)$, there exists $\mathfrak{g}_n \in \mathcal{S}_{G,\mathfrak{u}_n}$ such that, for each $\mathfrak{z} \in [0,1]$,

$$\hbar_n(\mathfrak{z}) = \frac{1}{\Theta_1}\left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2}\right)\Lambda_{\mathfrak{g}_n}(\eta) + \frac{1}{\Theta_2}\Delta_{\mathfrak{g}_n}(\eta).$$

We will show that there exists $\mathfrak{g}_* \in \mathcal{S}_{G,\mathfrak{u}_*}$ such that for each $\mathfrak{z} \in [0,1]$,

$$\hbar_*(\mathfrak{z}) = \frac{1}{\Theta_1}\left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2}\right)\Lambda_{\mathfrak{g}_*}(\eta) + \frac{1}{\Theta_2}\Delta_{\mathfrak{g}_*}(\eta).$$

Let us consider the linear operator $\Xi : L^1([0,1],\mathbb{R}) \to \mathfrak{X}$ given by

$$\mathfrak{g} \mapsto \Xi(\mathfrak{g})(\mathfrak{z}) = \frac{1}{\Theta_1}\left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2}\right)\Lambda_{\mathfrak{g}}(\eta) + \frac{1}{\Theta_2}\Delta_{\mathfrak{g}}(\eta),$$

then we note that $\|\hbar_n - \hbar_*\|_{\mathfrak{X}} \longrightarrow 0$, as $n \longrightarrow \infty$.

By using Lemma 1.3.5 that $\Xi \circ \mathcal{S}_{\mathcal{G}}$ is a closed graph operator. Further, we have $\hbar_n(\mathfrak{z}) \in \Xi(\mathcal{S}_{\mathcal{G},\mathfrak{u}_n})$. $\mathfrak{u}_n \to \mathfrak{u}_*$, we have that

$$\hbar_*(\mathfrak{z}) = \frac{1}{\Theta_1}\left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2}\right)\Lambda_{\mathfrak{g}_*}(\eta) + \frac{1}{\Theta_2}\Delta_{\mathfrak{g}_*}(\eta),$$

for some $\mathfrak{g}_* \in \mathcal{S}_{G,\mathfrak{u}_*}$. Hence B has a closed graph. Thus, the operator B is compact and upper semicontinuous.

Step 3: Now, we establish that $\mathsf{A}(\mathfrak{u}) + \mathsf{B}(\mathfrak{u}) \subset \mathbf{B}_{\mathsf{a}}$ for all $\mathfrak{u} \in \mathbf{B}_{\mathsf{a}}$.

Suppose $\mathfrak{u} \in \mathbf{B}_{\mathsf{a}}$, with $\mathsf{a} > \left(\Pi_1 + \frac{\Pi_2}{\Gamma(3-\mathsf{v})}\right)\|\mathscr{E}\|$ (Π_1, Π_2 defined by (3.3.2)).

For $\hbar \in \mathsf{A}, \mathsf{B}$ and $\mathscr{V} \in \mathcal{S}_{G,\mathfrak{u}}$, we have

$$\hbar(\mathfrak{z}) = \int_0^{\mathfrak{z}} e^{-\mathtt{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathscr{V}(\mathfrak{n})d\mathfrak{n} + \frac{1}{\Theta_1}\left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2}\right)\Lambda_{\mathscr{V}}(\eta) + \frac{1}{\Theta_2}\Delta_{\mathscr{V}}(\eta).$$

By Lemma 3.2.2, Corollary 3.2.3 and ($\mathfrak{B}_{10}$), we have

$$\begin{aligned}
|\hbar(\mathfrak{z})| &\leq \left|\int_0^{\mathfrak{z}} e^{-\mathtt{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathscr{V}(\mathfrak{n})d\mathfrak{n}\right| + \left|\frac{1}{\Theta_1}\left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2}\right)\Lambda_{\mathscr{V}}(\eta)\right| + \left|\frac{1}{\Theta_2}\Delta_{\mathscr{V}}(\eta)\right| \\
&\leq \left[\frac{(|\Theta_2| + |\mathfrak{p}|)(1-e^{-\mathtt{k}})}{|\Theta_2|\,\mathtt{k}\Gamma(\mathsf{v}+1)} + \frac{|\Theta_2| + |\Theta_3|}{|\Theta_1|\,|\Theta_2|}\left(\frac{\eta(e^{-\mathtt{k}\eta} + \mathtt{k}\eta - 1)\,|\mathtt{k}^2 - \mathtt{k}|}{\mathtt{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)}\right.\right. \\
&\quad \left.\left. + \frac{|1-\mathtt{k}|\,\eta^2 + \mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} + \frac{|\mathfrak{q}|\,(2-e^{-\mathtt{k}})}{\Gamma(\mathsf{v}+1)}\right) + \frac{(1-e^{-\mathtt{k}\eta})(\mathsf{v}\eta^{2\mathsf{v}-1} + \eta^{2\mathsf{v}})}{\mathtt{k}\,|\Theta_2|\,[\Gamma(\mathsf{v}+1)]^2}\right]\|\mathscr{E}\| \\
&\leq \Pi_1\,\|\mathscr{E}\|\,.
\end{aligned}$$

Also we have

$$\begin{aligned}\left|\hbar'(\mathfrak{z})\right| &= \left|-\mathtt{k}\int_0^{\mathfrak{z}} e^{-\mathtt{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathscr{V}(\mathfrak{n})d\mathfrak{n} + \mathcal{I}^{\mathsf{v}}\mathscr{V}(\mathfrak{z}) + \frac{1}{\Theta_1}\Lambda_{\mathscr{V}}(\eta)\right| \\ &\leq \mathtt{k}\left|\int_0^{\mathfrak{z}} e^{-\mathtt{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathscr{V}(\mathfrak{n})d\mathfrak{n}\right| + |\mathcal{I}^{\mathsf{v}}\mathscr{V}(\mathfrak{z})| + \frac{1}{|\Theta_1|}\left|\Lambda_{\mathscr{V}}(\eta)\right| \\ &\leq \left[\frac{2-e^{-\mathtt{k}}}{\Gamma(\mathsf{v}+1)} + \frac{1}{|\Theta_1|}\left(\frac{\left|\mathtt{k}^2-\mathtt{k}\right|\eta(\mathtt{k}\eta+e^{-\mathtt{k}\eta}-1)}{\mathtt{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{\left|1-\mathtt{k}\right|\eta^2+\mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})}\right.\right. \\ &\qquad \left.\left. + \frac{|\mathfrak{q}|\left(2-e^{-\mathtt{k}}\right)}{\Gamma(\mathsf{v}+1)}\right)\right]\|\mathscr{E}\| \\ &\leq \Pi_2\|\mathscr{E}\|.\end{aligned}$$

From Definition 1.2.4 with $\mathsf{v}\in(1,2]$, we get

$$\begin{aligned}\left|{}^c\mathcal{D}^{\mathsf{v}-1}\hbar(\mathfrak{z})\right| &\leq \int_0^{\mathfrak{z}}\frac{(\mathfrak{z}-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}\left|\hbar'(\mathfrak{n})\right|d\mathfrak{n} \\ &\leq \Pi_2\|\mathscr{E}\|\int_0^{\mathfrak{z}}\frac{(\mathfrak{z}-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}d\mathfrak{n} \\ &\leq \frac{\Pi_2}{\Gamma(3-\mathsf{v})}\|\mathscr{E}\|.\end{aligned}$$

From the above inequalities, we get

$$\begin{aligned}\|\hbar\|_{\mathfrak{X}} &= \|\hbar\| + \left\|{}^c\mathcal{D}^{\mathsf{v}-1}\hbar\right\| \\ &\leq \left(\Pi_1 + \frac{\Pi_2}{\Gamma(3-\mathsf{v})}\right)\|\mathscr{E}\| < \mathsf{a}.\end{aligned}$$

Then, $\mathsf{A}(\mathfrak{u})+\mathsf{B}(\mathfrak{u})\subset\mathbf{B}_{\mathsf{a}}$ for all $\mathfrak{u}\in\mathbf{B}_{\mathsf{a}}$.

Thus, A and B satisfy the hypothesis of Theorem 1.4.7 and hence its conclusion implies that $\mathfrak{u}\in\mathsf{A}(\mathfrak{u})+\mathsf{B}(\mathfrak{u})$ in $\mathbf{B}_{\mathsf{a}}$. Therefore problem (4.1.1) has a solution in $\mathbf{B}_{\mathsf{a}}$. □

4.2.2 Nonconvex set-valued case

In this subsection, we show the existence of solutions for the inclusion problem (4.1.1) with the right-hand side being nonconvex set-valued map by applying **Wegrzyk's fixed point theorem** (Theorem 1.4.8).

Theorem 4.2.2. *Postulate that:*

($\mathfrak{B}_{12}$) $G : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $G(.,\mathfrak{u}_1,\mathfrak{u}_2)$ is measurable for each $\mathfrak{u}_1,\mathfrak{u}_2 \in \mathbb{R}$;

($\mathfrak{B}_{13}$) $\mathcal{H}_d\left(G(\mathfrak{z},\mathfrak{u}_1,\mathfrak{u}_2), G(\mathfrak{z},\bar{\mathfrak{u}}_1,\bar{\mathfrak{u}}_2)\right) \leq \mathfrak{m}(\mathfrak{z})\mathfrak{F}\left(|\mathfrak{u}_1 - \bar{\mathfrak{u}}_1| + |\mathfrak{u}_2 - \bar{\mathfrak{u}}_2|\right)$ for almost all $\mathfrak{z} \in [0,1]$ and $\mathfrak{u}_i, \bar{\mathfrak{u}}_i \in \mathbb{R}\,(i = 1,2)$ with $\mathfrak{m} \in C([0,1],\mathbb{R}^+)$ and $d\left(0, G(\mathfrak{z},0,0)\right) \leq \mathfrak{m}(\mathfrak{z})$ for a.e $\mathfrak{z} \in [0,1]$, where $\mathfrak{F} : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing.

Then, problem(4.1.1) has at least one solution on $[0,1]$ if $\tilde{\rho}\mathfrak{F} : \mathbb{R}^+ \to \mathbb{R}^+$ is a strict comparison function, where

$$\tilde{\rho} = \left(\Pi_1 + \frac{\Pi_2}{\Gamma(3-\mathsf{v})}\right)\|\mathfrak{m}\|. \tag{4.2.2}$$

Proof. Suppose that $\tilde{\rho}\mathfrak{F} : \mathbb{R}^+ \to \mathbb{R}^+$ is a strict comparison function. By ($\mathfrak{B}_{12}$) and ($\mathfrak{B}_{13}$), we claim that $G(.,\mathfrak{u}(.),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(.))$ is measurable and has a measurable selection $\mathscr{V}(.)$ (see Theorem III.6 [12]). Also $\mathfrak{m} \in C([0,1],\mathbb{R})$ and

$$\begin{aligned}
|\mathscr{V}(\mathfrak{z})| &\leq d\left(0, G(\mathfrak{z},0,0)\right) + \mathcal{H}_d\left(G(\mathfrak{z},0,0), G\left(\mathfrak{z},\mathfrak{u}(\mathfrak{z}),{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z})\right)\right)\\
&\leq \mathfrak{m}(\mathfrak{z}) + \mathfrak{m}(\mathfrak{z})\mathfrak{F}\left(|\mathfrak{u}(\mathfrak{z})| + \left|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z})\right|\right)\\
&\leq \left(1 + \mathfrak{F}(\|\mathfrak{u}\|_{\mathfrak{X}})\right)\mathfrak{m}(\mathfrak{z}).
\end{aligned}$$

Then, $\mathcal{S}_{G,\mathfrak{u}}$ is nonempty for each $\mathfrak{u} \in \mathfrak{X}$. Now we show that the operator G defined by (4.2.1) fulfills the conditions of Theorem 1.4.8. To show that $\mathsf{G}(\mathfrak{u}) \in \mathcal{P}_{cl}(\mathfrak{X})$ for each $\mathfrak{u} \in \mathfrak{X}$. Let $\{\mathscr{Z}_n\}_{n\geq 0} \in \mathsf{G}(\mathfrak{u})$ be such that $\mathscr{Z}_n \to \mathscr{Z}$ in $\mathfrak{X}$ as $n \to \infty$. Then $\mathscr{Z} \in \mathfrak{X}$ and there exists $\mathscr{V}_n \in \mathcal{S}_{G,\mathfrak{u}}$ such that, for each $\mathfrak{z} \in [0,1]$,

$$\mathscr{Z}_n(\mathfrak{z}) = \int_0^{\mathfrak{z}} e^{-\mathsf{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathscr{V}_n(\mathfrak{n})d\mathfrak{n} + \frac{1}{\Theta_1}\left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2}\right)\Lambda_{\mathscr{V}_n}(\eta) + \frac{1}{\Theta_2}\Delta_{\mathscr{V}_n}(\eta).$$

Since G has compact values, we pass onto a subsequence to obtain that $\mathscr{V}_n$ converges to $\mathscr{V}$ in $L^1([0,1],\mathbb{R})$. Thus, $\mathscr{V} \in \mathcal{S}_{G,\mathfrak{u}}$ and for each $\mathfrak{z} \in [0,1]$, we have

$$\begin{aligned}
&\mathscr{Z}_n(\mathfrak{z}) \to \mathscr{Z}(\mathfrak{z})\\
&= \int_0^{\mathfrak{z}} e^{-\mathsf{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}\mathscr{V}(\mathfrak{n})d\mathfrak{n} + \frac{1}{\Theta_1}\left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2}\right)\Lambda_{\mathscr{V}}(\eta) + \frac{1}{\Theta_2}\Delta_{\mathscr{V}}(\eta).
\end{aligned}$$

Therefore, $\mathscr{Z} \in \mathsf{G}(\mathfrak{u})$.

Next we show that

$$\mathcal{H}_d\left(\mathsf{G}(\mathfrak{u}), \mathsf{G}(\overline{\mathfrak{u}})\right) \leq \tilde{\rho}\mathfrak{F}\left(\|\mathfrak{u}-\overline{\mathfrak{u}}\|_{\mathfrak{X}}\right) \quad \text{for each } \mathfrak{u}, \overline{\mathfrak{u}} \in \mathfrak{X}.$$

Let $\mathfrak{u}, \overline{\mathfrak{u}} \in \mathfrak{X}$ and $\hbar_1 \in \mathsf{G}(\mathfrak{u})$. Then there exists $\mathscr{V}_1 \in \mathcal{S}_{G,\mathfrak{u}}$ such that, for each $\mathfrak{z} \in [0,1]$

$$\hbar_1(\mathfrak{z}) = \int_0^{\mathfrak{z}} e^{-\mathsf{k}(\mathfrak{z}-\mathfrak{n})} \mathcal{I}^{\mathsf{v}} \mathscr{V}_1(\mathfrak{n}) d\mathfrak{n} + \frac{1}{\Theta_1}\left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2}\right) \Lambda_{\mathscr{V}_1}(\eta) + \frac{1}{\Theta_2}\Delta_{\mathscr{V}_1}(\eta).$$

By $(\mathfrak{B}_{13})$, we have

$$\begin{aligned} &\mathcal{H}_d\left(G\left(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z})\right), G\left(\mathfrak{z}, \overline{\mathfrak{u}}(\mathfrak{z}), {}^{c}\mathcal{D}^{\mathsf{v}-1}\overline{\mathfrak{u}}(\mathfrak{z})\right)\right) \\ &\leq \mathfrak{m}(\mathfrak{z})\mathfrak{F}\left(|\mathfrak{u}(\mathfrak{z}) - \overline{\mathfrak{u}}(\mathfrak{z})| + \left|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z}) - {}^{c}\mathcal{D}^{\mathsf{v}-1}\overline{\mathfrak{u}}(\mathfrak{z})\right|\right). \end{aligned}$$

So, there exists $\mathscr{W} \in G\left(\mathfrak{z}, \overline{\mathfrak{u}}(\mathfrak{z}), {}^{c}\mathcal{D}^{\mathsf{v}-1}\overline{\mathfrak{u}}(\mathfrak{z})\right)$ such that

$$|\mathscr{V}_1(\mathfrak{z}) - \mathscr{W}(\mathfrak{z})| \leq \mathfrak{m}(\mathfrak{z})\mathfrak{F}\left(|\mathfrak{u}(\mathfrak{z}) - \overline{\mathfrak{u}}(\mathfrak{z})| + \left|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z}) - {}^{c}\mathcal{D}^{\mathsf{v}-1}\overline{\mathfrak{u}}(\mathfrak{z})\right|\right), \quad \mathfrak{z} \in [0,1].$$

Define $\mathsf{K} : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$\mathsf{K}(\mathfrak{z}) = \left\{\mathscr{W} \in \mathbb{R} : |\mathscr{V}_1(\mathfrak{z}) - \mathscr{W}| \leq \mathfrak{m}(\mathfrak{z})\mathfrak{F}\left(|\mathfrak{u}(\mathfrak{z}) - \overline{\mathfrak{u}}(\mathfrak{z})| + \left|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z}) - {}^{c}\mathcal{D}^{\mathsf{v}-1}\overline{\mathfrak{u}}(\mathfrak{z})\right|\right)\right\}.$$

Since the multivalued operator $\mathsf{K}(\mathfrak{z}) \bigcap G\left(\mathfrak{z}, \overline{\mathfrak{u}}(\mathfrak{z}), {}^{c}\mathcal{D}^{\mathsf{v}-1}\overline{\mathfrak{u}}(\mathfrak{z})\right)$ is measurable (Proposition III.4 [12]), there exists a function $\mathscr{V}_2$ which is a measurable selection for K. Hence $\mathscr{V}_2(\mathfrak{z}) \in G\left(\mathfrak{z}, \overline{\mathfrak{u}}(\mathfrak{z}), {}^{c}\mathcal{D}^{\mathsf{v}-1}\overline{\mathfrak{u}}(\mathfrak{z})\right)$ and for each $\mathfrak{z} \in [0,1]$, we have

$$|\mathscr{V}_1(\mathfrak{z}) - \mathscr{V}_2(\mathfrak{z})| \leq \mathfrak{m}(\mathfrak{z})\mathfrak{F}\left(|\mathfrak{u}(\mathfrak{z}) - \overline{\mathfrak{u}}(\mathfrak{z})| + \left|{}^{c}\mathcal{D}^{\mathsf{v}-1}\mathfrak{u}(\mathfrak{z}) - {}^{c}\mathcal{D}^{\mathsf{v}-1}\overline{\mathfrak{u}}(\mathfrak{z})\right|\right).$$

For each $\mathfrak{z} \in [0,1]$, let us define

$$\hbar_2(\mathfrak{z}) = \int_0^{\mathfrak{z}} e^{-\mathsf{k}(\mathfrak{z}-\mathfrak{n})} \mathcal{I}^{\mathsf{v}} \mathscr{V}_2(\mathfrak{n}) d\mathfrak{n} + \frac{1}{\Theta_1}\left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2}\right) \Lambda_{\mathscr{V}_2}(\eta) + \frac{1}{\Theta_2}\Delta_{\mathscr{V}_2}(\eta).$$

From the above, applying Lemma 3.2.2 and Corollary 3.2.3, we find that

$$\begin{aligned}
&|\hslash_1(\mathfrak{z}) - \hslash_2(\mathfrak{z})| \\
&\leq \left|\int_0^{\mathfrak{z}} e^{-\mathtt{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}(\mathscr{V}_1(\mathfrak{n}) - \mathscr{V}_2(\mathfrak{n}))d\mathfrak{n}\right| + \left|\frac{1}{\Theta_1}\left(\mathfrak{z} + \frac{\Theta_3}{\Theta_2}\right)\Lambda_{\mathscr{V}_1-\mathscr{V}_2}(\eta)\right| + \left|\frac{1}{\Theta_2}\Delta_{\mathscr{V}_1-\mathscr{V}_2}(\eta)\right| \\
&\leq \left[\frac{(|\Theta_2| + |\mathfrak{p}|)(1-e^{-\mathtt{k}})}{|\Theta_2|\,\mathtt{k}\Gamma(\mathsf{v}+1)} + \frac{|\Theta_2| + |\Theta_3|}{|\Theta_1|\,|\Theta_2|}\left(\frac{\eta(e^{-\mathtt{k}\eta} + \mathtt{k}\eta - 1)\,|\mathtt{k}^2 - \mathtt{k}|}{\mathtt{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{|1-\mathtt{k}|\,\eta^2 + \mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})}\right.\right. \\
&\qquad\left.\left. + \frac{|\mathfrak{q}|\,(2-e^{-\mathtt{k}})}{\Gamma(\mathsf{v}+1)}\right) + \frac{(1-e^{-\mathtt{k}\eta})(\mathsf{v}\eta^{2\mathsf{v}-1} + \eta^{2\mathsf{v}})}{\mathtt{k}\,|\Theta_2|\,[\Gamma(\mathsf{v}+1)]^2}\right] \\
&\qquad\times \|\mathfrak{m}\|\,\mathfrak{F}\left(\|\mathfrak{u} - \overline{\mathfrak{u}}\| + \left\|{}^c\mathcal{D}^{\mathsf{v}-1}\mathfrak{u} -{}^c \mathcal{D}^{\mathsf{v}-1}\overline{\mathfrak{u}}\right\|\right) \\
&\leq \Pi_1\,\|\mathfrak{m}\|\,\mathfrak{F}\left(\|\mathfrak{u} - \overline{\mathfrak{u}}\|_{\mathfrak{X}}\right).
\end{aligned}$$

Also we have

$$\begin{aligned}
&\left|\hslash_1'(\mathfrak{z}) - \hslash_2'(\mathfrak{z})\right| \\
&\leq \mathtt{k}\left|\int_0^{\mathfrak{z}} e^{-\mathtt{k}(\mathfrak{z}-\mathfrak{n})}\mathcal{I}^{\mathsf{v}}(\mathscr{V}_1(\mathfrak{n}) - \mathscr{V}_2(\mathfrak{n}))\,d\mathfrak{n}\right| + |\mathcal{I}^{\mathsf{v}}(\mathscr{V}_1(\mathfrak{z}) - \mathscr{V}_2(\mathfrak{z}))| \\
&\quad + \frac{1}{|\Theta_1|}\,|\Lambda_{\mathscr{V}_1-\mathscr{V}_2}(\eta)| \\
&\leq \left[\frac{2-e^{-\mathtt{k}}}{\Gamma(\mathsf{v}+1)} + \frac{1}{|\Theta_1|}\left(\frac{|\mathtt{k}^2-\mathtt{k}|\,\eta(\mathtt{k}\eta + e^{-\mathtt{k}\eta} - 1)}{\mathtt{k}^2\Gamma(2-\mathsf{v})\Gamma(\mathsf{v}+1)} + \frac{|1-\mathtt{k}|\,\eta^2 + \mathsf{v}\eta}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})}\right.\right. \\
&\quad\left.\left. + \frac{|\mathfrak{q}|\,(2-e^{-\mathtt{k}})}{\Gamma(\mathsf{v}+1)}\right)\right]\|\mathfrak{m}\|\,\mathfrak{F}\left(\|\mathfrak{u} - \overline{\mathfrak{u}}\|_{\mathfrak{X}}\right) \\
&\leq \Pi_2\,\|\mathfrak{m}\|\,\mathfrak{F}\left(\|\mathfrak{u} - \overline{\mathfrak{u}}\|_{\mathfrak{X}}\right).
\end{aligned}$$

By Definition 1.2.4 with $\mathsf{v} \in (1, 2]$, we get

$$\begin{aligned}
\left|{}^c\mathcal{D}^{\mathsf{v}-1}(\hslash_1 - \hslash_2)(\mathfrak{z})\right| &\leq \int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}\left|(\hslash_1' - \hslash_2')(\mathfrak{n})\right| d\mathfrak{n} \\
&\leq \Pi_2\,\|\mathfrak{m}\|\,\mathfrak{F}\left(\|\mathfrak{u} - \overline{\mathfrak{u}}\|_{\mathfrak{X}}\right)\int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-\mathfrak{n})^{1-\mathsf{v}}}{\Gamma(2-\mathsf{v})}d\mathfrak{n} \\
&\leq \frac{\Pi_2}{\Gamma(3-\mathsf{v})}\,\|\mathfrak{m}\|\,\mathfrak{F}\left(\|\mathfrak{u} - \overline{\mathfrak{u}}\|_{\mathfrak{X}}\right).
\end{aligned}$$

The above inequalities imply

$$\begin{aligned}
\|\hslash_1 - \hslash_2\|_{\mathfrak{X}} &= \|\hslash_1 - \hslash_2\| + \left\|{}^c\mathcal{D}^{\mathsf{v}-1}(\hslash_1 - \hslash_2)(\mathfrak{z})\right\| \\
&\leq \left(\Pi_1 + \frac{\Pi_2}{\Gamma(3-\mathsf{v})}\right)\|\mathfrak{m}\|\,\mathfrak{F}\left(\|\mathfrak{u} - \overline{\mathfrak{u}}\|_{\mathfrak{X}}\right).
\end{aligned}$$

Analogously, interchanging the roles of $\mathfrak{u}$ and $\overline{\mathfrak{u}}$, we obtain

$$\mathcal{H}_d\left(\mathsf{G}(\mathfrak{u}), \mathsf{G}(\overline{\mathfrak{u}})\right) \leq \tilde{\rho}\mathfrak{F}\left(\|\mathfrak{u}-\overline{\mathfrak{u}}\|_{\mathfrak{X}}\right) = \left(\Pi_1 + \frac{\Pi_2}{\Gamma(3-\mathsf{v})}\right)\|\mathfrak{m}\|\,\mathfrak{F}\left(\|\mathfrak{u}-\overline{\mathfrak{u}}\|_{\mathfrak{X}}\right)$$

for each $\mathfrak{u}, \overline{\mathfrak{u}} \in \mathfrak{X}$. Therefore, G is a generalized contraction. Thus it follows by Theorem 1.4.8 that G has a fixed point $\mathfrak{u}$ which is a solution to problem (4.1.1). This completes the proof. □

4.3 Examples

Example 4.3.1. *Consider the following SFDI:*

$$\begin{cases} ({}^c\mathcal{D}^{\frac{5}{2}} + 4\,{}^c\mathcal{D}^{\frac{3}{2}})\mathfrak{u}(\mathfrak{z}) \in G(\mathfrak{z}, \mathfrak{u}(\mathfrak{z}), {}^c\mathcal{D}^{\frac{3}{2}}\mathfrak{u}(\mathfrak{z})), \quad \mathfrak{z} \in [0,1]; \\ \mathfrak{u}(0) + 2\mathfrak{u}(1) = \mathcal{I}^{\frac{1}{2}}\mathfrak{u}(\frac{1}{2}) + \mathcal{I}^{\frac{3}{2}}\mathfrak{u}(\frac{1}{2}); \\ \mathfrak{u}'(0) + 3\mathfrak{u}'(1) = {}^c\mathcal{D}^{\frac{1}{2}}\mathfrak{u}(\frac{1}{2}) + {}^c\mathcal{D}^{\frac{3}{2}}\mathfrak{u}(\frac{1}{2}); \\ \mathfrak{u}''(0) = 0, \end{cases} \tag{4.3.1}$$

here $\mathsf{v} = \frac{3}{2}$, $\mathtt{k} = 4$, $\mathfrak{p} = 2$, $\mathfrak{q} = 3$, $\eta = \frac{1}{2}$.

Hence

$\Theta_1 \approx 4.7978845608$, $\Theta_2 \approx 1.9361539188$, $\Theta_3 \approx -1.6808461757$ *and* $\Pi_1 \approx 1.9554626139$, $\Pi_2 \approx 2.7258795146$.

Let $G : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$*:*

$$G(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2) = \left[\frac{3}{10}, \frac{\tan^{-1}\mathfrak{u}_1}{4\sqrt{\mathfrak{z}}+676} + \frac{|\sin\mathfrak{u}_2|^{11}}{4\sqrt{\mathfrak{z}}+676\left(1+|\sin\mathfrak{u}_2|^{11}\right)} + \frac{3}{4}\right]. \tag{4.3.2}$$

Then

$$\|G(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2)\|_{\mathcal{P}} = \sup\left\{|\mathscr{V}| : \mathscr{V} \in G(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2)\right\} \leq \frac{\pi+2}{8\sqrt{\mathfrak{z}}+676} + \frac{3}{4} = \mathscr{E}(\mathfrak{z}),$$

and

$$\begin{aligned} \mathcal{H}_d\left(G(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2), G(\mathfrak{z}, \overline{\mathfrak{u}}_1, \overline{\mathfrak{u}}_2)\right) &\leq \frac{1}{4\sqrt{\mathfrak{z}}+676}\left(|\mathfrak{u}_1 - \overline{\mathfrak{u}}_1| + |\mathfrak{u}_2 - \overline{\mathfrak{u}}_2|\right) \\ &\leq \frac{1}{104}\left(\|\mathfrak{u}_1 - \overline{\mathfrak{u}}_1\| + \|\mathfrak{u}_2 - \overline{\mathfrak{u}}_2\|\right), \quad \textit{for } \mathfrak{u}_i,\ \overline{\mathfrak{u}}_i \in \mathbb{R}\,(i = 1, 2). \end{aligned}$$

With $\|\mathscr{S}\| = \frac{1}{104}$, *we find that*

$$\|\mathscr{S}\| \left(\frac{1-e^{-\mathtt{k}}}{\mathtt{k}\Gamma(\mathsf{v}+1)} + \frac{2-e^{-\mathtt{k}}}{\Gamma(\mathsf{v}+1)\Gamma(3-\mathsf{v})} \right) \approx 0.0179492753 < 1.$$

Clearly all the assumptions of Theorem 4.2.1 hold and consequently, problem (4.3.1) with $G(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2)$ *given by (4.3.2) has a solution by Theorem 4.2.1 on* $[0, 1]$.

Example 4.3.2. *Consider the problem (4.3.1) where* $G : [0, 1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ *defined by*

$$G(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2) = \left[0, \frac{1}{8(\mathfrak{z}^3+1)} \left(\tan^{-1} \mathfrak{u}_1 + \cos \mathfrak{u}_2 - 1 \right) + \sin^2 \mathfrak{z} \right], \tag{4.3.3}$$

and

$$\|G(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2)\|_{\mathcal{P}} = \sup \{ |\mathscr{V}| : \mathscr{V} \in G(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2) \} \leq \frac{\pi}{16(\mathfrak{z}^3+1)} + \sin^2 \mathfrak{z},$$

$$\mathcal{H}_d \left(G(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2), G(\mathfrak{z}, \bar{\mathfrak{u}}_1, \bar{\mathfrak{u}}_2) \right) \leq \frac{1}{8(\mathfrak{z}^3+1)} \left(|\mathfrak{u}_1 - \bar{\mathfrak{u}}_1| + |\mathfrak{u}_2 - \bar{\mathfrak{u}}_2| \right).$$

Fixing $\mathfrak{m}(\mathfrak{z}) = \frac{1}{8(\mathfrak{z}^3+1)}$ *such that* $d\left(0, G(\mathfrak{z}, 0, 0)\right) \leq \mathfrak{m}(\mathfrak{z})$ *for almost all* $\mathfrak{z} \in [0, 1]$ *and* $\tilde{\rho} = \left(\Pi_1 + \frac{\Pi_2}{\Gamma(3-\mathsf{v})} \right) \|\mathfrak{m}\| \approx 0.6289110338$ *(*$\tilde{\rho}$ *defined by (4.2.2)). Letting* $\mathfrak{F}(\mathfrak{u}) = \mathfrak{u}$, *all the conditions of Theorem 4.2.2 are fulfilled. Then, problem (4.3.1) with* $G(\mathfrak{z}, \mathfrak{u}_1, \mathfrak{u}_2)$ *given by (4.3.3) has has at least one solution on* $[0, 1]$ *by the conclusion of Theorem 4.2.2.*

Conclusion and Perspectives

In this thesis, we have investigated several classes of fractional differential equations and inclusions involving Caputo-type derivatives under various nonlocal and mixed boundary conditions. By employing different fixed point theorems, we established new existence and uniqueness results for both single-valued and multivalued fractional models.

The work presented here not only unifies and extends numerous results from the existing literature but also highlights the effectiveness of fixed point techniques in analyzing fractional systems. The obtained results provide a general analytical framework that can be adapted to a wide range of boundary value problems.

As future perspectives, further investigations could focus on the numerical approximation of solutions and the stability analysis of more complex fractional systems, including those affected by impulsive effects, stochastic perturbations, or time delays. Another promising direction is the study of fractional differential equations of variable order and models involving the conformable derivative, which offer greater flexibility and more accurate modeling of memory-dependent processes. Such extensions would deepen the theoretical framework and broaden the applicability of fractional differential equations in various scientific and engineering domains.

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