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Ministry of Higher Education and Scientific Research

RELIZANE UNIVERSITY

FACULTY OF SCIENCE AND TECHNOLOGY

Course handout

MECHANICS OF THE MATERIAL POINT

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-FOREWORD-

This handout of courses and exercises in mechanics of the material point is a teaching aid intended for students in the first year of Basic Science and Technology Education (Bachelor's – Master's – Doctorate) 1st. A. L.M.D/ST.

This work follows the national program for the Point Mechanics subject "Physics-1" proposed by the CPND intended for students in the first year of the common core and this present work covers the four chapters of the material point mechanics program.

In the spirit of being understood by everyone and without great difficulty. At the beginning of each chapter, a summary of the definitions is given, then the physical concepts are mathematically detailed.

The exercises proposed and their solutions at the end of each chapter have been chosen to provide a simple and clear interpretation of the principles.

SUMMARY

Chapter I: Mathematical Reminders

Chapter III: Dynamics of the material point

Chapter IV: Work, power and energy

Chapter I

Mathematical Reminders

I.1-General information on physical quantities:

A physical quantity is a quantity which relates to a property and which can be measured. However, to measure is to compare. It is to compare, using an instrument, an unknown physical quantity with a quantity of the same nature, we will say of the same dimension, taken as a reference which we call a standard.

There are two types of physical quantities:

- identifiable physical quantities
- measurable physical quantities

- **I.1.1-Recognizable physical quantities:**

A quantity is said to be identifiable if the sum and the product of this quantity have no meaning.

Example :

- Viscosity
- Hardness
- Dielectric strength

- **I.1.2-Measurable physical quantities:**

A quantity is said to be measurable if we can assign a value to it. In addition, the sum and product of measurable quantities have a meaning and can also have a numerical value associated with it. The number which measures this quantity is the ratio of this quantity to the quantity of the same space chosen as unit.

There are two types of measurable quantities: Scalars and Vectors.

Example of scalar quantities:

Length, time, mass, temperature… :d_1=14m ; d_2=0.8cm.

- Time : t_1=28s ;t_2=7min ; t3=2h.

- Mass : m_1=20g ; m_2=3kg.

Example of vector quantities:

Volume, surface area, density, speed, acceleration.

Speed

$\vec{V}_1$: $\|\vec{V}_1\| = 80 km/h$

C light=3.108m/s

Accélératio**n**

$\vec{\gamma}$: $\|\vec{\gamma}\| = 5 m.s^{-2}$

$\vec{g}$: $g = 9.81 ms^{-2}$

I.2-Systems of units in physics:

I.2.1-Basic units of the international system:

The international system (IS) is made up of the units of the MKSA system, hence:

(M: Meter, K: Kilogram, S: Second and A: Ampere).

In this unit system, the basic or fundamental units are defined as follows:

• **Length:** the S.I base unit of length is the meter (m). The meter is the length equal to 1650 763.73 wavelength, in the vacuum of the radiation corresponding to the transition between the 2p10 and 5d5 levels of the Krypton 86 atom.

• **Mass:** the basic SI unit of mass is the Kilogram (Kg). The Kilogram is the mass of the platinum prototype, which was sanctioned by the general conference of Weights and Measures, held in Paris in 1889, and which was deposited at the BIPM (International Bureau of Weights and Measures: it is housed at the Pavillon de Breteuil in Sèvres, in the Parc de Saint-Cloud near Paris).

• **Time:** the basic unit in the I.S. of time is the second (s).

• **Electric current intensity:** the basic unit in the I.S. of electric current intensity is the Ampere (A). The ampere is defined as the intensity of the constant current.

• **Thermodynamic temperature:** the basic unit in the SI of thermodynamic temperature is the Degree Kelvin (°K). The Degree Kelvin is defined as being the degree of the thermodynamic scale of absolute temperatures.

• **Luminous intensity:** the basic unit in the S. I of luminous intensity is the Candela (Cd). The Candela is defined as being the luminous intensity, in a specific direction, of an opening perpendicular to this direction, having an area of 1/60 of a square centimeter and radiating like an integral radiator (Black Body) at the solidification temperature of platinum.

The international system (SI) is made up of 7 basic (fundamental) units corresponding to 7 physical quantities as summarized in the following table:

PHYSICAL SIZE	**UNIT**	**SYMBOL OF UNITY**
Length	Metre	M
Mass	Kilogram	Kg
Time	Second	S
Electric current intensity	Ampere	A
Temperature	Kelvin	K
Light intensity	Candela	cd
Quantity of material	mole	mol

Table 1 physical quantities and their units and symbols.

Note: There are also other systems of units in physics, such as:

• The CGS system (Centimeter, Gram, Second);

• The MTS system (Meter, Ton, Second).

I.2.2-Units derived from the International System

From the base units previously defined, we can easily define resulting units:

- **Area**: square meter (m^2) . Area of a square of side one meter (1m)

-**Volume**: cubic meter (m3). Volume of a cube of one meter side (1m)

- **Angle plan** : radian (rd ou rad). Plane angle, having its vertex at the center of a circle, intercepting, on the circumference of this circle, an arc of a length equal to that of the radius.

- ***Angle solide*** : stéradian (sr)

Solid angle, having its apex at the center of a sphere, cutting out on the surface of this sphere, an area equal to that of a square having as side the radius of the sphere.

- **Speed**: meter per second (m/s) .Speed of a mobile which, driven by a uniform movement, travels in one second (1s), a distance of one meter (1m).

- **Acceleration:** meter per second, per second (m/s^2)

- **Angular Speed**: radian per second (rd/s)

- **Force**: Newton (N)

Force which communicates to a body, having a mass of one kilogram (1kg), an acceleration of one meter per second squared ($1m/s^2$).

- **Moment:** Meter. Newton (m.N)

Moment relative to an axis, with a force of one Newton (1N) whose support is one meter (1m) away from the axis and which is orthogonal to it.

- **Energy, Work, Quantity of Heat:** Joule (J)

Work produced by a force of one Newton (1N) whose point of application moves one meter (1m) in the direction of the force.

-**Power**: Watt (W) .Power of one Joule per second (J/s) (Work/Time).

-**Stress, Pressure:** Pascal (Pa) . Uniform pressure which, acting on a flat surface of one square meter (1m2), exerts perpendicular to this surface, a total force of one Newton (1N).

I.3- Multiples and submultiples of units:

To simplify the handling of measurements that have high unit ratios, we precede the unit with prefixes, thus obtaining multiples or submultiples. In the table below we summarize the multiples and submultiples of the units.

Multiple or	***Factor by***		

submultiple	***which the unit is multiplied***	***Prefix***	***Symbol***
Multiple	10^{18}	Exa	E
Multiple	10^{15}	Péta	P
Multiple	10^{12}	Téra	T
Multiple	10^{9}	Giga	G
Multiple	10^{6}	Méga	M
Multiple	10^{3}	Kilo	K
Multiple	10^{2}	Hecto	H
sous-multiple	10^{1}	Déca	Da
sous-multiple	10^{-1}	Déci	D
sous-multiple	10^{-2}	Cent	C
sous-multiple	10^{-3}	Milli	M
sous-multiple	10^{-6}	Micro	μ
sous-multiple	10^{-9}	Nano	N
sous-multiple	10^{-12}	Pico	P
sous-multiple	10^{-15}	Femto	F
sous-multiple	10^{-18}	Atto	A
sous-multiple	10^{-21}	Zepto	Z
sous-multiple	10^{-24}	Yocto	Y

Table 2 Multiples and submultiples of units

I.4-Dimension and Equation to dimensions:

I.4.1-Dimension:

Each physical quantity is characterized by its dimension which is a property associated with a unit. The dimension of the quantity G is denoted [G]. It tells us about the physical nature of greatness. For example, if G has the dimension of a mass, we say that it is homogeneous to a mass. The relation [G] = M corresponds to the equation with the dimensions of the size G.

Here are the symbols of the seven fundamental quantities of the international system (SI):

* Length → L

*Mass → M

* Time → T
* The intensity of the electric current → I
* Temperature → θ
* Light intensity → J
* The quantity of material → N

So, if G has the dimension of a: mass, we write [G]=M

Length, we note [G]=L

Time, we note [G]=T

Intensity of electric current, we note [G]=I

Temperature, we note [G]=θ

Light intensity, we note [G] = J Quantity of material, we note [G]= N

All other quantities are linked to these fundamental quantities. For example, the volume of a cube V being the product of three lengths, its dimension is [V] = L3.

Exemple :

-speed

$V = \frac{Longueur}{Temps} \Rightarrow [V]=LT^{-1}$; Speed unit: ms^{-1}

- *accélération :*

$\gamma = a = \frac{Longueur}{(Temps)^2} \Rightarrow [\gamma]=[a]=LT^{-2}$; Acceleration unit: ms^{-2}

- strength :

$[F]= M.\gamma = MLT^{-2}$; Force unit: N

- *work :*

$[W]= F.L = ML^2T^{-2}$; Unité du travail : J

- power :

$P = \frac{Travail}{Temps} \Rightarrow [P]=\frac{[w]}{[t]} = ML^2T^{-3}$; Power Unit: W

- quantity of heat:

$[Q] = ML^2T^{-2}$ (like a job); unit: J

- pressure *:*

Pressure unit Pascal (Pa).

I.4.2-Uses of dimensional equations:

The dimensional equations are used to check the homogeneity of the formulas.

Example:

Work: Is homogeneous to an energy (that is to say a work), the equation for the dimensions of a work is $W = ML^2T^{-2}$. So ; Is indeed an energy.

I.4.3-Applications:

Example: Dimensions of ε_0 and μ_0 (permittivity and permeability of vacuum).

The force exerted between two point electric charges q and q', separated by a distance r, is given, in modulus, by Coulomb's law:

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{r^2}$$

The Laplace force which is exerted between two currents I and I', parallel and of length l, separated by a distance r is given by:

$$F = \frac{\mu_0}{2\pi}\frac{I.I'}{r}l.$$

-Give the dimensions of ε_0 and μ_0.

-Check the homogeneity of the relationship: $\varepsilon_0\mu_0c^2 = 1$, where c being the speed of light.

Solution :

$$[\varepsilon_0] = \left[\frac{1}{4\pi}\right]\left[\frac{qq'}{Fr^2}\right] = 1.\frac{I.T.I.T}{MLT^{-2}.L^2} = L^{-3}M^{-1}T^4I^2$$

$$[\mu_0] = \left[2\pi\frac{F.r}{II'.l}\right] = 1.\frac{MLT^{-2}.L}{I.I.L} = MLT^{-2}I^{-2}$$

$$\left[c^2\right] = L^2T^{-2} \Rightarrow \left[\varepsilon_0\mu_0c^2\right] = (L^{-3}M^{-1}T^4I^2)(MLT^{-2}I^{-2})(L^2T^{-2}) = 1$$

I.5-Relationship between units:

The relationships between the units of different systems can be easily established using the dimensional equations.

Example :

Calculate the relationship between the Barye (unit of pressure in the CGSA system) and the pascal (unit of pressure Pa in the S. I. or MKSA).

Pressure: p = $ML^{-1}T^{-2}$

Solution :

$$\left[\frac{1Pa}{1Barye}\right]=\frac{M}{M}\frac{L^{-1}}{L^{-1}}\frac{T^{-2}}{T^{-2}}=\frac{MKSA}{CGSA}=\frac{kg}{g}\frac{m^{-1}}{cm^{-1}}\frac{s^{-2}}{s^{-2}}=\frac{1}{10^{-3}}\frac{1^{-1}}{\left(10^{-2}\right)^{-1}}\frac{1}{1}=10\Rightarrow 1Pa=10Baryes.$$

I.6-Uncertainties and error calculation:

I.6.1-Uncertainties:

When we measure any quantity G, we can never obtain an exact value,

but a more or less approximate value. We call error the difference between the value measured and the exact value. But since we do not know the exact value, we cannot know the error made. The result is therefore always uncertain (we are talking about measurement uncertainties).

Example: G = Gex± G

There are several types of errors:

- Systematic errors: remain approximately the same when operating under identical conditions.
- Accidental or fortuitous errors: always present; are often due to causes that are difficult to know: vibrations, temporary temperature variations, poor contacts whose electrical resistance is a function of current and time.

However, random errors can only be combated by repeating measurements with average calculations justified by statistical considerations.

- Personal errors: are attributable to all imperfections, both physical and intellectual, of the operator. For example, reading errors on the dial of an indicating device.

In addition to the reading error which exists in the value given by the measuring device, this error is given by the relationship:

$X_m = \frac{C\ Xm}{100}$ (%)

Where C, is the device class found on each device, Xm is the caliber used in the measurement and DXm is the largest absolute error made.

I.6.2-Error calculation:

To measure a quantity G; If we repeat the measurement several times, we obtain slightly different numbers. If X is the exact (true) value of G and Xex the result of the measurement (experimental), the difference δG = X – Xex is called absolute error of the measurement.

Since the absolute error is not known, we must content ourselves with looking for an upper limit ΔG, called absolute uncertainty, such that |δG| =ΔG. The ratio is called relative uncertainty.

I.6.3-Absolute uncertainty:

Since the absolute error δan' is not known, we simply give an upper limit Δa called absolute uncertainty such that: $|\delta a| \le \Delta a \Rightarrow \Delta a > 0$ (the absolute uncertainty of a quantity is always positive (Δa>0)). This means that the absolute uncertainty is the maximum value that the absolute error can reach.

I.6.4-Relative uncertainty:

It is defined by the ratio $\frac{\Delta a}{a}$, it is a positive number without dimensions. It represents the precision of the measurement and is given in percentage%. The more the value of n is small the more the precision is large and the opposite is true.

I.6.5- Calculations of errors and uncertainties:

Given a quantity g=f(x, y, z), its total differential is written:

$$dg = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \,. \qquad \text{(I.1)}$$

The absolute uncertainty on the variable g is obtained by passing to the variations of the variables which compose it, i.e.:

$$\Delta g = \left|\frac{\partial f}{\partial x}\right|\Delta x + \left|\frac{\partial f}{\partial y}\right|\Delta y + \left|\frac{\partial f}{\partial z}\right|\Delta z \,. \qquad \text{(I.2)}$$

Given a quantity g= f(x, y, z)= $x^m y^n z^p$, détermine the relative uncertainty .

$$\frac{\Delta g}{g} \cdot \Rightarrow dg = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \;= mx^{m-1}dx.y^n z^p + ny^{n-1}dy.x^m z^p + pz^{p-1}dz.x^m y^n \qquad \text{(I.3)}$$

Calculation of relative error

$$\frac{dg}{g}=\frac{(mx^{m-1}dx.y^{n}z^{p}+ny^{n-1}dy.x^{m}z^{p}+pz^{p-1}dz.x^{m}y^{n})}{x^{m}y^{n}z^{p}} \quad \text{(I.4)}$$

$$\frac{dg}{g}=m\frac{dx}{x}+n\frac{dy}{y}+p\frac{dz}{z}$$

Transition to the calculation of relative uncertainties

$$(n,m,p)\in R^{3}.\ \frac{dg}{g}\Rightarrow\frac{\Delta g}{g}=|m|\frac{\Delta x}{x}+|n|\frac{\Delta y}{y}+|p|\frac{\Delta z}{z} \quad \text{(I.5)}$$

We can find the same result using the logarithm function Ln.

Ln(g)= Ln ($x^m y^n z^p$)

$$Ln(g)=mLn(x)+nLn(y)+pLn(z)\ . \quad \text{(I.6)}$$

By differentiating both sides of the equation we obtain:

$$d\left[Ln(g)\right]=d\left[mLn(x)+nLn(y)+pLn(z)\right]$$

$$\frac{dg}{g}=m\frac{dx}{x}+n\frac{dy}{y}+p\frac{dz}{z} \quad \text{(I.7)}$$

Transition to the calculation of relative uncertainties

$$(n,m,p)\in R^{3}.\Rightarrow\frac{\Delta g}{g}=|m|\frac{\Delta x}{x}+|n|\frac{\Delta y}{y}+|p|\frac{\Delta z}{z} \quad .\text{(I.8)}$$

I.6.6-Calculation of uncertainties of an algebraic sum:

The absolute uncertainty of an algebraic sum is equal to the arithmetic sum of the absolute uncertainties of each of the members constituting this sum.

Consider the algebraic sum: S=nU+pV-qW+k where (n, p, q) are positive coefficients (parameters); k is a constant and knowing the absolute uncertainties on the quantities U, V, W are respectively ΔU, ΔV and ΔW. What is the absolute uncertainty on S?

Calculation of error on the quantity S: we proceed to the derivation of the quantity S taken as a function which depends on three (3) variables U, V and W.d(S)=d(nU+pV –qW+k)

dS =ndU+pdV –qdW+0.

Calculation of the absolute uncertainty on the quantity S

ΔS=nΔU +pΔV+ qΔW (I.9)

Important note:

The measurement results are written as follows:

$X_0=X\pm\Delta X$ *(unité).* (I.10)

X0: Actual value X: Approximate value

ΔX: Absolute uncertainty unit: unit of the measured quantity.

I.6.7-Applications:

Example:

- Calculate the relative uncertainty of the density of the substance of a homogeneous cube of edge a having an absolute uncertainty Δa and of mass m having absolute uncertainty Δm. [a=(10.00±0.05) cm,m=(15000±15) g]

Solution :

$$\rho=\frac{masse}{volume}=\frac{m}{a^3}=m.a^{-3}\Rightarrow Ln\rho=Ln(m.a^{-3})=Ln(m)-3Ln(a)\ .$$

By differentiating both sides of the equation we obtain:

$$d\left[Ln(\rho)\right]=d\left[Ln(m)\right]-3d\left[Ln(a)\right]=\frac{d\rho}{\rho}=\frac{dm}{m}-3\frac{da}{a}$$

Moving on to the uncertainties relating $\Rightarrow\frac{\Delta\rho}{\rho}=\frac{\Delta m}{m}+3\frac{\Delta a}{a}$.

$$\rho=\frac{15000}{1000}=15g.cm^{-3}$$

$$\frac{\Delta\rho}{\rho}=\frac{15}{15000}+3\frac{0.05}{10}=0.016\Rightarrow\Delta\rho=\rho*0.016=15*0.016=0.24g.cm^{-3}$$

$$\rho=(15.00\pm0.24)g.cm^{-3}$$

Exemple2 :

We want to determine the density μ of a solid body by application of Archimedes' theorem which is: $\mu=\frac{m_2-m_1}{m_3-m_1}$ where m1, m2 and m3 are the results of three mass measurements carried out, successively, with the same balance. Find the relative uncertainty on μ. We give m1=120g; m2=390g; m3=220g and Δm1 = Δm2= Δm3= Δm =0.2g.

Solution :

$$\mu = \frac{m_2 - m_1}{m_3 - m_1} = \frac{390-120}{220-120} = \frac{270}{100} = 2.7$$

$$\mu = \frac{m_2 - m_1}{m_3 - m_1} = (m_2 - m_1)(m_3 - m_1)^{-1} \Rightarrow Ln(\mu) = Ln(m_2 - m_1) - Ln(m_3 - m_1)$$

By deriving both sides of the equation we obtain:

$$\frac{d\mu}{\mu} = \frac{d(m_2 - m_1)}{(m_2 - m_1)} - \frac{d(m_3 - m_1)}{(m_3 - m_1)} = \frac{dm_2}{(m_2 - m_1)} - \frac{dm_1}{(m_2 - m_1)} - \frac{dm_3}{(m_3 - m_1)} + \frac{dm_1}{(m_3 - m_1)}$$

$$= dm_1\left[\frac{1}{(m_3 - m_1)} - \frac{1}{(m_2 - m_1)}\right] + \frac{dm_2}{(m_2 - m_1)} - \frac{dm_3}{(m_3 - m_1)}$$

-Transition to uncertainties:

$$\frac{\Delta\mu}{\mu} = \Delta m_1\left[\frac{1}{(m_3 - m_1)} - \frac{1}{(m_2 - m_1)}\right] + \frac{\Delta m_2}{m_2 - m_1} + \frac{\Delta m_3}{m_3 - m_1} = \Delta m_1 = \Delta m_2 = \Delta m_3 = \Delta m \ = 0.2g$$ On

we can then write:

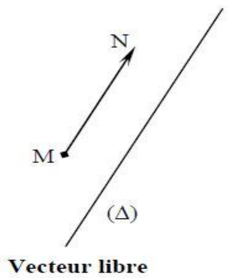
$$\frac{\Delta\mu}{\mu} = \Delta m\left[\frac{1}{m_3 - m_1} - \frac{1}{m_2 - m_1} + \frac{1}{m_2 - m_1} + \frac{1}{m_3 - m_1}\right] = 2\frac{\Delta m}{m_3 - m_1} = \frac{2*0.2}{}$$

$$\frac{\Delta\mu}{\mu} = \Delta m\left[\frac{1}{m_3 - m_1} - \frac{1}{m_2 - m_1} + \frac{1}{m_2 - m_1} + \frac{1}{m_3 - m_1}\right] = \frac{2\Delta m}{m_3 - m_1} = \frac{2*0.2}{100} = 0.004$$

I.7-Vecteurs :

I.7.1-Définitions :

A vector is a geometric object of the plane or space; it is an “arrow” which is characterized by its norm (its “length”) and its Orientation. We can define opérations on these vectors: an addition, a subtraction, a scalar product, a vector product.



FigI.2 free vector MN and its direction the line Δ

Afterwards, we define a vector as an element of a vector space (that is to say a set provided with two laws of composition and obeying certain properties.

It is completely defined if we give ourselves:

- its origin or point of application.

- its direction which is that of the right (Δ).
- Its direction which is the direction of the movement of a mobile going from M to N.

I.7.2-Operations on free vectors:

I.7.2.a-Addition:

By definition, the sum of a certain number of free vectors; ; ;…. is a free vector whose representation is obtained by constructing a polygonal contour of any origin and whose sides are respectively equal to the vectors:

$\vec{a}_1$; $\vec{a}_2$; $\vec{a}_3$;........ ; $\vec{a}_n$.

We write :

$$\vec{a} = \vec{a}_1 + \vec{a}_2 + \vec{a}_3 + \dots\dots + \vec{a}_n . \qquad \text{(I.11)}$$

Exemple :

Addition of vectors $\vec{V} = \vec{V}_1 + \vec{V}_2 + \vec{V}_3 + \vec{V}_4 + \vec{V}_5$

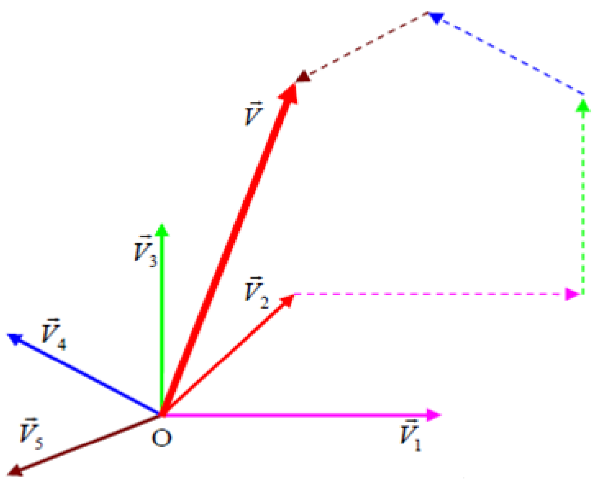


Fig. I.4 addition of vectors.

It is easily demonstrated, by considerations of elementary geometry, that this sum is not modified when we invert the order of the vectors, or when we replace several of them by their sum. The sum of the vectors is therefore commutative and associative.

On the other hand, the sum of two opposite vectors is zero. Which allows us to write?

$\vec{a} + op(\vec{a}) = \vec{a} + (-\vec{a}) = \vec{0}$

I.7.2.b-Différence de deux vecteurs :

The difference of two vectors $\vec{V}_1$ et $\vec{V}_2$: $\vec{V}=\vec{V}_1-\vec{V}_2$ is the sum of the first vector and the opposite of the second: $\vec{V}=\vec{V}_1+(-\vec{V}_2)$

Example :

Difference of vectors:

a) $\vec{V}=\vec{V}_1-\vec{V}_2$

b) $\vec{V}'=\vec{V}_2-\vec{V}_1$

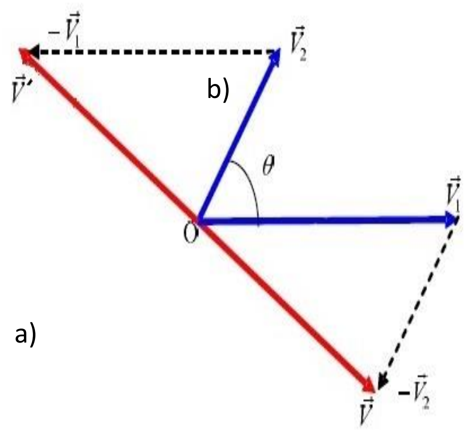


Fig. I.5 Difference of vectors

We notice that the vectors $\vec{V}$ et $\vec{V}'$ are opposites where their sum is zero $\Rightarrow \vec{V}+\vec{V}'=\vec{0}$.

I.7.3-Components of a vector:

Le vecteur $\vec{a}=\overrightarrow{OR}=\overrightarrow{OM}+\overrightarrow{ON}+\overrightarrow{OP}$

$\overrightarrow{OH}=\overrightarrow{OM}+\overrightarrow{ON}\Rightarrow\overrightarrow{OR}=\overrightarrow{OH}+\overrightarrow{HR}$

$\overrightarrow{HR}=\overrightarrow{OP}$

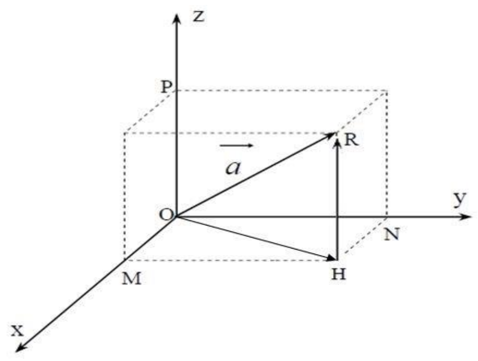


Fig. I.6 vector representation in space

The three projections OM, ON, OP of the vector = on the three coordinate axes Oxyz can be considered as three vectors; and carried respectively by the three axes ( ). The three vectors are called the 3 components of the vector = .

If we consider several vectors $\vec{U}_1, \vec{U}_2, \vec{U}_3$ of components ($\vec{X}_1$; $\vec{Y}_1$; $\vec{Z}_1$),($\vec{X}_2$; $\vec{Y}_2$; $\vec{Z}_2$) and

$(\vec{X}_3\ ;\vec{Y}_3;\vec{Z}_3)$, the general theorems of addition allow their sum to be written in the form:

$$\vec{U} = \vec{U}_1+\vec{U}_2+\vec{U}_3=\vec{X}_1+\vec{Y}_1+\vec{Z}_1+\vec{X}_2+\vec{Y}_2+\vec{Z}_2+\vec{X}_3+\vec{Y}_3+\vec{Z}_3$$

$$= (\vec{X}_1+\vec{X}_2+\vec{X}_3) + (\vec{Y}_1+\vec{Y}_2+\vec{Y}_3) + (\vec{Z}_1+\vec{Z}_2+\vec{Z}_3) \tag{I.12}$$

Thus the vector equality $\vec{U} = \vec{U}_1+\vec{U}_2+\vec{U}_3$ is equivalent to the following three algebraic equalities:

$$\left.\begin{array}{l} X = X_1+X_2+X_3 \\ Y = Y_1+Y_2+Y_3 \\ Z = Z_1+Z_2+Z_3 \end{array}\right\} \tag{I.13}$$

I.7.4-Multiplication d'un vecteur par une quantité scalaire :

The sum of n vectors, all equal to the same vector $\vec{U}$ is obviously a vector $\vec{V}$ having the same direction and same meaning as the vector $\vec{U}$ and whose magnitude V is equal to n times the magnitude U of the vector$\vec{U}$.

$$\vec{V} = n\vec{U} = \vec{U}+\vec{U}+\vec{U}+\ldots\ldots+\vec{U}\ \text{(n times)} \tag{I.14}$$

Likewise the report $\frac{\vec{W}}{\vec{U}}$ of two parallel vectors $\vec{W}$ and $\vec{U}$ is an algebraic number m, whose absolute value is the ratio $\frac{W}{U}$ of the magnitudes of the two vectors, and which is positive if the two vectors have the same direction, negative if the two vectors have the opposite direction.

Noticed :

The three components $\vec{X}\ ;\vec{Y}\ ;\vec{Z}$ of a vector $\vec{U}$ can be considered as the products by the three algebraic numbers X, Y, Z coordinates of this vector, of vectors ($\vec{i}\ ;\vec{j}\ ;\vec{k}$) de grandeurs of magnitudes equal to unity, respectively directed along the three axes Ox, Oy, Oz in the positive direction of these axes are $(\vec{i}\ ;\vec{j}\ ;\vec{k})$ called unit vectors of the axes. So the vector $\vec{U}$ coordinates X, Y, Z will be written:

$$\vec{U} = X\vec{i} + Y\vec{j} + Z\vec{k} \tag{I.15}$$

I.7.5- Modulus of a vector:

The module or norm of a vector $\overrightarrow{AB} = \vec{U} = X\vec{i} + Y\vec{j} + Z\vec{k}$ is equal to the length of the line segment AB, and is written in the following form:

$$\left\|\overrightarrow{AB}\right\| = \left\|\vec{U}\right\| = \sqrt{X^2 + Y^2 + Z^2} \tag{I.16}$$

I.7.6- Scalar product:

We call the scalar product of two vectors $\vec{U}_1$ et $\vec{U}_2$, making between them the angle θ, and is represented by the quantity m denoted : m $= \vec{U}_1 \cdot \vec{U}_2$ (Scalar)

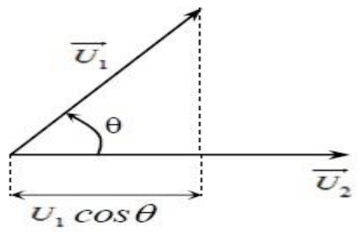


Fig. I.7 vector projection $\vec{U}_1$ on the direction of $\vec{U}_2$

Using the modules:

m $= U_1 \cdot U_2 \cdot \cos\theta = \left\|\vec{U}_1\right\| \cdot \left\|\vec{U}_2\right\| \cdot \cos\theta$. (I.17)

Using the components:

m $= \vec{U}_1 \cdot \vec{U}_2 = (X_1\vec{i} + Y_1\vec{j} + Z_1\vec{k}) \cdot (X_2\vec{i} + Y_2\vec{j} + Z_2\vec{k})$(I.18)

m $= X_1X_2 + Y_1Y_2 + Z_1Z_2$

m$= \left\|\vec{U}_1\right\| \cdot \left\|\vec{U}_2\right\| \cdot \cos\theta = X_1X_2 + Y_1Y_2 + Z_1Z_2$ (I.19)

According to the definition, it turns out that the dot product is commutative i.e.

$\vec{U}_1 . \vec{U_2} = \vec{U_2} . \vec{U}_1$

It is easy to see from the definition that:

$\vec{U} . \vec{U} = (\vec{U})^2 = U^2$ (I.20)

We can extend the various rules of algebraic calculation to the scalar product:

Dot product of vector sums

$$(\vec{U_1} + \vec{U_2}).(\vec{V_1} + \vec{V_2}) = \vec{U_1}.\vec{V_1} + \vec{U_1}.\vec{V_2} + \vec{U_2}.\vec{V_1} + \vec{U_2}.\vec{V_2} \qquad \text{(I.21).}$$

Likewise by designating by $X_1; Y_1; Z_1$ and $X_2; Y_2; Z_2$ the coordinates of two vectors with respect to the three rectangular axes of unit vectors ($\vec{i}$; $\vec{j}$; $\vec{k}$), we can write :

$$\vec{U}_1 . \vec{U_2} = (X_1\vec{i} + Y_1\vec{j} + Z_1\vec{k}).(X_2\vec{i} + Y_2\vec{j} + Z_2\vec{k})$$

$$= X_1X_2\vec{i}^{\,2} + X_1Y_2\vec{i}\vec{j} + X_1Z_2\vec{i}\vec{k} + Y_1X_2\vec{j}\vec{i} + Y_1Y_2\vec{j}^{\,2} + Y_1Z_2\vec{j}\vec{k} + Z_1X_2\vec{k}\vec{i} + Z_1Y_2\vec{k}\vec{j} + Z_1Z_2\vec{k}^{\,2}$$

$\vec{i}^{\,2} = \vec{j}^{\,2} = \vec{k}^{\,2} = 1$

$\vec{i} . \vec{j} = \vec{i} . \vec{k} = \vec{j} . \vec{i} = \vec{j} . \vec{k} = \vec{k} . \vec{i} = \vec{k} . \vec{j} = 0$

$$\left.\begin{aligned} &\Rightarrow \vec{U}_1 . \vec{U_2} = X_1X_2 + Y_1Y_2 + Z_1Z_2 \\ &\left(\vec{U}\right)^2 = U^2 = X^2 + Y^2 + Z^2 \end{aligned}\right\} \qquad \text{(I.22)}$$

In the same way we have:

Quantities $X^2 + Y^2 + Z^2$ et $X_1X_2 + Y_1Y_2 + Z_1Z_2$ represent scalar quantities defined independently of the Oxyz axes, they do not depend on the choice of axes, and therefore constitute what we call invariants.

I.7.7-Vector product:

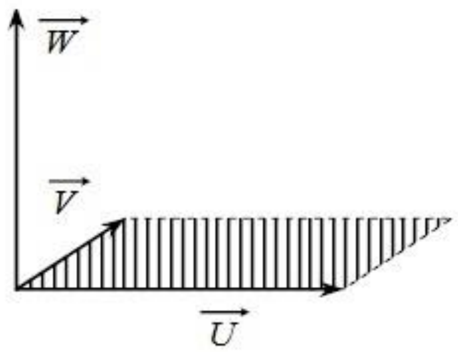
We call the vector product of a free vector $\vec{U}$ by a free vector $\vec{V}$ and which we note by:

$$\vec{W} = \vec{U} \wedge \vec{V} \tag{I.23}$$

A vector $\vec{W}$ free, perpendicular to the vector plane $\vec{U}$ and $\vec{V}$, of meaning such that the trihedron, , is direct and whose magnitude is given by:

$$W = U.V.\text{Sin}\theta \tag{I.24}$$

The module $\vec{W}$ therefore corresponds to the area of the parallelogram constructed on the two vectors $\vec{U}$ et $\vec{V}$ (figure.I.8). It follows from the definition that the vector product is not independent of the order of the two factors.



FigI.8.a Geometric representation of the vector $\vec{W}$

$\vec{V_1} \wedge \vec{V_2} = \vec{W}$ et $\vec{V_2} \wedge \vec{V_1} = -\vec{W}$ are two opposite vectors (see figure.I.8.b).

We can apply the ordinary rules of algebraic calculation to the vector product provided that we never invert the order of the factors.

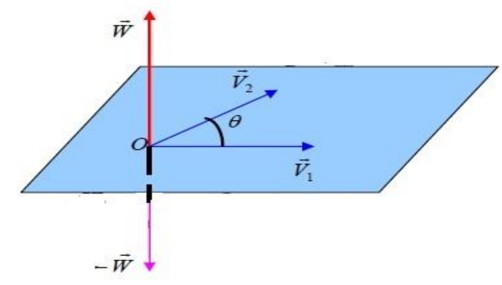


Fig I.8 b Geometric representation of vectors $\vec{W}$ and $\vec{W}'$

Distribution of cross product versus addition

$$\vec{U}\wedge(\vec{V_1}+\vec{V_2})=\vec{U}\wedge\vec{V_1}+\vec{U}\wedge\vec{V_2} \qquad \text{(I.25)}$$

Let us express in particular the cross product of two vectors $\vec{V}$ et $\vec{V'}$ coordinates (x, y, z) and (x’, y’, z’).

$$\vec{V}\wedge\vec{V'}=(x\vec{i}+y\vec{j}+z\vec{k})\wedge(x'\vec{i}+y'\vec{j}+z'\vec{k})$$

$$=xx'(\vec{i}\wedge\vec{i})+xy'(\vec{i}\wedge\vec{j})+xz'(\vec{i}\wedge\vec{k})+yx'(\vec{j}\wedge\vec{i})+yy'(\vec{j}\wedge\vec{j})+yz'(\vec{j}\wedge\vec{k})+$$
$$+zx'(\vec{k}\wedge\vec{i})+zy'(\vec{k}\wedge\vec{j})+zz'(\vec{k}\wedge\vec{k}). \qquad \text{(I.26)}$$

$$\begin{cases}(\vec{i}\wedge\vec{i})=(\vec{j}\wedge\vec{j})=(\vec{k}\wedge\vec{k})=\vec{0}\\[2ex](\vec{i}\wedge\vec{j})=\vec{k}\ \text{et}\ (\vec{j}\wedge\vec{i})=-\vec{k}\quad(\vec{j}\wedge\vec{k})=\vec{i}\ \text{et}\ (\vec{k}\wedge\vec{j})=-\vec{i}\quad(\vec{k}\wedge\vec{i})=\vec{j}\ \text{et}\ (\vec{i}\wedge\vec{k})=-\vec{j}\end{cases}$$

from where $\Rightarrow \vec{V}\wedge\vec{V'}=xy'\vec{k}-xz'\vec{j}-yx'\vec{k}+yz'\vec{i}+zx'\vec{j}-zy'\vec{i}$

$$=(yz'-zy')\vec{i}+(zx'-xz')\vec{j}+(xy'-yx')\vec{k}\ . \qquad \text{(I.27)}$$

Practical layout of calculating the cross product of two vectors $\vec{V_1}\wedge\vec{V_2}$:

$$\vec{V_1}\wedge\vec{V_2}=\begin{vmatrix}\vec{i} & -\vec{j} & \vec{k}\\ x & y & z\\ x' & y' & z'\end{vmatrix}=\begin{vmatrix}y & z\\ y' & z'\end{vmatrix}\vec{i}-\begin{vmatrix}x & z\\ x' & z'\end{vmatrix}\vec{j}+\begin{vmatrix}x & y\\ x' & y'\end{vmatrix}\vec{k}=(yz'-zy')\vec{i}+(zx'-xz')\vec{j}+(xy'-yx')\vec{k} \qquad \text{(I.28)}$$

I.7.8-Applications:

Example1:

Calculate the cross product of the two vectors $\overrightarrow{W} = \overrightarrow{V_1} \wedge \overrightarrow{V_2}$ knowing that $\overrightarrow{V_1} = 2\vec{i} + \vec{j} - \vec{k}$ et $\overrightarrow{V_2} = \vec{i} + 0\vec{j} - 2\vec{k}$

- Deduce the angle θ between them.

Solution :

$$\overrightarrow{W} = \left[(1\times -2)-(-1\times 0)\right]\vec{i} + \left[(-1\times 1)-(2\times -2)\right]\vec{j} + \left[(2\times 0)-(1\times 1)\right]\vec{k} = -2\vec{i} + 3\vec{j} - \vec{k}$$

$$\left\|\overrightarrow{V_1}\right\| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$$

$$\left\|\overrightarrow{V_2}\right\| = \sqrt{1^2 + 0^2 + (-2)^2} = \sqrt{5}.$$

$$\left\|\overrightarrow{W}\right\| = \sqrt{(-2)^2 + 3^2 + (-1)^2} = \sqrt{14} = 3.74$$

$$\left\|\overrightarrow{W}\right\| = \left\|\overrightarrow{V_1}\right\|.\left\|\overrightarrow{V_2}\right\|.\sin\theta \Rightarrow \sin\theta = \frac{\left\|\overrightarrow{W}\right\|}{\left\|\overrightarrow{V_1}\right\|.\left\|\overrightarrow{V_2}\right\|} = \frac{\sqrt{14}}{\sqrt{6}.\sqrt{5}} 0.683 \Rightarrow \theta = 43.09°$$

Example2:

Calculate the dot product of the two vectors $m = \overrightarrow{V_1}.\overrightarrow{V_2}$ and the angle θ between them knowing that $\overrightarrow{V_1} = 2\vec{i} + \vec{j} - \vec{k}$ et $\overrightarrow{V_2} = \vec{i} + 0\vec{j} - 2\vec{k}$

Solution :

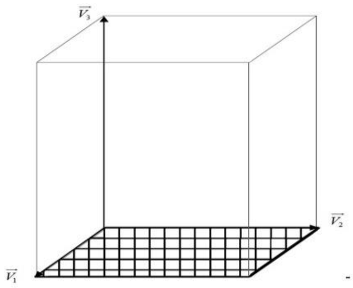
$$m = (2\vec{i} + \vec{j} - \vec{k})(\vec{i} + 0\vec{j} - 2\vec{k}) = 2 + 0 + 2 = 4$$

$$m = \left\|\overrightarrow{V_1}\right\|.\left\|\overrightarrow{V_2}\right\|.\cos\theta \Rightarrow \cos\theta = \frac{m}{\left\|\overrightarrow{V_1}\right\|.\left\|\overrightarrow{V_2}\right\|} = \frac{4}{\sqrt{6}.\sqrt{5}} = 0.73 \Rightarrow \theta = 43.09°$$

I.7.9-Mixed product:

We call the mixed product of three vectors $\overrightarrow{V_1}$;$\overrightarrow{V_2}$ et $\overrightarrow{V_3}$ a scalar quantity m equal to the scalar product of the third vector and the vector product of the first two: $m = (\overrightarrow{V_1} \wedge \overrightarrow{V_2}).\overrightarrow{V_3}$

This mixed product gives the volume of the parallelepiped constructed on the three vectors $\vec{V_1}$; $\vec{V_2}$ et $\vec{V_3}$ (see figure I.9)



FigI.9 Representation of the mixed product

As the volume of the parallelepiped can be evaluated from any of the faces, we have :

$$m = (\vec{V_1} \wedge \vec{V_2}).\vec{V_3} = (\vec{V_2} \wedge \vec{V_3}).\vec{V_1} = (\vec{V_3} \wedge \vec{V_1}).\vec{V_2} \quad \text{(I.29)}$$

As the dot product is commutative, we can therefore interchange scalar multiplication and vector multiplication.

Practical calculation of the mixed product of vectors $\vec{V_1}$; $\vec{V_2}$ and $\vec{V_3}$

$$\vec{V_1}.(\vec{V_2} \wedge \vec{V_3}) = \begin{vmatrix} x_1 & -y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = (y_2 z_3 - y_3 z_2)x_1 - (x_2 z_3 - x_3 z_2)y_1 + (x_2 y_3 - x_3 y_2)z_1 \quad \text{(I.30)}$$

I.7.10-Applications:

Exercise 1 :

a- Find the sum of the following three vectors such that:

$\vec{V_1} = 5\vec{i} - 2\vec{j} + 2\vec{k}$; $\vec{V_2} = -3\vec{i} + \vec{j} - 7\vec{k}$; $\vec{V_3} = 4\vec{i} + 7\vec{j} + 6\vec{k}$; $\vec{R} = \vec{V_1} + \vec{V_2} + \vec{V_3}$.

b- Calculate the modulus of the resultant as well as the angles it forms with the axes ox; oy and oz.

Solution :

a- $\vec{R}=\vec{V_1}+\vec{V_2}+\vec{V_3}=(5-3+4)\vec{i}+(-2+1+7)\vec{j}+(+2-7+6)\vec{k}=6\vec{i}+6\vec{j}+\vec{k}$

b- $\|\vec{R}\|=R=\sqrt{6^2+6^2+1^2}=\sqrt{73}=8.544$

$$\cos\alpha_x=\frac{R_x}{R}=\frac{6}{8.544}=0.702\Rightarrow\alpha_x=45.39°$$

$$\cos\alpha_y=\frac{R_y}{R}=\frac{6}{8.544}=0.702\Rightarrow\alpha_y=45.39°$$

$$\cos\alpha_z=\frac{R_z}{R}=\frac{1}{8.544}=0.117\Rightarrow\alpha_z=83.29°$$

Exercice2 :

We consider, in a reference frame (oxyz), three vectors

$\vec{V_1}=-3\vec{i}-4\vec{j}+4\vec{k}$, $\vec{V_2}=2\vec{i}+3\vec{j}-4\vec{k}$, $\vec{V_3}=5\vec{i}-\vec{j}+3\vec{k}$

a- Calculate the modules of $\vec{V_1}$; $\vec{V_2}$ and $\vec{V_3}$.

b- Calculate the components as well as the modules of the vectors:

$\vec{A}=\vec{V_1}+\vec{V_2}+\vec{V_3}$;

$\vec{B}=2\vec{V_1}-\vec{V_2}+\vec{V_3}$

c- Determine the unit vector carried by the vector $\vec{C}=\vec{V_1}+\vec{V_3}$.

d- Calculate the dot product $\vec{V_1}.\vec{V_3}$ and deduce the angle θ formed by the two vectors.

e- Calculate the vector product $\vec{V_1}\wedge\vec{V_3}$

Solution :

a- $\|\vec{V_1}\|=\sqrt{41}=6.40$; $\|\vec{V_2}\|=\sqrt{29}=5.38$; $\|\vec{V_3}\|=\sqrt{35}=5.916$

b- $\vec{A}=10\vec{i}-2\vec{j}+3\vec{k}$; $\vec{B}=9\vec{i}-15\vec{j}+15\vec{k}$

c- $\vec{C}=8\vec{i}-5\vec{j}+7\vec{k}$; $\vec{u}=\dfrac{\vec{C}}{\|\vec{C}\|}=\dfrac{8}{\sqrt{35}}\vec{i}-\dfrac{5}{\sqrt{35}}\vec{j}+\dfrac{7}{\sqrt{35}}\vec{k}$

d- $\vec{V_1}.\vec{V_3}=x_1x_3+y_1y_3+z_1z_3=15+4+12=31.$

$$\cos\theta = \frac{\vec{V_1}.\vec{V_3}}{\|\vec{V_1}\|.\|\vec{V}\|} = \frac{31}{\sqrt{41}.\sqrt{35}} = \frac{31}{37.88} \approx 0.818 \Rightarrow \theta = 35.08°.$$

e- $\vec{V_1} \wedge \vec{V_3} = 5\vec{i} - 26\vec{j} - 17\vec{k}$

I.8-Derivative of a vector:

I.8.1-Definition:

Given a variable t, suppose that for each value of t we know how to correspond to a certain vector $\vec{V}$, we say that this vector is a function of t: $\vec{V}(t)$. Analytically, this means that we give ourselves three functions x(t), y(t), z(t) of the variable t which are the coordinates of the vector $\vec{V}$.

Consider two vectors $\vec{V}(t)$ et $\vec{V}(t+\Delta t)$ of the variable t; two values of the vector correspond to them $\vec{V}(t)$, we can form their difference which is a certain vector $\Delta\vec{V}$. This vector $\Delta\vec{V}$ generally tends towards zero at the same time as , but the vector $\frac{\Delta\vec{V}}{\Delta t}$ generally tends towards a limit. This limit is a vector $\vec{V'}$ derivative of the vector $\vec{V}$ *;* we write:

$$\vec{V'} = \frac{d\vec{V}}{dt} \tag{I.31}$$

We can define in the same way $(\vec{V}')'$ which we call second derivative of the vector $\vec{V}$, and we write:

$$\vec{V}'' = \frac{d^2\vec{V}}{dt^2} \tag{I.32}$$

We immediately see that the components of the vector $\vec{V'}$ are given by:

$$x' = \frac{dx}{dt}\ ;\ y' = \frac{dy}{dt}\ ;\ z' = \frac{dz}{dt} \tag{I.33}$$

It is the same for the second derivative of the vector $\vec{V}$. The components of $\vec{V''}$ are given by:

$$x'' = \frac{d^2x}{dt^2}\ ;\ y'' = \frac{d^2y}{dt^2}\ ;\ z'' = \frac{d^2z}{dt^2} \tag{I.34}$$

I.8.2- Various applications:

Derivative of a scalar product:

$$(\vec{U}.\vec{V})' = \vec{U}'.\vec{V} + \vec{V}'.\vec{U} \qquad \text{(I.35)}$$

- If $\vec{V}$ is constant $\Rightarrow \vec{V}' = \vec{0}$ from where $(\vec{U}.\vec{V})' = \vec{U}'\vec{V}$
- if $\vec{V}$ is a constant modulus vector: $\left\|\vec{V}\right\| = V = C^{st}$,

Its square $(\vec{V})^2 = V^2 = C^{st}$,

Its derivative which is given by: $\left[\vec{V}^2\right]' = 2\vec{V}.\vec{V}' = 0$, on a $\vec{V} \neq \vec{0}$ and $\vec{V}' \neq \vec{0}$.

The scalar product of the two vectors:

$\vec{V}.\vec{V}' = 0 = \left\|\vec{V}\right\|.\left\|\vec{V}'\right\|\cos\alpha = 0 \Rightarrow \cos\alpha = 0 \Rightarrow \alpha = \frac{\pi}{2} = 90°$.

$\Rightarrow$ The vector $\vec{V}'$ is perpendicular to the vector $\vec{V}$.$\vec{V}' \perp \vec{V}$.

- We can always define a vector by $\vec{a} = a\vec{u}$ or $\vec{u}$ is a unitary vector. Its derivative is obtained by:

$$\vec{a}' = (a\vec{u})' = a'\vec{u} + a\vec{u}' \qquad \text{(I.36)}$$

We therefore see that the derivative of is the sum of two vectors, the first of which is parallel to the vector, the second of which is perpendicular to it.

Example :

Let be the orthonormal coordinate system ***(o, x, y), of unit vectors*** $(\vec{i}, \vec{j})$ such as : $\left\|\vec{i}\right\| = \left\|\vec{j}\right\| = 1$. Let the unit vectors $\vec{U};\vec{V}$ such as $\left\|\vec{U}\right\| = \left\|\vec{V}\right\| = 1$ and $\vec{U} \perp \vec{V}$

Let's project the vectors $\vec{U}$ et $\vec{V}$ on the axes $\overrightarrow{ox}$ et $\overrightarrow{oy}$.

$$\vec{U} = U_x\vec{i} + U_y\vec{j} = \left\|\vec{U}\right\|\cos\theta.\vec{i} + \left\|\vec{U}\right\|\sin\theta.\vec{j} = \cos\theta.\vec{i} + \sin\theta.\vec{j}$$
$$\vec{V} = -V_x\vec{i} + V_y\vec{j} = -\left\|\vec{V}\right\|\sin\theta.\vec{i} + \left\|\vec{V}\right\|\cos\theta.\vec{j} = -\sin\theta.\vec{i} + \cos\theta.\vec{j} .$$

Let's calculate the derivative of the vector $\vec{U}$ compared to θ.

$$\frac{d\vec{U}}{d\theta} = \frac{d(\cos\theta.\vec{i} + \sin\theta.\vec{j})}{d\theta} = -\sin\theta.\vec{i} + \cos\theta.\vec{j}$$

$$\frac{d\vec{U}}{d\theta} = -\sin\theta.\vec{i} + \cos\theta.\vec{j} = \vec{V}$$

$\Rightarrow$ The derivative of a vector $\vec{U}$ is another vector $\vec{V}$ which is directly perpendicular to it and belonging to the same plane. That is to say we obtain the vector $\vec{V}$ by turning in the positive trigonometric direction of an angle $\alpha = +\frac{\pi}{2} = 90°$ from vector $\vec{U}$.

I.9-Moment of a vector relative to a point:

Consider a linked vector of origin A and end B carried by a line (Δ) and a point (O). We call moment of the vector with respect to the point O, a vector equal to the vector product of the vector $\overrightarrow{OA}$ by the vector $\vec{a}$:

$$\mathrm{M}^{t}_{/o}\vec{a} = \overrightarrow{OA} \wedge \vec{a} \qquad \text{(I.37)}$$

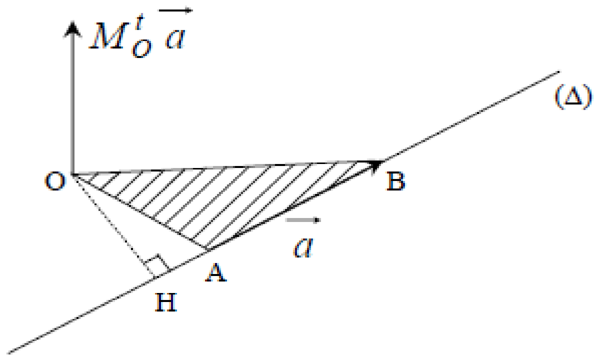


Fig I.10 Representation of the vector moment $\vec{a}$

Its module is twice the area of triangle OAB, its module is therefore equal to the product of the quantity a=AB by the distance OH from point O to the line (Δ) (figure I.10). We see that its magnitude, its direction and its sense are independent of the position of AB on the right (Δ). The notion of moment therefore relates to a sliding vector.

We can note two theorems relating to the calculation of moments:

1. the moment of a vector a with respect to a point (O') is equal to the sum of its moment with respect to the point (O) and the moment with respect to (O'), of an equal original vector (O):

$$\mathrm{M}^{t}_{/o'}\vec{a} = \overrightarrow{O'A} \wedge \vec{a} = (\overrightarrow{O'O} + \overrightarrow{OA}) \wedge \vec{a} = \overrightarrow{O'O} \wedge \vec{a} + \overrightarrow{OA} \wedge \vec{a} \qquad \text{(I.38)}$$

2. le moment de la somme de plusieurs vecteurs concourants est égal à la somme de leurs moments (Varignon's theorem).

En désignant par A le point de concours des vecteurs glissants $\vec{a}, \vec{b}, \vec{c} \ldots$

This theorem translates geometric equalities:

$$\mathrm{M}^{t}_{/o}(\vec{a} + \vec{b} + \vec{c} + \ldots) = \overrightarrow{OA} \wedge (\vec{a} + \vec{b} + \vec{c} + \ldots$$

$$= \overrightarrow{OA} \wedge \vec{a} + \overrightarrow{OA} \wedge \vec{b} + \overrightarrow{OA} \wedge \vec{c} + \ldots = \mathrm{M}^{t}_{/o}\vec{a} + \mathrm{M}^{t}_{/o}\vec{b} + \mathrm{M}^{t}_{/o}\vec{c} + \ldots \qquad \text{(I.39)}$$

Chapter II

Kinematics concepts

II.1- General:

It is the study of the movement of a body without worrying about what caused it (the force). That is to say the apparent modification of its position over time. The study of the movement of a body

involves determining its position over time, its speed and acceleration, knowing that this movement only has a relative meaning: in relation to a spatial reference and a temporal origin.

The word kinetic is the historical stage of the Greek language that extends from the 9th century BC. BC refers to movement. By extension, it also relates to the speed of various processes as well as the mechanisms that explain it.

In mechanics is a branch of physics whose object is the study of movement, deformations or states of equilibrium of physical systems. This science thus aims to describe the movements of different kinds of bodies, from subatomic particles with quantum mechanics, to galaxies with celestial mechanics. Until the 19th century, the notion of mechanics included both the scientific study of bodies in motion and the theory of machines. In the 21st century, although mechanics as a science does not lose sight of the question of its practical application, it is no longer primarily an activity aimed at designing machines.

In mechanics, so that there is no possible doubt, it is imperative to specify in relation to which the study of the movement will be carried out, that is to say, to indicate the chosen frame of reference.

II.1.1- Concept of référence:

Rest and movement are two relative concepts. Indeed, a stationary observer A sees a tree in a fixed position while driver B of a car driving nearby sees it moving backwards. The study of a movement requires knowing the position in space at any time t in relation to a frame of reference. This frame of reference can be linked to an orthonormal reference frame R (O, x, y, z) in which the position M(x,y,z) of a mobile is located. The body is at rest relative to this benchmark if its coordinates are constant over time.

However, if at least one of them varies the body is in motion with respect to R.

A space reference: Example the Cartesian reference R(O, x, y, z) is defined by a fixed origin O in the reference frame and axes and provided with a direct orthonormal base A clock: At each instant, we associate a real number t called date which corresponds to the duration elapsed since the original instant.

The movement of a point is a relative concept. In other words, we cannot say that a body is “in motion” (or “at rest”) without specifying in relation to what. Hence the need to define a benchmark equipped with a stopwatch, to know the position of the point in relation to this benchmark and the instant corresponding to this position (measurement of time). This is an inertia reference point called a frame of reference.

Depending on the nature of the movement of the point, its position will be located by one of the systems namely: Cartesian, polar, cylindrical or spherical.

This movement can be:

- A translation
- A rotation
- A vibration
- Combination of these movements

II.1.2- Trajectory:

The trajectory of a moving point M in a given frame of reference is the curve formed by all the successive positions of the point M in this frame of reference.

The trajectory of a moving point depends on the chosen reference frame.

II.1.3- Velocity vector of a material point:

Since the trajectory of a moving point depends on the chosen frame of reference, the characteristics of the movement must change from one frame of reference to another. One of these characteristics is the velocity vector of the moving point. This is why we use the notation $\vec{V}(M/R)$ to indicate that it is the speed of the point M relative to the frame of reference R. We will use the same notation for the two types of speed that we will deal with in the following, the average speed and the instantaneous speed.

Average speed:

Consider a material point describing a trajectory (C) in a frame of reference R. The material point occupies position M at time t and position M' at time t'=t+ Δt.

II.2- Point kinematics:

II.2.1- Définitions :

Kinematics is the study of the movements of masses, quantity of matter, independently of the causes which generate them.

II.2.1.1-Material point:

By a material point we mean matter which is concentrated at its center of gravity, without geometric dimension whose rotational movement around itself is neglected.

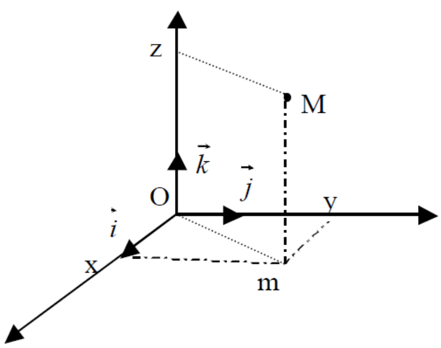
A- Reference:

The movement of a point is a relative concept. In other words, we cannot say that a body is "in motion" (or "at rest") without specifying in relation to what. Hence the need to define a benchmark equipped with a stopwatch, to know the position of the point in relation to this benchmark and the instant corresponding to this position (measurement of time). This is an inertia reference point called a frame of reference.

Depending on the nature of the movement of the point, its position will be located by one of the systems: Cartesian, polar, cylindrical or spherical.

B- Position vector:

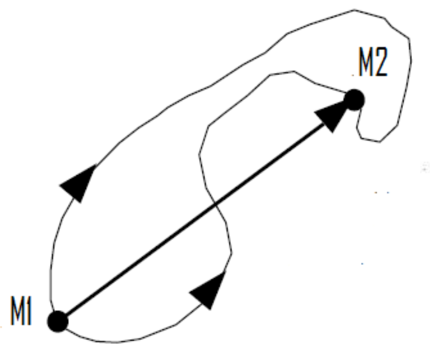
* Varies over time, these coordinates are a function of time: X=f(t) y=g(t) z=h(t). These are called hourly equations

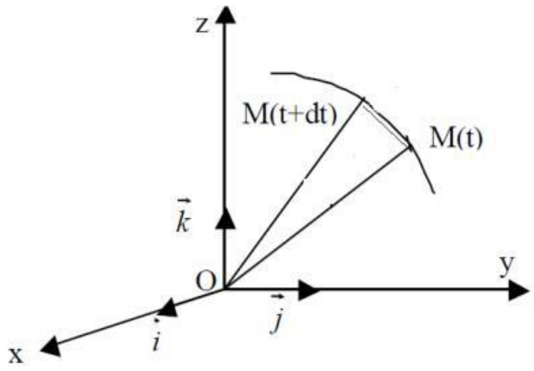


$$\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$$

C- Displacement vector:

Let there be two points M1 at time t and another M2 at time (t+dt), we can define three different paths between these two points, which correspond to the same vector





It is the displacement vector formed by the origin M1 and the end M2, which defines a movement which takes place from point M1 to point M2.

$$\overrightarrow{M_1M_2} = \overrightarrow{OM_2} - \overrightarrow{OM_1}$$

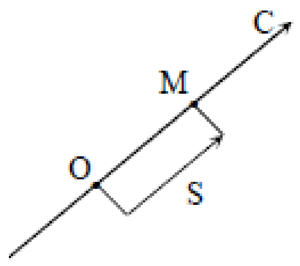
$$\overrightarrow{M_1M_2} = \left|\overrightarrow{M_1M_2}\right|\vec{U}$$

$\vec{U}$ is the unit vector carried by the vector $\overrightarrow{M_1M_2}$

II.2.1.2-Trajectory:

This is the set of points occupied by a mobile at all times. Mathematically it is a relationship linking the x, y and z coordinates to each other independently of time. This equation is obtained by eliminating the time between different coordinates or time equations.

y=f(x,z) or x=g(y,z) otherwise z=h(x,y)



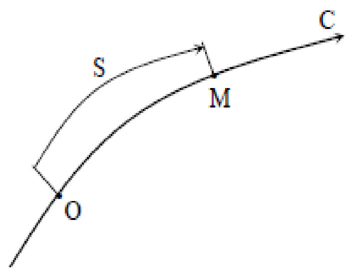


Fig II.1.a-Open rectilinear trajectory **Fig II.1.b-Open curvilinear trajectory**

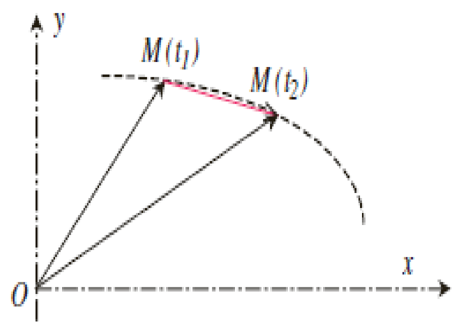
II.2.1.3-Speed vector:

Speed is a quantity that characterizes a movement; it is the variation of position in relation to time. Furthermore, this quantity is vector because the movement of a point is characterized by a direction and a direction. There are two speeds, an average speed and an instantaneous speed:

A- Average speed:

Average speed is the change in overall distance relative to elapsed time. A mobile traveling from Sidi Bel Abbes to Algiers by highway, it travels 500km during 5 hours. We define an average speed of 100 km/h in module, from Sidi Bel Abbes towards Algiers in direction. This average speed only takes into consideration the start and finish point. To summarize, the average speed vector in this case is defined by a module of 100Km/h, a support which is the highway and a direction from Sidi Bel Abbes towards Algiers.

Consider point M1 at time t1 and point M2 at time t2.



$$\vec{V}_m = \frac{\overrightarrow{M_1M_2}}{\Delta t}$$

With $\Delta t = t_2 - t_1$

B- Instantaneous speed:

La vitesse instantanée (vitesse à un instant t, celle par exemple qui apparait sur le compteur de vitesse d'un véhicule) peut se définir comme une vitesse moyenne entre la position $M_1 = M(t)$ du point mobile à la date t et la position $M_2 = M(t+\Delta t)$ de ce même point à la date $t+\Delta t$ où Δt représente une durée très faible. Cette vitesse moyenne tend d'autant vers la vitesse instantanée à la date t que la durée Δt tend vers zéro $(\Delta t \to 0)$ .Le vecteur position $\overrightarrow{OM} = \overrightarrow{OM}(t)$ is a function of time and the instantaneous speed then corresponds to the derivative with respect to time of the position vector:

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\overrightarrow{OM}(t+\Delta t) - \overrightarrow{OM}(t)}{\Delta t} \qquad \text{(II.1)}$$

When we consider an elementary time duration dt "infinitely small" the mobile point passes from a positionM to a positionM' "infinitely close" to M. The corresponding elementary displacement can be written: $\overrightarrow{MM'} = \overrightarrow{OM'} - \overrightarrow{OM} = d\overrightarrow{OM}$. The elementary duration is chosen sufficiently small so that the average speed over the elementary displacement coincides with the instantaneous speed.

With these notations, the velocity vector $\vec{v}(t)$, derived from the position vector $\overrightarrow{OM}$, is written:

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt} \qquad \text{(II.2)}$$

When the point M' tends towards the point M, the chord MM' tends towards the tangent to thetrajectory at the point M. The speed vector is a vector tangent to the trajectory considered.

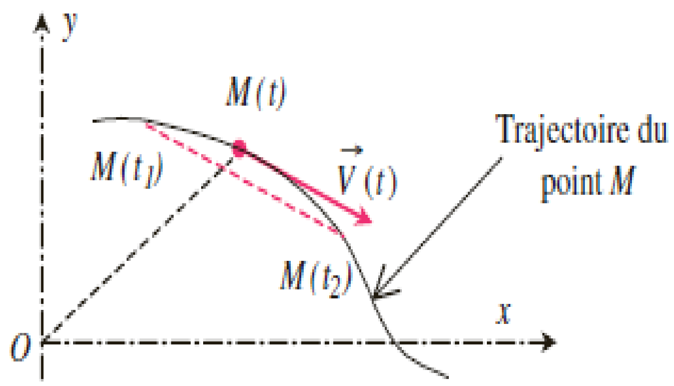


Fig II.2 Representation of the speed vector

II.2.1.5- Acceleration vector:

The acceleration vector is an evolution quantity which measures the variation of the vector speed, standard and direction**.**

$$\vec{\gamma} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2}$$

The norm of the acceleration vector, called acceleration and denoted γ, has the dimension of a ratio between a distance by the square of time LT^{-2} :

$$\vec{\gamma} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$$

Example of movement:

A movement can be made following rectilinear or curvilinear trajectories or following the combination of the two.

A- Rectilinear movement:

In the study frame of reference, the trajectory is a straight line portion. It is then obvious to locate the point M on this line merging, for example, with the axis Ox of the coordinates cartesian. There is then only one time equation x(t) and a single component for the
speed vectors

$$\overrightarrow{OM} = x\vec{\imath}$$

$$\vec{v} = \frac{dx}{dt}\vec{i}$$

Where the module of the speed is expressed:

$$v = \frac{dx}{dt}$$

It is a differential equation which allows us to provide information on the movement and its nature. If the speed is constant then :

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} v\,dt = v\int_{t_0}^{t} dt$$

This is the time equation for uniform rectilinear motion.

- In the case where the speed varies as a function of time, it is expressed as follows:

If γ is constant then:

$$\gamma = \frac{dv}{dt}$$

$$\int_{v_0}^{v} dv = \int_{t_0}^{t} \gamma dt = \gamma\int_{t_0}^{t} dt$$

$$\Longrightarrow v = \gamma(t - t_0) + v_0$$

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} v\,dt = \int_{t_0}^{t} (\gamma(t - t_0) + v_0)dt$$

$$x - x_0 = \frac{1}{2}\gamma(t^2 - t_0^2) - \gamma t_0(t - t_0) + v_0(t - t_0)$$

It is the uniformly varied movement. This movement can be accelerated or delayed.
In the first case the product of the speed and the acceleration must be positive; in the second case the same product must be negative

II.2.2-Velocity vector and Frenet's base:

II.2.2.1-Curvilinear abscissa:

Consider a circle with radius R, center O and a vertex angle α expressed in radians. The measure s of the length of the arc of a circle intercepted by this angle is given by: s = R α. L'angle α appears as the ratio of 2 lengths and is therefore dimensionless.
We can express the distance traveled by the mobile M on the circumference of the circle as being the length of the arc M1M2. The oriented arc M1M2

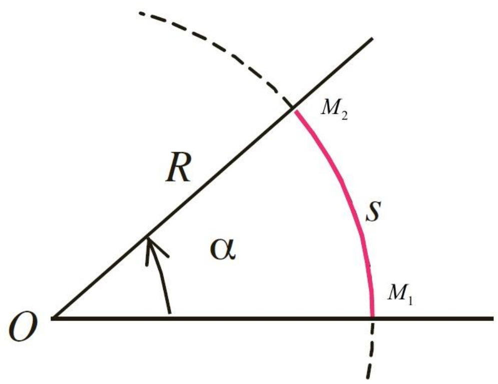


Fig II.3 Representation of the arc M1M2 designates the curvilinear abscissa which is written under the following form M1M2=s(t).

In the case of a plane movement, and by defining at any point M, a unit vector tangent to the trajectory $\vec{T}$ and oriented like this, the speed vector, itself tangent to the trajectory at point M (see figure) can be written:

$\vec{V}(t) = v.\vec{T}$ and $\|\vec{V}\| = V = |v|$ (II.3)

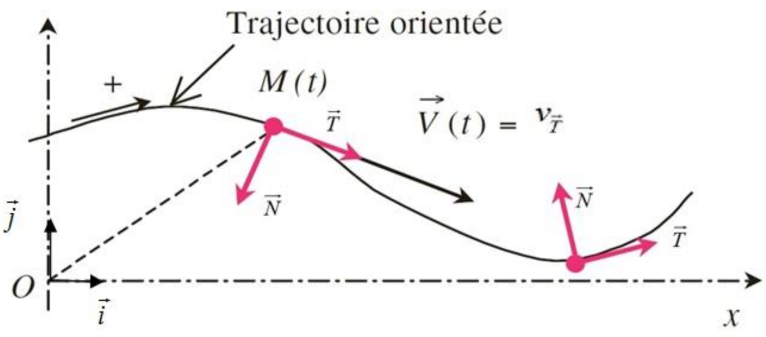


Fig. II.4 Representation of in the Frenet base

To obtain a new base in the plane, simply define a unit vector $\vec{N}$ perpendicular to $\vec{T}$: $(\vec{T}, \vec{N})$ is called the Frenet base. The vector $\vec{T}$ is tangent to the trajectory in the chosen positive direction and the vector $\vec{N}$ is normal to the trajectory and always turned towards the concavity as indicated in figure II.4. This base is a mobile base in the study reference frame since the direction of the base vectors depends on the point considered on the trajectory. It can be practical to use if the trajectory is known.

II.2.3-Acceleration:

II.2.3.1-Definition:

Just like the speed vector which accounts for the variation of the position vector with respect to time, the acceleration vector will account for the variations of the speed vector with respect to time. The acceleration vector therefore corresponds to the derivative with respect to time of the speed vector c' that is to say also to the second derivative of the position vector.

$$\vec{a} = \vec{\gamma} = \frac{d\vec{v}_M}{dt} = \frac{d^2(\overrightarrow{OM})}{dt^2} \qquad \text{(II.4)}$$

The acceleration vector corresponds to the variation of the velocity vector per unit of time. Acceleration is expressed, in the international system, in meters divided by seconds squared: symbol ms^{-2}

A- In Cartesian coordinates:

$$\vec{a}=\vec{\gamma}=\frac{d\vec{v}_M}{dt}=\frac{d\left(\vec{v}_x+\vec{v}_y+\vec{v}_z\right)}{dt}=\frac{d\vec{v}_x}{dt}+\frac{d\vec{v}_y}{dt}+\frac{d\vec{v}_z}{dt}=\frac{d^2x\vec{i}}{dt^2}+\frac{d^2y\vec{j}}{dt^2}+\frac{d^2z\vec{k}}{dt^2}$$

$$=\vec{a}_x+\vec{a}_y+\vec{a}_z=x''\vec{i}+y''\vec{j}+z''\vec{k}=\ddot{x}\vec{i}+\ddot{y}\vec{j}+\ddot{z}\vec{k} \qquad \text{(II.5)}$$

$$\vec{a}_M=\vec{\gamma}_M=\begin{cases}\vec{a}_x=\vec{\gamma}_x=\dfrac{dv_x}{dt}\vec{i}=\dfrac{d^2x}{dt^2}\vec{i}=x''\vec{i}=\ddot{x}\vec{i}\\ \vec{a}_y=\vec{\gamma}_y=\dfrac{dv_y}{dt}\vec{j}=\dfrac{d^2y}{dt^2}\vec{j}=y''\vec{j}=\ddot{y}\vec{j}\\ \vec{a}_z=\vec{\gamma}_z=\dfrac{dv_z}{dt}\vec{k}=\dfrac{d^2x}{dt^2}\vec{i}=z''\vec{k}=\ddot{z}\vec{k}\end{cases} \qquad \text{(II.6)}$$

Its module: $\|\vec{a}_M\|=\|\vec{\gamma}_M\|=\sqrt{a_x^2+a_y^2+a_z^2}=\sqrt{x''^2+y''^2+z''^2}=\sqrt{\ddot{x}^2+\ddot{y}^2+\ddot{z}^2}$ (II.7)

B- In polar coordinates:

$$\vec{a}_M=\vec{\gamma}_M=\frac{d\vec{v}_M}{dt}=\frac{d\left(\dot{r}.\vec{u}_r+r\dot{\theta}\vec{u}_\theta\right)}{dt}=\frac{d\dot{r}}{dt}\vec{u}_r+\dot{r}\frac{d\vec{u}_r}{dt}+\frac{dr}{dt}\dot{\theta}\vec{u}_r+r\frac{d\dot{\theta}}{dt}\vec{u}_\theta+r\dot{\theta}\frac{d\vec{u}_\theta}{dt}$$

avec : $\dfrac{d\dot{r}}{dt}=\ddot{r}$; $\dfrac{d\vec{u}_r}{dt}=\dfrac{d\theta}{dt}\vec{u}_\theta=\dot{\theta}\vec{u}_\theta$; $\dfrac{d\vec{u}_\theta}{dt}=-\dfrac{d\theta}{dt}\vec{u}_r=-\dot{\theta}\vec{u}_r$ and $\dfrac{d\dot{\theta}}{dt}=\ddot{\theta}$

$$\vec{a}_M=\vec{\gamma}_M=\ddot{r}\vec{u}_r+\dot{r}\dot{\theta}\vec{u}_\theta+\dot{r}\dot{\theta}\vec{u}_r+r\ddot{\theta}\vec{u}_\theta-r\dot{\theta}^2\vec{u}_r=\left(\ddot{r}-r\dot{\theta}^2\right)\vec{u}_r+\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\vec{u}_\theta$$

$$\vec{a}_M=\vec{\gamma}_M=\left(\ddot{r}-r\dot{\theta}^2\right)\vec{u}_r+\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\vec{u}_\theta=a_r\vec{u}_r+a_\theta\vec{u}_\theta \qquad \text{(II.8)}$$

$$\begin{cases}a_r=\left(\ddot{r}-r\dot{\theta}^2\right) \text{ : Following } \vec{u}_r\text{ ; radial component with: } \vec{u}_r\perp\vec{u}_\theta\\ a_\theta=(r\ddot{\theta}+2\dot{r}\dot{\theta})\text{ : Following } \vec{u}_\theta\text{ ; orthoradial or tangential component}\end{cases}$$

c-In semi-polar or cylindrical coordinates:

$$\vec{a}_M=\vec{\gamma}_M=\frac{d\vec{v}_M}{dt}=\frac{d\left(\dot{r}.\vec{u}_r+r\dot{\theta}\vec{u}_\theta+\dot{z}\vec{k}\right)}{dt}=\frac{d\dot{r}}{dt}\vec{u}_r+\dot{r}\frac{d\vec{u}_r}{dt}+\frac{dr}{dt}\dot{\theta}\vec{u}_r+r\frac{d\dot{\theta}}{dt}\vec{u}_\theta+r\dot{\theta}\frac{d\vec{u}_\theta}{dt}+\frac{d\dot{z}}{dt}\vec{k}$$

$$\vec{a}_M=\vec{\gamma}_M=\ddot{r}\vec{u}_r+\dot{r}\dot{\theta}\vec{u}_\theta+\dot{r}\dot{\theta}\vec{u}_r+r\ddot{\theta}\vec{u}_\theta-r\dot{\theta}^2\vec{u}_r+\ddot{z}\vec{k}=\left(\ddot{r}-r\dot{\theta}^2\right)\vec{u}_r+\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\vec{u}_\theta+\ddot{z}\vec{k}$$

$$\vec{a}_M=\vec{\gamma}_M=\left(\ddot{r}-r\dot{\theta}^2\right)\vec{u}_r+\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\vec{u}_\theta+\ddot{z}\vec{k} \qquad \text{(II.9)}$$

II.2.3.2-Acceleration in the Frenet base:

In the case of a plane movement, it is possible to express the acceleration vector using the Frenet basis. The derivative of the velocity vector in this basis leads to the following result:

The acceleration vector can be written:

$$\vec{a}_M=\vec{a}_T+\vec{a}_N \qquad \text{(II.10)}$$

$\vec{a}_T$: Tangential component

Où $\vec{a}_N$: Normal component

$a_T=\frac{dv}{dt}$: represents the derivative of the algebraic value of the speed $a_N=\frac{v^2}{R_c}$: where Rc represents the radius of curvature of the trajectory

The radius of curvature corresponding to the radius of the circle tangent to the trajectory at the point considered. The component is always positive and therefore the acceleration vector $\vec{a}_M$ is always turned towards concavity.

$$\vec{a}_M=\vec{a}_T+\vec{a}_N=\frac{dv}{dt}\vec{T}+\frac{v^2}{R_c}\vec{N} \qquad \text{(II.11)}$$

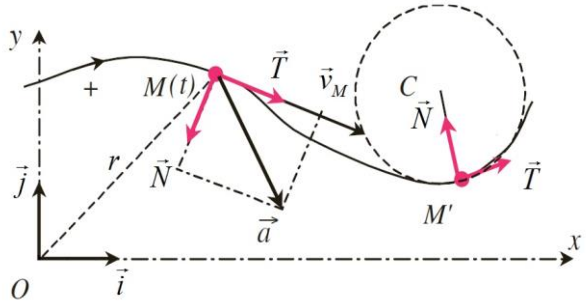


Fig II.5 Representation of an osculating circle on a trajectory.

II.2.4-Applications:

Exercise 1:

A material point is identified by its Cartesian coordinates: $\vec{x}(t) = 3\cos 2\pi t\vec{i}$ and $\vec{y}(t) = 3\sin 2\pi t\vec{j}$

1-Calculate the components $\vec{v}_x(t)$ and $\vec{v}_y(t)$; deduce the modulus of the speed $\left\|\overrightarrow{v_M}(t)\right\|$.

2- Calculate the components $\vec{\gamma}_x(t)$ and $\vec{\gamma}_y(t)$; deduce the module of the acceleration $\left\|\overrightarrow{\gamma_M}(t)\right\|$.

Solution:

Calculation of components $\vec{v}_x(t)$ and $\vec{v}_y(t)$:

$$\vec{v}_x(t) = \frac{dx}{dt}\vec{i} = \frac{d\left(3cos2\pi t\vec{i}\right)}{dt} = -6\pi(\sin 2\pi t)\vec{i}$$

$$\vec{v}_y(t) = \frac{dy}{dt}\vec{j} = \frac{d\left(3\sin 2\pi t\vec{j}\right)}{dt} = 6\pi(\cos 2\pi t)\vec{j}$$

Modulus calculation $\left\|\overrightarrow{v_M}(t)\right\|$

$$\Rightarrow \left\|\vec{v}_M\right\| = v = \sqrt{v_x^2 + v_y^2} = \sqrt{(-6\pi)^2\cos^2 2\pi t + (6\pi)^2\sin^2 2\pi t} = 6\pi(ms^{-1})$$

Calculation of components $\vec{\gamma}_x(t)$ et $\vec{\gamma}_y(t)$ and the module

$$\vec{\gamma}_x(t) = \frac{dv_x}{dt}\vec{i} = \frac{d\left(-6\pi\sin 2\pi t\right)}{dt}\vec{i} = -12\pi^2(\cos 2\pi t)\vec{i}$$

$$\vec{\gamma}_y(t)=\frac{dv_y}{dt}\vec{j}=\frac{d\left(6\pi\cos 2\pi t\right)}{dt}\vec{j}=-12\pi^2(\sin 2\pi t)\vec{j}$$

Calculation of the acceleration modulus $\left\|\overrightarrow{\gamma_M}(t)\right\|$:

$$\Rightarrow\left\|\vec{\gamma}_M\right\|=\gamma=\sqrt{\gamma_x^2+\gamma_y^2}=\sqrt{(-12\pi^2)^2\cos^2 2\pi t+(-12\pi^2)^2\sin^2 2\pi t}=12\pi^2(ms^{-2}).$$

Exercise 2:

A material point moves in the plane (xOy) according to the following time equations:

$x(t)=t$ and $y(t)=t^2$

-give the equation of motion of the mobile

- calculate the speed as well as the acceleration of point M.

Solution

- $y=x^2$ trajectory equation

- $\vec{v}=\vec{i}+2t\vec{j}$

- $\vec{a}=2\vec{j}$

Exercise 3:

Consider a point M in an orthonormal coordinate system, we represent it by its position vector $\overrightarrow{OM}=\vec{r}(t)=at\vec{i}-bt^2 2\vec{j}$ such that a and b are two positive constants.

1) What is the dimension of a and b?

2) Calculate the relative uncertainty on the coordinates of M knowing that the relative uncertainty on a and b is 10^{-2}?

Exercise4:

A baseball player hits a ball which reaches a speed of 14m/s and makes an angle α=30° with the horizontal. Another player distant x=30.5m from the first and in the same plane of the trajectory, starts running when the first one hits the ball.

1- Calculate the maximum speed so that the second player can catch the ball, when it is at a height of 2.44m, knowing that this ball was 0.6m at the time of its strike.

2- What is the distance that must be covered.

Solution :

1- The time equations of the ball are:

$$x_1 = v_0 \cos(\alpha) t$$

$$y_1 = -\frac{1}{2} g t^2 + v_0 \sin(\alpha) t + 0.9$$

For the second player to be able to catch the ball, it is necessary that $x_1=x_2$ and $y_1=y_2$:

$-vt+30.5 = v_0 \cos(\alpha)t$

$2.44 = -½ gt^2+v_0\sin(\alpha) t +0.9$

From these last two equations we deduce that time has two values t=1.17s and t=0.25s

so we will have two speeds v=13.4m/s and v=109.36m/s. and the minimum speed is 13.4m/s.

2- The minimum distance traveled is d=15.7m

Exercise 5:

A material point moves in a straight line following the following time equation:

$X(t)=-6t^2+16t$ (t in seconds)

-What is the position of this body at t=1s ?

-At what time t, it passes through the position O(origin)?

-What is the average speed in the time interval between 0s and 2s ?

-What is the expression of the average speed in the time interval included $t0<t<\Delta t+t0$?

-Give the expression for the instantaneous speed, deduce its value at t=0s .

-What is the expression for the average acceleration during the time $t0 <t<\Delta t+t0$?

-Give the expression for the instantaneous acceleration.

Solution :

X(1)=10

X=0 => $6t^2+16t=0$

it passes through the origin to t=0s et t=8/3=2.7s

Vmoy = x (t=2)- x(t=0) = 4 m/s
2 - 0

$$v_{moy} = \frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t} = 16 - 12t_0 - 6\Delta t$$

$$v(t) = \lim_{\Delta t \to 0} v_{moy} = 16 - 12t$$

V(0) = 16 m/s

$$\gamma_{moy} = \frac{v(t=2) - v(t=0)}{2-0}$$

$= -12\ m/s^2$

Exercise 6:

We give the parametric equations of the plane trajectory of a moving point with respect to a reference frame: x =2t and y =4t2- 4t.

1-Determine the equation of the trajectory, What is its speed?

2-Calculate the speed of the mobile.

3-Show that its acceleration is constant.

4-Determine the tangential and normal components of the acceleration in a Frenet coordinate system.

5-Deduce the radius of curvature Rc(ρ).

Solution :

Solution :

1-Equation of the trajectory and its pace:

x=2t; y=4t2-4t. We draw t from the equation of x which we replace in the equationdey

The equation obtained is the equation of a parabola.

2- Calculation of the speed of the mobile:

$$\left.\begin{aligned} v_x &= \frac{dx}{dt} = 2 \\ \|\vec{v}\| &= v = \sqrt{v_x^2 + v_y^2} = \sqrt{2^2 + (8t-4)^2} = \sqrt{(8t-4)^2 + 4}.(ms^{-1}) \\ v_y &= \frac{dy}{dt} = 8t - 4 \end{aligned}\right\}$$

3-Calculation of the acceleration of the mobile:

$$\gamma_x = \frac{dv_x}{dt} = \frac{d(2)}{dt} = 0.$$

$$\|\vec{\gamma}\| = \gamma = \sqrt{\gamma_x^2 + \gamma_y^2} = \sqrt{0^2 + 8^2} = 8.(ms^{-2}) = \text{constant}$$

$$\gamma_y = \frac{dv_y}{dt} = \frac{d(8t-4)}{dt} = 8.$$

4-Determination of the tangential and normal components of the acceleration in a Frenet coordinate system:

a/ Tangential component $\vec{\gamma}_T$:

$$\gamma_T = \frac{dv}{dt} = \frac{d\left[(8t-4)^2 + 4\right]^{\frac{1}{2}}}{dt} = \frac{1}{2}.2.8.(8t-4).\left[(8t-4)^2 + 4\right]^{-\frac{1}{2}}$$

$$\gamma_T = \frac{8(8t-4)}{\sqrt{(8t-4)^2 + 4}}$$

b/Normal component $\vec{\gamma}_N$;

$$\vec{\gamma} = \vec{\gamma}_T + \vec{\gamma}_N \Rightarrow \gamma^2 = \gamma_T^2 + \gamma_N^2 \Rightarrow \gamma_N = \sqrt{\gamma^2 - \gamma_T^2} = 8^2 - \left(\frac{8(8t-4)}{\sqrt{(8t-4)^2 + 4}}\right)^2$$

$$\gamma_N = \frac{16}{\sqrt{(8t-4)^2 + 4}}$$

5-Determination of the radius of curvature Rc(ρ) of the trajectory.

$$\gamma_N = \frac{16}{\sqrt{(8t-4)^2 + 4}}$$

$$\gamma_N = \frac{v^2}{R_c} = \frac{\left(\sqrt{(8t-4)^2+4}\right)^2}{R_c} = \frac{(8t-4)^2+4}{R_c}$$

$$\Rightarrow \frac{16}{\sqrt{(8t-4)^2+4}} = \frac{(8t-4)^2+4}{R_c} \Rightarrow R_c = \frac{\left((8t-4)^2+4\right).\left((8t-4)^2+4\right)^{\frac{1}{2}}}{16} = \frac{\left[(8t-4)^2+4\right]^{\frac{3}{2}}}{16}$$

$$R_c = \frac{\left[(8t-4)^2+4\right]^{\frac{3}{2}}}{16}$$

Exercise 7:

The plane is related to an orthonormal coordinate system xOy of origin O and base ($\vec{i}$, $\vec{j}$).The x and y coordinates of a moving point M in the plane ($O,\vec{i}$, $\vec{j}$)vary over time following the law:

1-Determine the nature of the trajectory.

2- Determine the components of the speed vector $\vec{v}_M$.

3- as well as that of the curvilinear abscissa s of point M at time t, taking as initial condition when S_0=0 and t_0 = 0

4- Determine the tangential components γ_T and normal γ_N acceleration $\vec{\gamma}_M$ in a Frenet landmark,

5-Deduce the radius of curvature of the trajectory.

6- The trajectory remains the same, but now the point M undergoes an angular acceleration. On what date will point M reach a speed of , knowing that it started from rest. How far did he then travel?

Solution :

1- Nature of the trajectory: $x = 2\cos\frac{t}{2} \Rightarrow x^2 = \left(2\cos\frac{t}{2}\right)^2 = 4\cos^2\frac{t}{2}$ $\Rightarrow x^2 + y^2 = 4$ This is the equation of the circle $y = 2\sin\frac{t}{2} \Rightarrow y^2 = \left(2\sin\frac{t}{2}\right)^2 = 4\sin^2\frac{t}{2}$ with center O and radius r=2.

2- Components of the velocity vector $\vec{v}_M$:

$$v_x = \frac{dx}{dt} = \frac{d\left(2\cos\frac{t}{2}\right)}{dt} = -\sin\frac{t}{2} \Rightarrow v_M^2 = v_x^2 + v_y^2 = \left(-\sin\frac{t}{2}\right)^2 + \left(\cos\frac{t}{2}\right)^2 = 1$$

$$v_y = \frac{dy}{dt} = \frac{d\left(2\sin\frac{t}{2}\right)}{dt} = \cos\frac{t}{2} \Rightarrow \left\|\vec{v}_M\right\| = v = 1.ms^{-1} \; ; v = \frac{ds}{dt} = 1.$$

3-Determine the expression for the speed $\frac{ds}{dt}$ and the curvilinear abscissa of point M at time t:

$$v = \frac{ds}{dt} = 1.ms^{-1} \Rightarrow ds = v.dt \Rightarrow \int ds = \int v.dt = \int 1.dt$$

$$\int_{s_0=0}^{s(t)} ds = \int_{t_0=0}^{t} dt = s(t) - s_0 = t - t_0 \Rightarrow s(t) = t$$

$$s(t) = t$$

4- Determination of tangential components γ_T and normal γ_N :

$$\gamma_x = \frac{dv_x}{dt} = \frac{d\left(-\sin\frac{t}{2}\right)}{dt} = -\frac{1}{2}\cos\frac{t}{2} \Rightarrow \left\|\vec{\gamma}_M\right\| = \gamma = \sqrt{\gamma_x^2 + \gamma_y^2} = \sqrt{\left(-\frac{1}{2}\cos\frac{t}{2}\right)^2 + \left(-\frac{1}{2}\sin\frac{t}{2}\right)^2} = \frac{1}{2}$$

$$\gamma_y = \frac{dv_y}{dt} = \frac{d\left(\cos\frac{t}{2}\right)}{dt} = -\frac{1}{2}\sin\frac{t}{2} \quad \vec{\gamma}_M = \vec{\gamma}_T + \vec{\gamma}_N \Rightarrow \gamma^2 = \gamma_T^2 + \gamma_N^2 = \frac{1}{4} = 0.25$$

$$\gamma_T = \frac{dv}{dt} = \frac{d(1)}{dt} = 0 \Rightarrow \gamma^2 = 0 + \gamma_N^2 = \frac{1}{4} = 0.25 \Rightarrow \gamma_N = \sqrt{0.25} = 0.5.ms^{-2}.$$

5-Deduce the radius of curvature of the trajectory:

$$\gamma_N = \frac{v^2}{R_c} = \frac{1}{R_c} \text{ et } \gamma_N = 0.5 \Rightarrow R_c = \frac{1}{0.5} = 2 \Rightarrow R_c = 2.m$$

6-The date when point M will reach speed of:

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} \Rightarrow d\dot{\theta} = \ddot{\theta}dt \Rightarrow \int_{\dot{\theta}_0}^{\dot{\theta}(t)} d\dot{\theta} = \int_{t_0=0}^{t} \ddot{\theta}dt = \int_0^t 0.2tdt = \dot{\theta}(t) - \dot{\theta}_0$$

we know that: $v = R\dot{\theta}(t) = \dot{s} = \frac{ds}{dt}$ à $t_0 = 0 \Rightarrow s_0 = 0 \Rightarrow \dot{s}_0 = R\dot{\theta}_0 = 0 \Rightarrow \dot{\theta}_0 = 0$

$$\dot{\theta}(t) = 0.1t^2 \Rightarrow v = R\dot{\theta} = 2\times 0.1t^2 = 0.2t^2 = 10$$

$$\Rightarrow t = \sqrt{\frac{10}{0.2}} \approx 7.1s$$

The distance traveled is equal to the length of the arc which is equal to the algebraic value of the curvilinear abscissa s(t).

$$s(t) = R.\theta(t).\text{avec}\,\dot{\theta} = \frac{d\theta}{dt} \Rightarrow d\theta = \dot{\theta}dt \Rightarrow \int_{\theta_0}^{\theta(t)} d\theta = \int_0^t \dot{\theta}dt = \int_0^t 0.1t^2 dt = \theta(t) - \theta_0 = \frac{0.1}{3}t^3$$

$s_0 = R\theta_0 = 0 \Rightarrow \theta_0 = 0\grave{a}t_{0=0}$.. The length of the arc $s = R\theta = 2\times\left(\frac{0.1}{3}\times(7.1)^3\right) \approx 23.9m$

II.3- RELATIVE MOVEMENT:

II.3.I- INTRODUCTION:

The movement of a material point can be divided into two distinct movements:

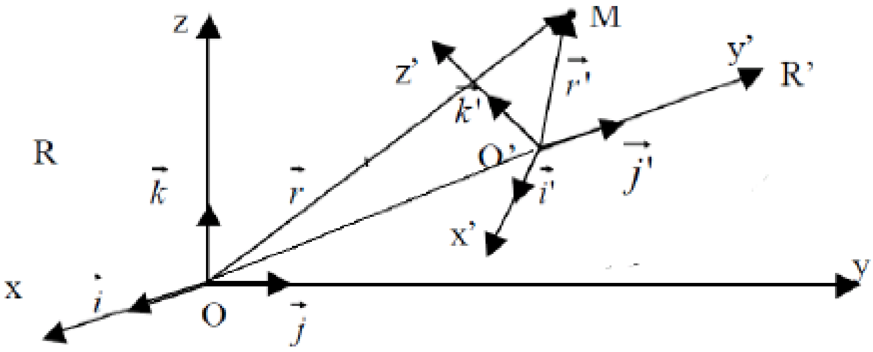
-A movement in relation to a fixed reference point which we will call the Absolute reference point.

-A movement in relation to a mobile reference point which we will call a relative reference point.

All quantities (position, speeds and acceleration) will be identified in relation to the appropriate benchmark.

II.3.2- Absolute and relative quantities:

Let a material point M be in motion relative to a mobile reference frame R'(o',x',y',z'), itself in motion relative to a fixed reference frame R (O,x,y,z).



A- The position:

The position of M in R is the absolute position and its position in R' is its relative position $\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$

$\overrightarrow{O'M} = x'\vec{i}' + y'\vec{j}' + z'\vec{k}'$

$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$

C- Speed:

Absolute speed is the speed of M relative to R

$$\vec{V}_a = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

Cette vitesse peut etre calculée d'une autre façon :

$$\frac{d\overrightarrow{OM}}{dt} = \frac{d\overrightarrow{OO'}}{dt} + \frac{d\overrightarrow{O'M}}{dt}$$

$$\frac{d\overrightarrow{OM}}{dt} = \frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{i}'}{dt} + y'\frac{d\vec{j}'}{dt} + z'\frac{d\vec{k}'}{dt} + \frac{dx'}{dt}\vec{i}' + \frac{dy'}{dt}\vec{j}' + \frac{dz'}{dt}\vec{k}'$$

We pose:

$$\vec{V}_e = \frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{i}'}{dt} + y'\frac{d\vec{j}'}{dt} + z'\frac{d\vec{k}'}{dt} \quad \vec{V}r = \frac{dx'}{dt}\vec{i}' + \frac{dy'}{dt}\vec{j}' + \frac{dz'}{dt}\vec{k}'$$

From where :

$$\boxed{\vec{V}_a = \vec{V}_e + \vec{V}_r}$$

Vr: this is the relative speed, that is to say the speed of the mobile M relative to the reference R’

Ve: represents the training speed, that is to say the speed of the reference R’ relative to the reference R.

Two cases of movement of R' can be, in translation and in rotation, the absolute and relative speed keep the same expression however the training speed is set differently.

-Translation case:

R’ in translation relative to R

$$\vec{V}_e = \frac{d\overrightarrow{OO'}}{dt}$$

The unit vectors of the reference frame R’ do not change; they keep the same meaning and direction, so their derivatives with respect to time are zero. There is only the origin O’ which varies over time.

$$\vec{i} = \vec{i}' \quad \vec{j} = \vec{j}' \quad \vec{k} = \vec{k}'$$

-Rotation case:

We assume that the reference R' is rotating relative to R along the perpendicular axis, therefore the angular velocity vector is carried by this axis. In this case the axis (oz) $\vec{\Omega} = w\vec{k}$.

We know that any vector in rotation with respect to the perpendicular axis its derivative in time is: $\frac{d\overrightarrow{OM}}{dt} = \vec{\Omega} \wedge \overrightarrow{OM}$ donc $\frac{d\vec{i}'}{dt} = (\vec{\Omega} \wedge \vec{i}')$

$$\vec{V}_e = \frac{d\overrightarrow{OO'}}{dt} + x'(\vec{\Omega} \wedge \vec{i}') + y'(\vec{\Omega} \wedge \vec{j}') + z'(\vec{\Omega} \wedge \vec{k}')$$

$$\vec{V}_e = \frac{d\overrightarrow{OO'}}{dt} + (\vec{\Omega} \wedge x'\vec{i}') + (\vec{\Omega} \wedge y'\vec{j}') + (\vec{\Omega} \wedge z'\vec{k}')$$

$$\vec{V}_e = \frac{d\overrightarrow{OO'}}{dt} + (\vec{\Omega} \wedge (x'\vec{i}' + y'\vec{j}' + z'\vec{k}')$$

$$\vec{V}_e = \frac{d\overrightarrow{OO'}}{dt} + (\vec{\Omega} \wedge \overrightarrow{O'M})$$

D- acceleration

The absolute acceleration is the acceleration of the point M in the frame R:

$$\vec{\gamma}_a = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$$

$$\frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2\overrightarrow{OO'}}{dt^2} + x'\frac{d^2\vec{i}'}{dt^2} + y'\frac{d^2\vec{j}'}{dt^2} + z'\frac{d^2\vec{k}'}{dt^2} + 2\left(\frac{dx'}{dt}\frac{d\vec{i}'}{dt} + \frac{dy'}{dt}\frac{d\vec{j}'}{dt} + \frac{dz'}{dt}\frac{d\vec{k}'}{dt}\right) +$$
$$\frac{d^2x'}{dt^2}\vec{i}' + \frac{d^2y'}{dt^2}\vec{j}' + \frac{d^2z'}{dt^2}\vec{k}'$$

$$\overrightarrow{\gamma_a} = \overrightarrow{\gamma_r} + \overrightarrow{\gamma_e} + \overrightarrow{\gamma_c}$$

The absolute acceleration is the sum of three accelerations:

Relative acceleration:

$$\overrightarrow{\gamma_r} = \frac{d^2x'}{dt^2}\vec{i}' + \frac{d^2y'}{dt^2}\vec{j}' + \frac{d^2z'}{dt^2}\vec{k}'$$

Training acceleration:

$$\overrightarrow{\gamma_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + x'\frac{d^2\vec{i}'}{dt^2} + y'\frac{d^2\vec{j}'}{dt^2} + z'\frac{d^2\vec{k}'}{dt^2}$$

and **Coriolis acceleration :**

$$\overrightarrow{\gamma_c} = 2\left(\frac{dx'}{dt}\frac{d\vec{i}'}{dt} + \frac{dy'}{dt}\frac{d\vec{j}'}{dt} + \frac{dz'}{dt}\frac{d\vec{k}'}{dt}\right)$$

- Translation case:

$\vec{\imath} = \vec{\imath}' = cte \qquad \vec{\jmath} = \vec{\jmath}' = cte \qquad \vec{k} = \vec{k}' = cte$

$$\overrightarrow{\gamma_a} = \overrightarrow{\gamma_r} + \overrightarrow{\gamma_e}$$

$$\overrightarrow{\gamma_e} = \frac{d^2\overrightarrow{OO'}}{dt^2}$$

- Rotation case:

$$\frac{d\vec{i'}}{dt} = (\vec{\Omega} \wedge \vec{i'})$$

$$\frac{d^2\vec{i'}}{dt^2} = (\frac{\overrightarrow{d\Omega}}{dt} \wedge \vec{i'}) + \vec{\Omega} \wedge \frac{d\vec{i'}}{dt}$$

$$\overrightarrow{\gamma_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + x'\left((\frac{\overrightarrow{d\Omega}}{dt} \wedge \vec{i'}) + \vec{\Omega} \wedge \frac{d\vec{i'}}{dt} \right) + y'\left((\frac{\overrightarrow{d\Omega}}{dt} \wedge \vec{j'}) + \vec{\Omega} \wedge \frac{d\vec{j'}}{dt} \right) + z'\left((\frac{\overrightarrow{d\Omega}}{dt} \wedge \vec{k'}) + \vec{\Omega} \wedge \frac{d\vec{k'}}{dt} \right)$$

$$\overrightarrow{\gamma_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \left((\frac{\overrightarrow{d\Omega}}{dt} \wedge x'\vec{i'}) + \vec{\Omega} \wedge x'\frac{d\vec{i'}}{dt} \right) + \left((\frac{\overrightarrow{d\Omega}}{dt} \wedge y'\vec{j'}) + \vec{\Omega} \wedge y'\frac{d\vec{j'}}{dt} \right) +$$

$$\left((\frac{\overrightarrow{d\Omega}}{dt} \wedge z'\vec{k'}) + \vec{\Omega} \wedge z'\frac{d\vec{k'}}{dt} \right)$$

$$\overrightarrow{\gamma_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \left(\frac{\overrightarrow{d\Omega}}{dt} \wedge (x'\vec{i'} + y'\vec{j'} + z'\vec{k'}) + \vec{\Omega} \wedge (x'\frac{d\vec{i'}}{dt} + y'\frac{d\vec{j'}}{dt} + z'\frac{d\vec{k'}}{dt}) \right)$$

$$\overrightarrow{\gamma_e} = \frac{d^2\overrightarrow{OO'}}{dt^2} + \left(\frac{\overrightarrow{d\Omega}}{dt} \wedge \overrightarrow{O'M} + \vec{\Omega} \wedge (\vec{\Omega} \wedge \overrightarrow{O'M}) \right)$$

$$\overrightarrow{\gamma_c} = 2\left(\frac{dx'}{dt}(\vec{\Omega} \wedge \vec{i'}) + \frac{dy'}{dt}(\vec{\Omega} \wedge \vec{j'}) + \frac{dz'}{dt}(\vec{\Omega} \wedge \vec{k'}) \right)$$

$$\overrightarrow{\gamma_c} = 2\vec{\Omega} \wedge \left(\frac{dx'}{dt}\vec{i'} + \frac{dy'}{dt}\vec{j'} + \frac{dz'}{dt}\vec{k'} \right) = 2(\vec{\Omega} \wedge \vec{V_r})$$

II.3.3-Applications:

Exercise 1 :

Consider a mobile point M having a rectilinear movement which has the equation:

$x(t) = 4\sin(0.1t + 0.5)$

1-Give the values of the amplitude; of the period; of the frequency and initial phase of the movement.

2-Calculate the speed and acceleration of the movement.

3- What are the initial conditions of the movement?

4-Find the position (abscissa); the speed and acceleration of the point M at t=5s.

5-Trace on the same graph the curves x(t), v(t) and a(t) in an interval [0.2T].

Solution :

1-We notice that the equation of motion of point M is of the form:

$x(t) = X_m \sin(\omega.t + \varphi)$ by analogy $x(t) = 4\sin(0.1t + 0.5)$ we can draw:

The amplitude: $X_m = 4$.

The period: $\omega.t = 0.1t$ knowing that $\omega = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{0.1} = 20\pi \approx 62.83s$.

Frequency: $f = \frac{1}{T} = \frac{1}{20\pi} \approx 0.016 Hertz$.

The initial phase: φ =0.5 rad

2-Calculation of speed and acceleration

a- Speed.

$$v(t) = \frac{dx(t)}{dt} = \frac{d\left(4\sin(0.1t + 0.5)\right)}{dt} = 0.1 \times 4.\cos(0.1t + 0.5) = 0.4\cos(0.1t + 0.5).$$

b-Accélération.

$$\gamma(t) = \frac{dv(t)}{dt} = \frac{d\left(0.4\cos(0.1t + 0.5)\right)}{dt} = -0.04\sin(0.1t + 0.5) = -0.01x.$$

3-The initial conditions of the movement of M.

at t = 0 $\Rightarrow x_0 = 4\sin 0.5 \approx 1.92m$

$v_0 = 0.4\cos 0.5 = 0.35ms^{-1}$

$\gamma_0 = -0.01x = -0.0192ms^{-2}$

4-The position (abscissa); the speed and acceleration of point M at t=5s.

a- The position:

$x(t = 5) = 4\sin(0.1 \times 5 + 0.5) = 4\sin(1) \approx 3.36m$

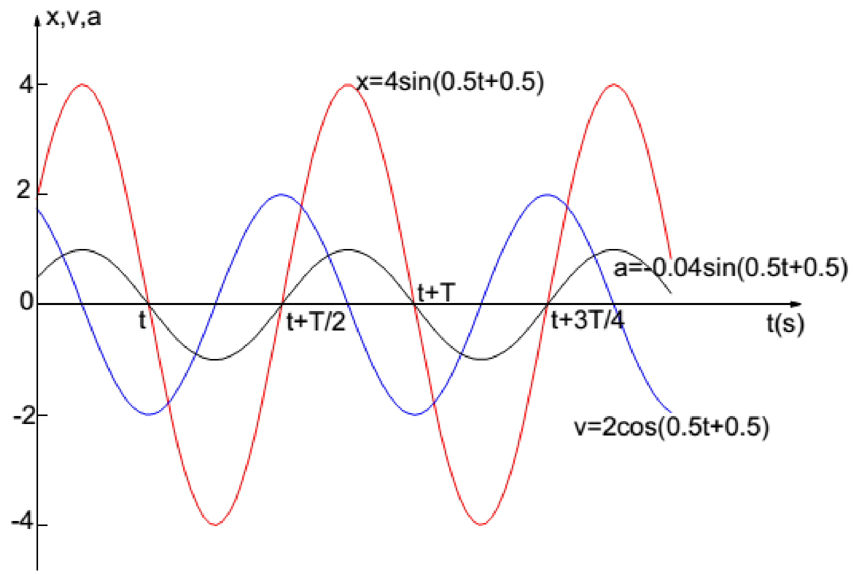
b-Speed:

$v(t = 5) = 0.4\cos(0.1 \times 5 + 0.5) = 0.4\cos(1) \approx 0.22ms^{-1}$

c-Acceleration:

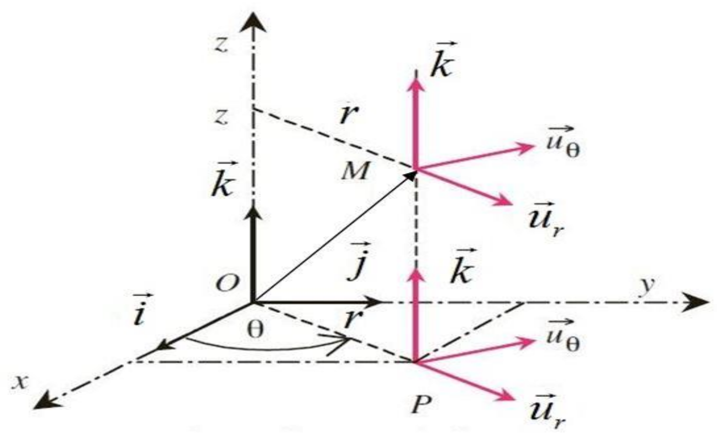
$\gamma(t=5) = -0.01 \times x = 0.0336 \approx 0.034 ms^{-2}$

5-Plotting the graph of the curves x(t), v(t) and a(t).



Exercise 2:

The parametric equations of the movement of a material point M in the frame (Oxyz) are as follows: x(t) = 0.3cos $\omega t\vec{i}$; y(t) =0.3cos $\omega t\vec{j}$; $z(t) = 0.1\omega t\vec{k}$ avec $\omega = 2\pi$ rad/s. $\theta(t) = \omega t$.



1- What is the nature of the movement of point M:

a-In the (Oxy) plane.

b-along the axis $\overrightarrow{Oz}$.

c- In the system (Oxyz) ?

2- Calculate the components:

a- of the velocity vector $\vec{V}_M(t)$.

b- of the acceleration vector $\vec{\gamma}_M(t)$.

3- Calculate the module

a-of the velocity vector $\vec{V}_M(t)$ et calculate the angle α that the speed vector makes with the axis $\overrightarrow{Oz}$

b- of the acceleration vector $\vec{\gamma}_M(t)$.

4.1

a-write the expression of the position vector according to the semi-polar coordinates $(\vec{u}_r;\vec{u}_\theta;\vec{k})$.

b-Calculate the components of the polar vector $\vec{u}_r$ and its module.

4.2 Calculate the components in the basis of semi-polar coordinates

a -velocity vector

b- of the acceleration vector

c- the modules of the speed and acceleration vectors.

5.

a-Give the expression for the acceleration vector $\vec{\gamma}_M(t)$ in function $\vec{\gamma}_N$ normal acceleration and $\vec{\gamma}_T$ tangential acceleration in the Frenet base.

b - Calculate the modules of $\vec{\gamma}_N$ et de $\vec{\gamma}_T$.

c - Calculate the radius of curvature RC of the trajectory of the movement of point M.

6- Give the expression for the curvilinear abscissa $\vec{S}(t)$ knowing that at t=0 we have $S_0 = 0$

Solution :

1-Nature of the movement of the moving point a- In the plane (xOy).

$\vec{x}(t) = 0.3\cos\omega t\vec{i}$; $\vec{y}(t) = 0.3\sin\omega t\vec{j}$

$x^2+y^2 = (0.3\cos\omega t\vec{i})^2 + (0.3\sin\omega t\vec{j})^2 = (0.3)^2$, it is the equation of the circle (c) with center O and radius r =0.3 and angular velocity $\omega = 2\pi$ rd/s constant $\Rightarrow$ the movement of point M in the plane (xOy) is circular and uniform.

b - Along the axis $\overrightarrow{Oz}$.

$\vec{z}(t) = 0.1\omega t\vec{k}$ it's form: z(t)= v_0t. Avec v_0= 0.1ω =constant$\Rightarrow$ The movement of point M along the axis $\overrightarrow{Oz}$ is a rectilinear and uniform movement.

c -In the reference (Oxyz)

It is the combination of two (2) movements, the 1st uniform circular in the plane (xOy), the 2nd uniform rectilinear along the axis $\overrightarrow{Oz} \Rightarrow$ Movement resulting in the reference frame (Oxyz) is a HELICOIDAL movement of steps *h=0.1*ω.

2-Calculation of components of:

a - the velocity vector $\vec{V}_M(t)$ **:**

$$\vec{V}_M(t) = \frac{d\overrightarrow{OM}(t)}{dt} = \frac{d(\vec{x}(t)+\vec{y}(t)+\vec{z}(t))}{dt} = \frac{d(0.3\cos\omega t\vec{i} + 0.3\sin\omega t\vec{j} + 0.1\omega t\vec{k})}{dt}$$

$$\vec{V}_M(t) = -0.3\omega\sin\omega t\vec{i} + 0.3\omega\cos\omega tj + 0.1\omega\ \vec{k} = \vec{v}_x + \vec{v}_y + \vec{v}_z = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

b - the acceleration vector $\vec{\gamma}_M(t)$:

$$\vec{\gamma}_M(t) = \frac{d\vec{V}_M(t)}{dt} = \frac{d(-0.3\omega\sin\omega t\vec{i} + 0.3\omega\cos\omega t\vec{j} + 0.1\omega\ \vec{k})}{dt}$$

$$\vec{\gamma}_M(t) = -0.3(\omega)^2\cos\omega t\vec{i} - 0.3(\omega)^2\sin\omega t\vec{j} + 0\vec{k} = \vec{\gamma}_x + \vec{\gamma}_y + \vec{\gamma}_z = \gamma_x\vec{i} + \gamma_y\vec{j} + \gamma_z\vec{k}$$

3-Calculation of the module of :

a - The velocity vector $\vec{V}_M(t)$:

$$\left\|\vec{V}_M(t)\right\| = \sqrt{(\vec{v}_x)^2 + (\vec{v}_y)^2 + (\vec{v}_z)^2} = \sqrt{(0.3\omega)^2(\cos^2(\omega t\vec{i}) + \sin^2(\omega t\vec{j})) + (0.1\omega\ \vec{k})^2}$$

$$= \sqrt{\omega^2((0.3)^2 + (0.1)^2} = \omega\sqrt{0.1}(m/s) = 1.987\text{m/s}$$

b-The value of the angle α that the vector makes $\vec{V}_M(t)$ with the axis $\overrightarrow{Oz}$:

To be able to calculate the value of the angle α that the direction of the speed vector makes $\vec{V}_M(t)$ with that of the axis $\overrightarrow{Oz}$, we proceed to calculate the scalar product of these two (2) vectors.

$$\vec{V}_M(t)\cdot O\vec{z} = \left\|\vec{V}_M(t)\right\|.\left\|\vec{k}\right\|.\cos\alpha = \omega\sqrt{0.1}\cos\alpha$$

$$= (-0.3\omega\sin\omega t\vec{i} + 0.3\omega\cos\omega t j + 0.1\omega\ \vec{k}).\vec{k} \quad = 0.1\omega \quad \text{from where}$$

$$\Rightarrow \cos\alpha = \frac{0.1\omega}{\omega\sqrt{0.1}} = \sqrt{0.1} = 0.316 \Rightarrow \alpha = 1.249\text{rd} = 71{,}56°$$

c-The vector acceleration $\vec{\gamma}_M(t)$:

$$\left\|\vec{\gamma}_M(t)\right\| = \sqrt{(\vec{\gamma}_x)^2 + (\vec{\gamma}_y{}^2) + (\vec{\gamma}_z)^2} = \sqrt{(-0.3(\omega)^2)^2(\cos^2(\omega t\vec{i}) + \sin^2(\omega t\vec{j})) + 0^2} = 0.3(\omega)^2(m/s^2)$$

$$\left\|\vec{\gamma}_M(t)\right\| = 11.84 m/s^2$$

4.1.a-Writing the position vector in semi-polar coordinates $(\vec{u}_r;\vec{u}_\theta;\vec{k})$:

$$\overrightarrow{OM}(t) = \overrightarrow{OP} + \overrightarrow{PM} = r\vec{u}_r + z(t)\vec{k}$$

$r = \|\vec{r}\| = \left\|\overrightarrow{OP}\right\| = \sqrt{(\vec{x}_m)^2 + (\vec{y}_m)^2}$ =0.3 (radius of the circle (c) found in 1-a).

$$\overrightarrow{OM}(t) = 0.3\vec{u}_r + 0.1\omega t\vec{k}\ \vec{u}_r$$

4.1.b -Calculation of the components of the vector $\vec{u}_r$ and its module:

$$\vec{u}_r = \frac{\overrightarrow{OP}}{\left\|\overrightarrow{OP}\right\|} = \frac{x(t)\vec{i} + y(t)\vec{j}}{\left\|\overrightarrow{OP}\right\|} = \frac{0.3\cos(\omega t)\vec{i} + 0.3\sin(\omega t)\vec{j}}{0.3} = \cos(\omega t)\vec{i} + \sin(\omega t)\vec{j}$$

$$\|\vec{u}_r\| = \sqrt{(x_{u_r})^2 + (y_{u_r})^2} = \sqrt{(\cos(\omega t)\vec{i})^2 + (\sin(\omega t)\vec{j})^2} = 1$$

4.2-Calculation in the base of the semi-polar coordinates of the components:

a- of the speed vector $\vec{V}_M(t)$:

$$\vec{V}_M(t) = \frac{d\overrightarrow{OM}(t)}{dt} = \frac{d(0.3\vec{u}_r + 0.1\omega t\vec{k})}{dt} = \frac{0.3d\vec{u}_r}{dt} + 0.1\omega\vec{k}$$

$$\frac{d\vec{u}_r}{dt} = \frac{d\theta(t)}{dt}\vec{u}_\theta = \dot{\theta}\vec{u}_\theta = \omega\vec{u}_\theta \Rightarrow \vec{V}_M(t) = 0.3\omega\vec{u}_\theta + 0.1\omega\vec{k} = v\vec{u}_\theta + v\vec{k}$$

b-of the acceleration vector $\vec{\gamma}_M(t)$:

$$\vec{\gamma}_M(t)=\frac{d\vec{V}_M(t)}{dt}=\frac{d(0.3\omega\vec{u}_\theta+0.1\omega\vec{k})}{dt}=\frac{0.3\vec{u}_\theta d\omega}{dt}+\frac{0.3\omega d\vec{u}_\theta}{dt}+\frac{d(0.1\omega\vec{k})}{dt}$$

ω=constant $\Rightarrow \frac{d\omega}{dt}=0$ et $\frac{d\vec{u}_\theta}{dt}=-\frac{d\theta(t)}{dt}\vec{u}_r=-\frac{d(\omega t)}{dt}\vec{u}_r=-\omega\vec{u}_r$

$$\vec{\gamma}_M(t)=-0.3\omega^2\vec{u}_r$$

c - Calculation of the speed modules $\vec{V}_M(t)$ and acceleration $\vec{\gamma}_M(t)$:

- Speed $\vec{V}_M(t)$ **:**

$$\left\|\vec{V}_M(t)\right\|=\sqrt{(v\vec{u}_\theta)^2+(v\vec{k})^2}=\sqrt{(0.3\omega\vec{u}_\theta)^2+(0.1\omega\vec{k})^2}=\sqrt{(0.3\omega)^2+(0.1\omega)^2}=\omega\sqrt{0.1}=1.987(m/s)$$

- Acceleration $\vec{\gamma}_M(t)$ **:**

$$\left\|\vec{\gamma}_M(t)\right\|=\sqrt{(-0.3\omega^2\vec{u}_r)^2}=0.3(\omega)^2=11.84(m/s^2)$$

5-Expression of the acceleration vector in the FRENET base

a- Expression of the vector $\vec{\gamma}_M(t)$ in terms of $\vec{\gamma}_T$ and of $\vec{\gamma}_N$:

$$\vec{\gamma}_M(t)=\vec{\gamma}_T+\vec{\gamma}_N$$

Tangential acceleration $\vec{\gamma}_T$:

$$\vec{\gamma}_T=\frac{d\left(\left\|\overrightarrow{V_M}\right\|\right)}{dt}\vec{T}=\frac{d\left(\omega\sqrt{0.1}\right)}{dt}\vec{T}=\vec{0}=0\vec{T}\,,\qquad \vec{V}_M(t)=\frac{d\vec{S}(t)}{dt}\vec{T}=\dot{S}(t)\vec{T}$$

Normal acceleration $\vec{\gamma}_N$:

$$\vec{\gamma}_N=\frac{(\vec{V}_M(t))^2}{R_c}\vec{N}=\frac{(\omega\sqrt{0.1})^2}{R_c}\vec{N}$$

-Calculation of modules $\vec{\gamma}_T$ and $\vec{\gamma}_N$:

$$\vec{\gamma}_M(t)=\vec{\gamma}_N\Rightarrow(\vec{\gamma}_M(t))^2=(\vec{\gamma}_N)^2\Rightarrow\left\|\vec{\gamma}_M(t)\right\|=\left\|\vec{\gamma}_N\right\|$$

$$\left\|\vec{\gamma}_M\right\|=0.3\omega^2$$

$$\left\|\vec{\gamma}_N\right\|=\frac{0.1\omega^2}{R_c}$$

-Calculation of the radius of curvature RC:

$$0.3\omega^2 = \frac{0.1\omega^2}{R_c} \Rightarrow R_c = \frac{0.1\omega^2}{0.3\omega^2} = \frac{1}{3} = 0.333m$$

6-Expression of the curvilinear abscissa $\vec{S}(t)$:

a-Velocity in curvilinear coordinates:

$$\vec{V}_M(t) = \frac{d\vec{S}(t)}{dt}\vec{T} = \dot{S}(t)\vec{T} = \left\|\vec{V}_M(t)\right\|\vec{T}$$

$$\frac{ds}{dt} = \dot{s} = \left\|\vec{V}_M(t)\right\| \Rightarrow ds = \left\|\vec{V}_M(t)\right\|.dt \quad \Rightarrow \int_{s_0}^{s(t)} ds = \int_{t_{0=0}}^{t} \left\|\vec{V}_M(t)\right\|.dt = \int_0^t \omega\sqrt{0.1}dt$$

$s(t) - s_0 = \omega\sqrt{0.1}t = 1.987t + s_0$ with $s_0 = 0$

$s(t) = 1.987t$

II.13.Change of the reference system

II.13.1.Composition of movements (relative movement)

Consider:

- A reference R, materialized by a basic trihedron (Oxyz) $(\vec{i};\vec{j};\vec{k})$.

- A reference R1, materialized by a basic trihedron (O1x1y1z1) $(\vec{i}_1;\vec{j}_1;\vec{k}_1)$ moving relative to R-

A point M, in motion, defined by x,y,z in the frame R and by x1, y1, z1
in the R1 refference .

By changing coordinates we can go from the movement of M relative to R1 to the movement of M relative to R.

Just apply the vector relationship: $\overrightarrow{OM} = \overrightarrow{OO_1} + \overrightarrow{O_1M}$

To study a well-defined movement, we use the following definitions:

➢the Oxyz trihedron (reference R) is the absolute reference or absolute reference frame;

➢the trihedron O1x1y1z1 (reference R1) is the relative reference or relative frame of reference;

➢the movement of point M relative to "R" is called absolute movement;

➢the movement of point M relative to "R1" is called relative movement;

➢the movement of "R1" relative to "R" is called drive movement;

II.13.2.Composition of speeds

$\overrightarrow{OM} = \overrightarrow{OO_1} + \overrightarrow{O_1M}$ (II.34)

a-absolute speed:

$\vec{v}_a = \frac{d\overrightarrow{OM}}{dt} = \frac{d\overrightarrow{OO_1}}{dt} + \frac{d\overrightarrow{O_1M}}{dt}$ (II.35)

In the reference R we have: $\overrightarrow{OM} = x.\vec{i} + y.\vec{j} + z.\vec{k}$

$\vec{v}_a = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = \dot{x}.\vec{i} + \dot{y}.\vec{j} + \dot{z}.\vec{k}$. (II.36)

In the reference R1 we have: $\overrightarrow{O_1M} = x_1.\vec{i}_1 + y_1.\vec{j}_1 + z_1.\vec{k}_1$.

$$\vec{v}_a = \frac{d\overrightarrow{OO_1}}{dt} + \frac{dx_1}{dt}.\vec{i}_1 + \frac{dy_1}{dt}.\vec{j}_1 + \frac{dz_1}{dt}.\vec{k}_1 + x_1\frac{d\vec{i}_1}{dt} + y_1\frac{d\vec{j}_1}{dt} + z_1\frac{d\vec{k}_1}{dt} .$$

$$\vec{v}_a = \underbrace{\dot{x}_1.\vec{i}_1 + \dot{y}_1.\vec{j}_1 + \dot{z}_1.\vec{k}_1}_{\vec{v}_r} + \underbrace{\frac{d\overrightarrow{OO_1}}{dt} + x_1.\frac{d\vec{i}_1}{dt} + y_1.\frac{d\vec{j}_1}{dt} + z_1.\frac{d\vec{k}_1}{dt}}_{\vec{v}_e} \qquad \text{(II.37)}$$

$\vec{v}_a = \vec{v}_r + \vec{v}_e$.

$$\left.\begin{aligned} \vec{v}_r &= \frac{dx_1}{dt}.\vec{i}_1 + \frac{dy_1}{dt}.\vec{j}_1 + \frac{dz_1}{dt}.\vec{k}_1 = \dot{x}_1.\vec{i}_1 + \dot{y}_1.\vec{j}_1 + \dot{z}_1.\vec{k}_1. \qquad \text{(II.38)} \\ \vec{v}_e &= \frac{d\overrightarrow{OO_1}}{dt} + x_1.\frac{d\vec{i}_1}{dt} + y_1.\frac{d\vec{j}_1}{dt} + z_1.\frac{d\vec{k}_1}{dt} \end{aligned}\right\}$$

Special cases :

➢If the reference R1 is in rectilinear and uniform movement with respect to the reference R then , $\vec{v}_e = \frac{d\overrightarrow{OO_1}}{dt}$ there is no rotational movement.

➢Si the reference R1 is fixed (No rotation and no translation of R1 relative to R)$\Rightarrow \vec{v}_e = \vec{0}$ from where $\vec{v}_a = \vec{v}_r$.

Noticed :

The derivative of a rotating (moving) vector. $\frac{d\vec{u}_r}{dt} = \frac{d\theta}{dt}.\vec{u}_\theta = \dot{\theta}.\vec{u}_\theta$ and knowing that

$\dot{\theta} = \omega \Rightarrow \frac{d\vec{u}_r}{dt} = \omega.\vec{u}_\theta$.

We consider the mobile base $(\vec{i}_1;\vec{j}_1;\vec{k}_1)$.

$\frac{d\vec{i}_1}{dt}=\frac{d\theta}{dt}.\vec{j}_1=\dot{\theta}.\vec{j}_1=\omega.\vec{j}_1$; $\vec{k}_1\wedge\vec{i}_1=\vec{j}_1$; we replace in the previous expression

$$\left.\begin{array}{l}\frac{d\vec{i}_1}{dt}=\omega.\vec{j}_1=\omega(\vec{k}_1\wedge\vec{i}_1)=\vec{\omega}\wedge\vec{i}_1\Rightarrow\frac{d\vec{i}_1}{dt}=\vec{\omega}\wedge\vec{i}_1\\ \frac{d\vec{j}_1}{dt}=\frac{d\theta}{dt}\vec{k}=\omega.\vec{k}_1=\omega(\vec{i}_1\wedge\vec{j}_1)=\vec{\omega}\wedge\vec{j}_1.\text{(II.39)}\\ \frac{d\vec{k}}{dt}=\frac{d\theta}{dt}\vec{i}=\omega\vec{i}=\omega(\vec{j}\wedge\vec{k})=\vec{\omega}\wedge\vec{k}.\end{array}\right\}$$

$$\vec{v}_e=\frac{d\overrightarrow{OO_1}}{dt}+x_1.\frac{d\vec{i}_1}{dt}+y_1.\frac{d\vec{j}_1}{dt}+z_1.\frac{d\vec{k}_1}{dt}=\frac{d\overrightarrow{OO_1}}{dt}+x_1.\vec{\omega}\wedge\vec{i}_1+y_1.\vec{\omega}\wedge\vec{j}_1+z_1.\vec{\omega}\wedge\vec{k}_1$$

$$\vec{v}_e=\frac{d\overrightarrow{OO_1}}{dt}+\vec{\omega}\wedge(x_1\vec{i}_1+y_1\vec{j}_1+z_1\vec{k}_1)=\frac{d\overrightarrow{OO_1}}{dt}+\vec{\omega}\wedge\overrightarrow{O_1M}.$$

$$\left.\begin{array}{l}\vec{v}_a=\vec{v}_r+\vec{v}_e.\\ \vec{v}_r=\frac{dx_1}{dt}.\vec{i}_1+\frac{dy_1}{dt}.\vec{j}_1+\frac{dz_1}{dt}.\vec{k}_1=\dot{x}_1.\vec{i}_1+\dot{y}_1.\vec{j}_1+\dot{z}_1.\vec{k}_1.\text{ (II.40)}\\ \vec{v}_e=\frac{d\overrightarrow{OO_1}}{dt}+x_1.\frac{d\vec{i}_1}{dt}+y_1.\frac{d\vec{j}_1}{dt}+z_1.\frac{d\vec{k}_1}{dt}=\frac{d\overrightarrow{OO_1}}{dt}+\vec{\omega}\wedge\overrightarrow{O_1M}\end{array}\right\}$$

II.13.2.Compositions of accelerations

$$\vec{\gamma}_a=\vec{a}_a=\frac{d\vec{v}_a}{dt}=\frac{d(\dot{x}.\vec{i}+\dot{y}.\vec{j}+\dot{z}.\vec{k})}{dt}=\ddot{x}.\vec{i}+\ddot{y}.\vec{j}+\ddot{z}.\vec{k}$$

$$\vec{\gamma}_a=\vec{a}_a=\frac{d\vec{v}_a}{dt}=\frac{d\vec{v}_r}{dt}+\frac{d\vec{v}_e}{dt}=\frac{d(\dot{x}_1.\vec{i}_1+\dot{y}_1.\vec{j}_1+\dot{z}_1.\vec{k}_1)}{dt}+\frac{d}{dt}\left(\frac{d\overrightarrow{OO_1}}{dt}+\vec{\omega}\wedge\overrightarrow{O_1M}\right).\text{ (II.41)}$$

$$\frac{d\vec{v}_r}{dt}=\frac{d(\dot{x}_1.\vec{i}_1+\dot{y}_1.\vec{j}_1+\dot{z}.\vec{k}_1)}{dt}=\ddot{x}_1.\vec{i}_1+\ddot{y}_1.\vec{j}_1+\ddot{z}_1.\vec{k}_1+\dot{x}_1(\vec{\omega}\wedge\vec{i}_1)+\dot{y}_1(\vec{\omega}\wedge\vec{j}_1)+\dot{z}_1(\vec{\omega}\wedge\vec{k}_1)$$

$$\frac{d\vec{v}_r}{dt}=\ddot{x}_1.\vec{i}_1+\ddot{y}_1.\vec{j}_1+\ddot{z}_1.\vec{k}_1+\vec{\omega}\wedge\vec{v}_r.$$

➢Relative acceleration : $\vec{a}_r=\ddot{x}.\vec{i}_1+\ddot{y}_1.\vec{j}_1+\ddot{z}_1.\vec{k}_1$ (II.42)

$$\frac{d\vec{v}_e}{dt}=\frac{d\left(\frac{d\overrightarrow{OO_1}}{dt}+\vec{\omega}\wedge\overrightarrow{O_1M}\right)}{dt}=\frac{d^2\overrightarrow{OO_1}}{dt^2}+\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O_1M}+\vec{\omega}\wedge\frac{d\overrightarrow{O_1M}}{dt}$$

$$\frac{d\overrightarrow{O_1M}}{dt} = \frac{d(x_1.\vec{i}_1 + y_1.\vec{j}_1 + z_1.\vec{k}_1)}{dt} = \dot{x}_1.\vec{i}_1 + \dot{y}_1.\vec{j}_1 + \dot{z}_1.\vec{k}_1 + \vec{\omega} \wedge \overrightarrow{O_1M} = \vec{v}_r + \vec{\omega} \wedge \overrightarrow{O_1M}$$

$$\frac{d\vec{v}_e}{dt} = \frac{d^2\overrightarrow{OO_1}}{dt^2} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O_1M} + \vec{\omega} \wedge (\vec{v}_r + \vec{\omega} \wedge \overrightarrow{O_1M})$$

$$\frac{d\vec{v}_e}{dt} = \frac{d^2\overrightarrow{OO_1}}{dt^2} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O_1M} + \vec{\omega} \wedge \vec{v}_r + \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{O_1M})$$

$$\vec{a}_a = \ddot{x}_1.\vec{i}_1 + \ddot{y}_1.\vec{j}_1 + \ddot{z}_1.\vec{k}_1 + \vec{\omega} \wedge \vec{v}_r + \frac{d^2\overrightarrow{OO_1}}{dt^2} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O_1M} + \vec{\omega} \wedge \vec{v}_r + \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{O_1M})$$

$$\vec{a}_a = \underbrace{\ddot{x}_1.\vec{i}_1 + \ddot{y}_1.\vec{j}_1 + \ddot{z}_1.\vec{k}_1}_{\vec{a}_r} + \underbrace{2(\vec{\omega} \wedge \vec{v}_r)}_{\vec{a}_c} + \underbrace{\frac{d^2\overrightarrow{OO_1}}{dt^2} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O_1M} + \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{O_1M})}_{\vec{a}_e} \quad .(II.43)$$

$$\vec{a}_a = \vec{a}_r + \vec{a}_c + \vec{a}_e$$

➢Absolute acceleration: $\vec{a}_a = x.\vec{i} + \ddot{y}.\vec{j} + \ddot{z}.\vec{k}$.

➢Relative acceleration: $\vec{a}_r = \ddot{x}.\vec{i}_1 + \ddot{y}_1.\vec{j}_1 + \ddot{z}_1.\vec{k}_1$.

➢Coriolis acceleration: $\vec{a}_c = 2(\vec{\omega} \wedge \vec{v}_r)$.

➢Training acceleration: $\vec{a}_e = \frac{d^2\overrightarrow{OO_1}}{dt^2} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O_1M} + \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{O_1M})$

II.13.3.Applications :

Exercise 1 :

Two cars A and B are driving on two lanes of a supposedly flat and horizontal highway with the respective speeds .

1-Indicate the relative speed vector of car A compared to car B in the following two cases:

a-The two cars are traveling in the same direction.

b-The two cars are traveling in two opposite directions.

2-If the cars drove with the same speeds as previously on two lanes making an angle α=30°. What is then the relative speed of car B compared to car A?

Solution :

a- The relative speed of car A compared to car B (same direction):

$\vec{v}_r = \vec{V}_{A/B} = \vec{V}_A - \vec{V}_B$.

Using the unit vector $\vec{i}$ carried by the axis Ox, and the speeds have the same direction we therefore obtain: $\vec{v}_r = \vec{V}_{A/B} = \vec{V}_A - \vec{V}_B = 110\vec{i} - 90\vec{i} = 20\vec{i} \Rightarrow \vec{V}_{A/B} = 20 km.h^{-1}$.

b- The relative speed of car A compared to car B (opposite directions): $\vec{v}_r = \vec{V}_{A/B} = \vec{V}_A - \vec{V}_B$

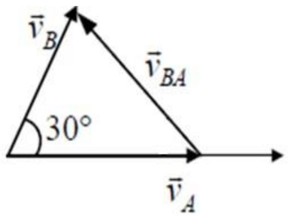
Using the unit vector $\vec{i}$ porté par l'axe Ox, and the speeds have opposite directions so we get

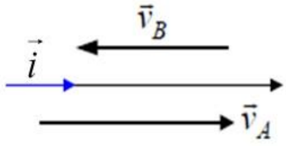
$\vec{v}_r = \vec{V}_{A/B} = \vec{V}_A - \vec{V}_B = 110\vec{i} - (-90\vec{i}) = 200\vec{i} \Rightarrow \vec{V}_{A/B} = 200 km.h^{-1}$

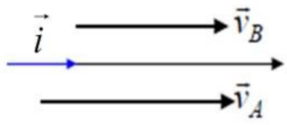
2-The two corridors make an angle α=30° between them. $\vec{V}_{B/A} = \vec{V}_B - \vec{V}_A \Rightarrow V_{BA}^2 = (V_B^2 + V_A^2 - 2.V_A.V_B.\cos 30°)$.

$V_{BA} = \left[90^2 + 110^2 - 2\times110\times90\times0.87\right]^{\frac{1}{2}} \Rightarrow V_{BA} = 54.5 km.h^{-1}$

π







We can solve questions a and b using the relation of relative speed $\vec{v}_r = \vec{V}_{BA} = \vec{V}_B - \vec{V}_A \Rightarrow (\vec{V}_B - \vec{V}_A)^2 = V_{BA}^2 = (V_B^2 + V_A^2 - 2\vec{V}_B.\vec{V}_A) = V_B^2 + V_A^2 - 2V_B V_A.cos\alpha$

With the angle $\alpha = (0 ou \pi)$ si α=0⇒cos0=1 we find

$v_r = V_{A/B} = \sqrt{(V_A - V_B)^2} = V_A - V_B = 110 - 90 = 20$km/h

si α=π⇒cosπ=-1 we find

$v_r = V_{A/B} = \sqrt{(V_A + V_B)^2} = V_A + V_B = 110 + 90 = 200 km/h$

Exercise 2

A material point M (a fly) is located at the center O of a wall clock. At the instant when the second hand (second hand) passes over the number (3) III of the clock face (t = 0) , point M moves towards the end A of this hand moving at constant speed v0= 3.33 cm/s. The seconds hand has a length of 2 m

1) What is the nature of the movement of point M in relation to the seconds hand? Give the expression for the position vector in the polar basis $(\vec{u}_\rho, \vec{u}_\theta)$

2) a-After how long will point M have reached the end of needle A.b-What will be the distance traveled by end A?

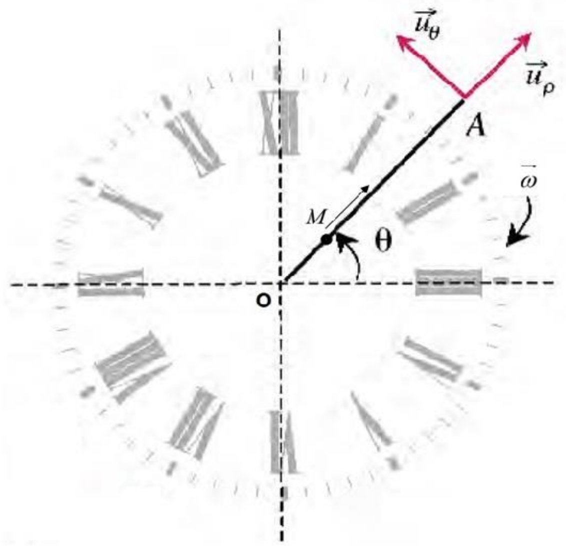
3) Express the velocity vector $\vec{v}(M)$ in the base $(\vec{u}_\rho, \vec{u}_\theta)$ mobile linked to ***M***.

4) Express the acceleration vector $\vec{a}(M)$ in the base $(\vec{u}_\rho, \vec{u}_\theta)$ mobile linked to ***M***

5) Calculate the components of the vectors $\vec{v}(M)$ For t = 0 s, 15 s, 30 s, 45 s et 60s in the fixed base ($\vec{i}$; $\vec{j}$).

6) Represent to scale on a diagram the positions and the speed vectors of the spider for the times t = 0 s, 15 s, 30 s, 45 s and 60 s. We will take 1 cm which will correspond to 0.1 ms–1.

7) Knowing that the velocity vectors are always tangent to the trajectory, trace the trajectory of the spider for an observer located at the base of the clock.



Solution :

1-Nature of the movement of the stitch along the needle.

Point M (the fly) advances along the needle at constant speed: the movement of point M relative to the needle is therefore rectilinear and uniform. : $\overrightarrow{OM}(t) = \vec{\rho}(t) = \rho.\vec{u}_\rho$

2a- The second hand measures 2 m and point M moves at constant speed $v_0 = v_M$, we deduce the travel time of the second hand $t = \frac{OA}{v_0} = \frac{d}{v_0} = \frac{200}{3.33} = 60.s$

b-the distance traveled by end A.

End A returns to the same point of its departure by describing a circle of radius R=2m, from where the distance traveled is :AA= $2.\pi.R = 2.\pi.2 \approx 12.57m$

3- the speed vector $\vec{v}(M)$ in the base $\left(\vec{u}_\rho, \vec{u}_\theta\right)$ mobile linked to M.

The position vector: $\overrightarrow{OM}(t) = \vec{\rho}(t) = \rho.\vec{u}_\rho$.

The speed vector: $\vec{v}_M = \frac{d\overrightarrow{OM}}{dt} = \frac{d(\rho.\vec{u}_\rho)}{dt} = \frac{d\rho}{dt}.\vec{u}_\rho + \rho\frac{d\vec{u}_\rho}{dt} = \dot{\rho}.\vec{u}_\rho + \rho.\frac{d\theta}{dt}\vec{u}\theta$

$\vec{v}_M = \dot{\rho}.\vec{u}_\rho + \rho.\dot{\theta}.\vec{u}_\theta$.

The distance OA corresponds to: $OA = \rho = v_M.t = v_0.t \Rightarrow \dot{\rho} = v_M = v_0$

The angular velocity corresponds to: $\dot{\theta} = \omega = \frac{1tour}{60} = \frac{2\pi}{60} = \frac{\pi}{30} rad.s^{-1}$

The needle rotates in the opposite direction to the trigonometric direction. Algebraic angular velocity $\dot{\theta} = \omega = -\frac{\pi}{30} rad.s^{-1}$ from which we can write the expression for the speed.

$\vec{v}_M = \dot{\rho}.\vec{u}_\rho + \rho.\dot{\theta}.\vec{u}_\theta = \quad v_0.\vec{u}_\rho - v_0.t.\dot{\theta}.\vec{u}_\theta = 3.3(\vec{u}_\rho - \frac{\pi}{30}.t.\vec{u}_\theta)$ (cm/s)

4-the acceleration vector $\vec{a}(M)$ in the base $\left(\vec{u}_\rho, \vec{u}_\theta\right)$ mobile linked to M the acceleration vector is obtained by deriving the velocity vector. $\vec{a}_M = \frac{d\vec{v}_M}{dt} = \frac{d\left(3.3(\vec{u}_\rho - \frac{\pi}{30}.t.\vec{u}_\theta)\right)}{dt} = 3.3\frac{d\vec{u}_\rho}{dt} - \frac{\pi}{30}\vec{u}_\theta - 3.3\frac{\pi}{30}.t\frac{d\vec{u}_\theta}{dt}$

$\vec{a}_M = 3.3\times\dot{\theta}\times\vec{u}_\theta - 3.3\times\frac{\pi}{30}\vec{u}_\theta + 3.3(\frac{\pi}{30})^2.t\vec{u}_\rho$

$\vec{a}_M = -3.3\times\frac{\pi}{30}\times\vec{u}_\theta - 3.3\times\frac{\pi}{30}\times\vec{u}_\theta + 3.3\left(\frac{\pi}{30}\right)^2.t.\vec{u}_\rho$.

$\vec{a}_M = -0.11\pi\left(-\frac{\pi}{30}.t.\vec{u}_r + 2.\vec{u}_\theta\right)$

5) Vectors $\vec{u}_\rho;\vec{u}_\theta$ have the expression in the base ($\vec{i};\vec{j}$) :

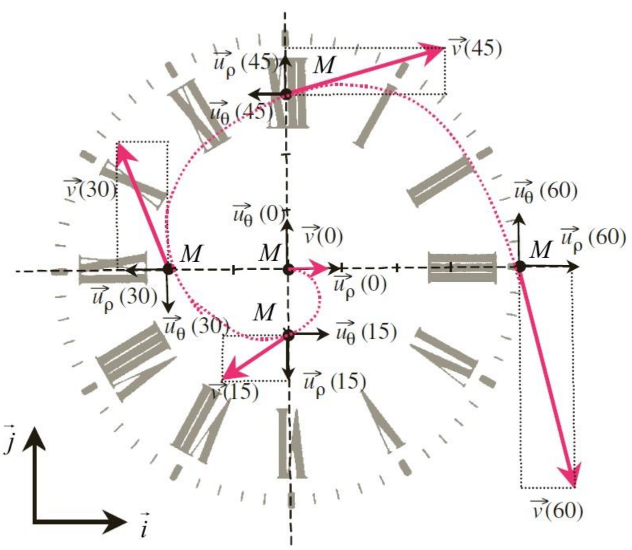
$\vec{u}_{\rho} = \cos\theta.\vec{i} + \sin\theta.\vec{j}$ et $\vec{u}_{\theta} = -\sin\theta.\vec{i} + \cos\theta.\vec{j}$.

Sachant qu'à t = 0 l'aiguille est sur le chiffre 3 (c'est-à-dire θ(0) = 0),l'angle θ s'exprime en fonction de la vitesse angulaire par :θ = –ω*t*.

The following table brings together the results obtained for the dates t = 0, 15, 30, 45 and 60 s

t (secondes)	$\theta = -\frac{\pi}{30}t$	$\vec{u}_{\beta}$	$\vec{u}_{\theta}$	$\vec{v}_M = 3.3(\vec{u}_{\rho} - \frac{\pi}{30}.t.\vec{u}_{\theta})$
0	0	$\vec{i}$	$\vec{j}$	$\vec{v}_M = \vec{v}(0) = 3.3\vec{i}$
15	$-\frac{\pi}{2}$	$-\vec{j}$	$\vec{i}$	$\vec{v}_M = \vec{v}(15) = 3.3(-\frac{\pi}{2}\vec{i} - \vec{j})$
30	$-\pi$	$-\vec{i}$	$-\vec{j}$	$\vec{v}_M = \vec{v}(30) = -3.3(\vec{i} - \pi\vec{j})$
45	$-\frac{3}{2}\pi$	$\vec{j}$	$-\vec{i}$	$\vec{v}_M = \vec{v}(45) = 3.3(\frac{3}{2}\pi\vec{i} + \vec{j})$
60	-2π	$\vec{i}$	$\vec{j}$	$\vec{v}_M = \vec{v}(60) = 3.3(\vec{i} - 2\pi\vec{j})$

For these dates, the unit vectors $\vec{u}_{\rho};\vec{u}_{\theta}$ are simply expressed in terms of vectors $\vec{i};\vec{j}$: le The result can be verified graphically.

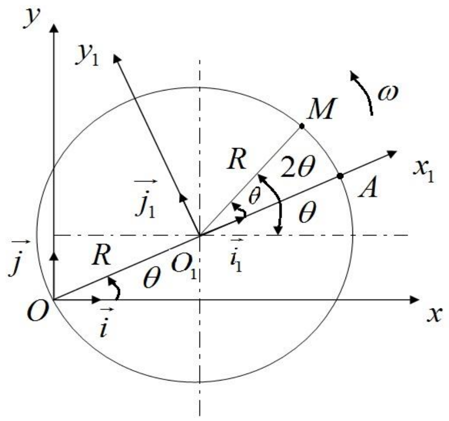


Exercice 3 :

In the plane, a circle of diameter OA rotates at constant angular speed around the point O origin of the fixed frame R(Oxy). We link to its (mobile) center O1 two axes O1x1 and O1y1 which form the mobile reference R1 (O1x1y1). The axis O1x1 is directed along OA see figure. At the initial instant t0=0 we have θ0=0, the point A is on the Ox axis (the Ox and O1x1 axes are collinear). A point M, initially at A, travels the circumference in the positive direction with the same angular speed knowing that

1- Directly calculate the components of the velocity and acceleration vectors of the point M in the fixed frame Oxy by deriving the position vector ($\overrightarrow{OM} = \overrightarrow{OO_1} + \overrightarrow{O_1M}$).

2- Calculate the components of the relative speed and acceleration of the point M in the moving frame (O1x1y1).



Solution :

1-a. Expression of the position vector in R.

$$\overrightarrow{OM} = \overrightarrow{OO_1} + \overrightarrow{O_1M} \;,\; \overrightarrow{OO_1} = R\cos\theta.\vec{i} + R\sin\theta.\vec{j} \;;\overrightarrow{O_1M} = R\cos 2\theta.\vec{i} + R\sin 2\theta.\vec{j}$$

$$\overrightarrow{OM} = R(\cos\theta + \cos 2\theta).\vec{i} + R(\sin\theta + \sin 2\theta).\vec{j}$$

b. Expression of the velocity vector in R.

$$\vec{v}_{M/R} = \frac{d(\overrightarrow{OM})}{dt} = -R\omega(\sin\omega.t + 2\sin 2\omega.t).\vec{i} + R\omega(\cos\omega.t + 2\cos 2\omega.t).\vec{j}$$

c. Expression of the acceleration vector in

R. $\vec{a}_M = \frac{d\vec{v}_M}{dt} = -R\omega^2(\cos\omega.t + 4\cos 2\omega.t).\vec{i} - R\omega^2(\sin\omega.t + 4\sin 2\omega.t).\vec{j}$

2-a. Expression of the vectorposition in R1.

$$\overrightarrow{O_1M} = R\cos\theta.\vec{i} + R\sin\theta.\vec{j}$$

b. Expression of the velocity vector in R1.

$$\vec{v}_r = \vec{v}_{M/R_1} = \frac{d(\overrightarrow{O_1M})}{dt} = -R\omega \sin \omega.t.\vec{i} + R\omega \cos \omega.t.\vec{j}$$

c. Expression of the acceleration vector in R1.

$$\vec{a}_r = \vec{a}_{M/R_1} = \frac{d\vec{v}_r}{dt} = -R\omega^2 \cos \omega.t.\vec{i} - R\omega^2 \sin \omega.t.\vec{j}$$

Chapter III

Dynamics of the material point

I-Introduction :

Kinematics aims to study the movements of bodies as a function of time, without taking into account the causes which provoke them.

Dynamics is the science that studies (or determines) the causes of the movements of these bodies.

III.1-General:

III.1.2-Absolute and Galilean reference frames:

The frame of reference is a set of observers who measure position and time. The absolute frame of reference is a frame of reference considered to be fixed, on the other hand the Galilean frame of reference is in uniform translation movement in relation to an absolute frame of reference.

A frame of reference is defined either by its name (example: terrestrial frame of reference) or by one of its marks R(O, x, y, z). Example:

The Copernican frame of reference is the best approximation of the Galilean frame of reference. Its origin is the center of mass of the solar system, and its axes are directions towards three fixed stars.

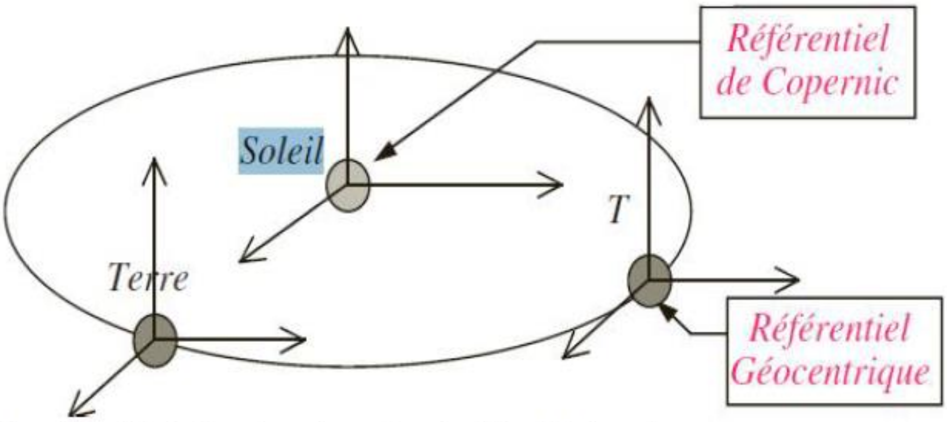


Figure II.1 Les référentiels de Copernic et géocentrique : le référentiel géocentrique est en mouvement circulaire uniforme par rapport au référentiel de Copernic.

III.1.2 Concept of mass:

The speed of a body is not enough to describe its movement. It is necessary to introduce a quantity characterizing the “repugnance” of the body to any modification of its movement, that is to say its inertia, it is possible to associate with a material point a positive scalar, m, which is its mass. It is expressed in kilograms (kg).

It is invariable in Newtonian mechanics, in relativistic mechanics, it dépends on the speed through the expression:

$$\boxed{m = \frac{m_0}{\sqrt{1+\frac{v^2}{c^2}}}} \qquad (III.1)$$

with:

m0: the mass at rest

m: mass at speed v

c: the speed of light, c =3.10^{8}m/s

III.1.3-Notion of Force:

In physics, a force designates the interaction between two objects or systems. There are 4 universal interactions: Gravitational interaction, Electromagnetic interaction, Weak nuclear interaction and Strong nuclear interaction.

III.2- Vector quantity of movement:

The momentum vector $\vec{P}$ (sometimes called impulse) for a material point

of mass m and speed $\vec{v}$ with respect to R is:

$$\boxed{\vec{P} = m\vec{v}} \qquad (III.2)$$

The unit of momentum in the international system is kg.m.s-1.
For a system made up of n material points the momentum vector is the Resulting vector:

$$\boxed{\vec{P} = \sum_{i=1}^{n} \vec{P}_i = \sum_{i=1}^{n} m_i \vec{v}_i} \qquad (III.3)$$

III.3-Newton's laws:

III.3.1-1st law of Newton, "Principle of inertia":

Galileo was the first to suggest this principle. It constitutes Newton's first law and which is stated as follows:
"Every object retains its state of rest or uniform rectilinear movement in the absence of forces acting on him" This 1st law can also be stated:
In a Galilean frame of reference (R), any mechanically isolated (or pseudo-isolated) material point A is either at rest or in uniform rectilinear motion.

$$\boxed{\frac{d\vec{P}}{dt} = \vec{0} \; ou \; bien \; \sum \vec{F}_{ext} = \vec{0}} \qquad (III.\ 4)$$

This principle leads to the law of conservation of the total momentum of a system

isolated or pseudo isolated. The above property constitutes a definition of the Galilean coordinate systems and

the principle of inertia postulates their existence. A Galilean frame of reference is a translational frame of reference rectilinear and uniform in the Copernicus benchmark.

III.3.2- Newton's 2nd law: Fundamental principle of point dynamics

Compared to a Galilean frame of reference R, the movement of a material point of mass m and subject to several external forces, the sum of which is $\boxed{\vec{f}_{ext}}$

$$\boxed{\frac{dP}{dt} = \vec{f}\,ext} \qquad \text{(III.5)}$$

The fundamental principle of dynamics is then written using the acceleration vector in the form:

$$\boxed{\vec{f}\,ext = m\vec{a}} \qquad \text{(III.6)}$$

Noticed :

This formula only applies to systems of constant mass. The case of a rocket which consumes fuel and sees its mass decrease or that of a drop of water which falls into an atmosphere containing water vapor and grows during the movement cannot be treated with this relation.

III.3.3- 3éme loi de Newton : Principe des actions réciproques

Newton's third and final law is stated as follows: If a material point A exerts on a material point B a force $\vec{f}_{A/B}$ alors le point B exerce sur le point A une force $\vec{f}_{B/A}$ such as :

$$\boxed{\vec{f}_{A/B} = -\vec{f}_{B/A}} \qquad \text{(III .7)}$$

III.4.Examples of forces:

III.4.1 Remote force:

III.4.1.a-The gravitational interaction force:

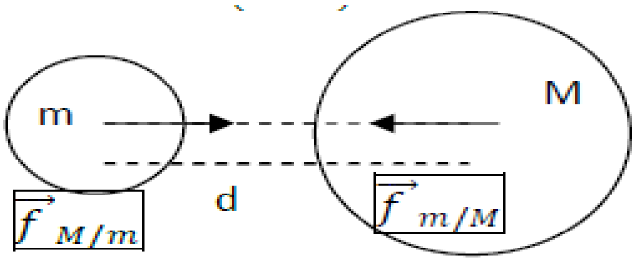
This interaction force follows a law stated by Newton in 1650

$$\boxed{\vec{f}_{M/m} = -\vec{f}_{m/M} = G\frac{M.m}{d^2}\vec{u}} \qquad \text{(III.8)}$$

$\vec{u}$: unit vector

G : a constant

G= 6,67 x 10-11 m3Kg^{-1}s^{-2}



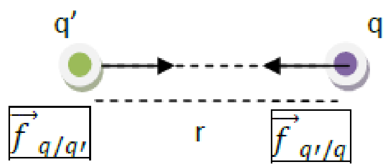
M and m the 2 interacting masses.

III.4.1.b-The Colombian force of interaction:

The Coulomb interaction, Columbus's law for electric charges.

$$\boxed{\vec{f}_{q/q'} = -\vec{f}_{q'/q} = k\frac{q.q'}{r^2}\vec{u}} \qquad \text{(III.9)}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 9\text{x } 10^9 \quad \text{(USI)}$$

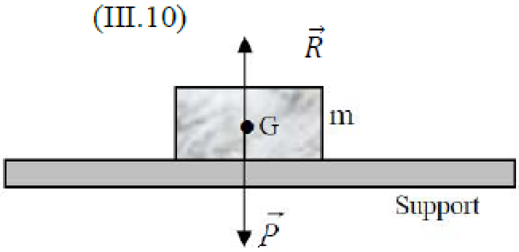


III.4.1.c-Contact force:

. Support reaction:

The force that an object, placed on a horizontal support, experiences from the support is called support reaction. The reaction of the support on the object m is distributed over the entire surface of contact support-object $\vec{R}$, represents the result of all the actions exerted on the contact surface.

L'objet étant en équilibre $\vec{R} + \vec{P} = \vec{0} \Rightarrow \vec{R} = -\vec{P}$ (III.10)



-Friction force:

It is the resistance which appears during contact between two rough surfaces and which opposes the relative movement of the two surfaces. There are several types of friction: Friction between solid bodies and friction in fluids:

- Solid/solid friction:

The nature of these forces results in a friction coefficient, the value of which depends on the materials in contact.

The expression for the norm of the friction force is written:

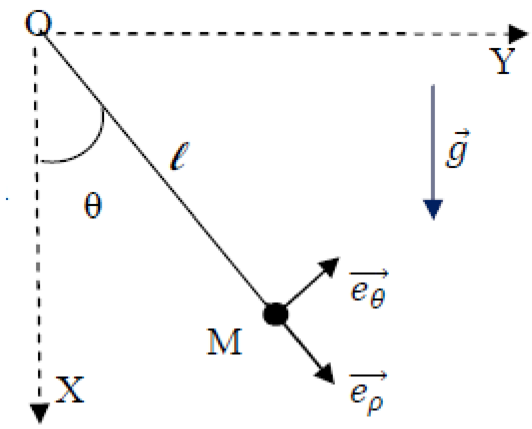
$$\boxed{\vec{f}_d = \mu_d . \vec{R}} \qquad \text{(III.11)}$$

Static friction occurs between surfaces that are stationary relative to each other. However, dynamic friction occurs during movement. We distinguish: the coefficient of static friction μ_s which represents the maximum value, and the coefficient of dynamic friction (or sliding) μ_d avec $\mu_d < \mu_s$.

Example 1:

A simple pendulum consisting of a point object M of mass m, attached to an extensible wire of length L and negligible mass. Its movement takes place in the vertical plane (xOy) of the fixed frame of reference R(O,xyz).

We move the pendulum by an angle θ from its equilibrium position $(\theta=0)$ and we release it without initial speed. Friction forces are assumed to be non-existent. (the Earth's gravity field g is considered uniform.) Exprimer les forces appliquées au point M dans la base $(\overrightarrow{e_\rho}, \overrightarrow{e_\theta}, \vec{k})$.



2) By applying the PDF in the Galilean repository $\Re$ establish the differential equation of motion in the case of weak oscillations.

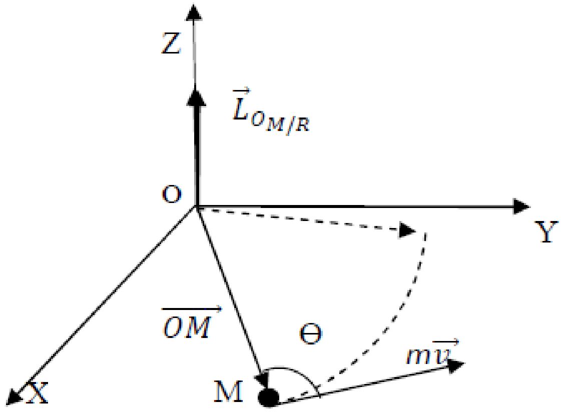
3) Establish the expression for the tension T of the wire.

We give : $a_r = \ddot{\rho} - \rho\,\dot{\theta}^2$; $a_\theta = 2\,\dot{\rho}\,\dot{\theta} + \rho\,\ddot{\theta}$

III.5-Theorem of angular momentum:

III.5.1- Angular momentum relative to a point: The angular momentum vector of the material point M $\vec{L}_{O_{M/R}}$ with respect to O moving at speed $\vec{v}_{M/R}$ in the frame of reference R is defined by the vector product:

$$\vec{L}_{O_{M/R}} = \overrightarrow{OM} \Lambda m\vec{v}_{M/R} \tag{III.12}$$



Avec : $\left\|\vec{L}_{O_{M/R}}\right\| = \left\|\overrightarrow{OM}\right\| . m \left\|\vec{v}_{M/R}\right\| \sin\theta$

m: is the mass of the point

$\overrightarrow{OM}$: is the position vector

$\vec{p} = m\vec{v}_{M/R}$: is The quantity of movement relative to the reference frame R

III.5.2-Theorem of angular momentum:

«The derivative with respect to time of the angular momentum vector is the moment of the force Fext with respect to a point”:

$$\boxed{\frac{d\vec{L}_{O_{M/R}}(M)}{dt} = \overrightarrow{OM} \Lambda \vec{f}_{\text{ext}}} \tag{III.13}$$

Demonstration:

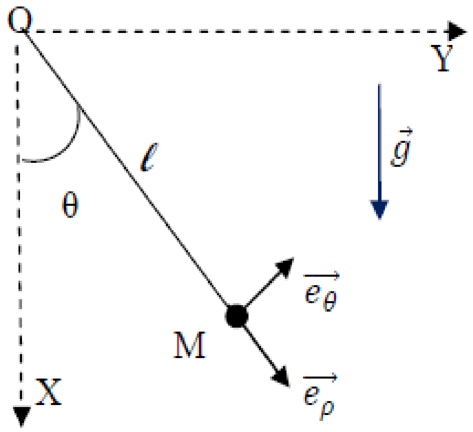
$$\frac{d\vec{L}_{O_{M/R}}(M)}{dt} = \underbrace{\left(\frac{d\overrightarrow{OM}}{dt} \Lambda\, m\vec{v}_{M/R}\right)}_{=0} + \overrightarrow{OM} \Lambda\, m\frac{d\vec{v}_{M/R}}{dt}$$

$\vec{\delta}_{O_{M/R}}(\vec{f}_{ext}) = \overrightarrow{OM} \Lambda\, \vec{f}_{ext} = \overrightarrow{OM} \Lambda\, m\frac{d\vec{v}_{M/R}}{dt}$ est le m***oment de la force F_{ext} par rapport à O***:

Example 2:

By applying the angular momentum theorem in the Galilean frame of reference, establish the differential equation of motion of a simple pendulum in the case of low Oscillations.

Réponse III.2:



Answer 2: The angular momentum theorem is written:

$$\frac{d\vec{L}_{O_{M/R}}}{dt} = \overrightarrow{OM}\Lambda \sum \vec{f}_{ext} = \overrightarrow{OM}\Lambda(\vec{p} + \vec{T})$$

With:

$$\vec{L}_{O_{M/R}} = \overrightarrow{OM}\Lambda\, m\vec{v} = l\vec{e}_\rho\Lambda\, ml\dot{\theta}\vec{e}_\theta = ml^2\dot{\theta}\vec{k} \;\Rightarrow \frac{d\vec{L}_{O_{M/R}}}{dt} = ml^2\ddot{\theta}\vec{k}$$

$$\overrightarrow{OM}\Lambda\vec{p} = l\vec{e}_\rho\Lambda\left(mgcos\theta\vec{e}_\rho - mgsin\theta\vec{e}_\rho\right) = -mglsin\theta\vec{k}$$

$$\overrightarrow{OM}\Lambda\vec{T} = \vec{0}$$

Ce qui implique que : $ml^2\ddot{\theta}\vec{k} = -mglsin\theta\vec{k} \Rightarrow \;\; \ddot{\theta} + \frac{g}{l}sin\theta = 0$

$On\ obtient\ finalement\ pour\ \ les\ faibles\ oscillations$

$$\boxed{\ddot{\theta} + \frac{g}{l}\theta = 0}$$

Exercice 1 :

A car, assimilated to a material point M of mass m = 1000 kg, begins a descent in A at speed V0 = 125 km.h in the figure opposite The trajectory of the descent from A to B is an arc of a circle, with center O, of radius R = 130 m and angle θ = 15°. It is assumed that during the descent; driving force $\vec{F}$ of the car is tangent to the road and the algebraic value F is constant. (We neglect friction.)

- • Determine the equations of motion of $M(R\,\theta)$ in projection on the polar base
- • After multiplying the equation on $\overrightarrow{e_\theta}$, integrate it.
- • Determine the normal reaction N as a function of, R, θ , g (gravity), m, v0 and F.
- a) Determine the expression verified by the angle for which the car would leave the ground (Take-off).
-
- b) Calculate this angle in the case where the driver cuts the motor F=0 at A (we will take g =9.8 m.s-2).

Exercice 2 :

We consider a track ABC inclined at an angle and made up of a perfectly smooth part AB = 16m, and a rough part BC. We launch from A a cube assimilated to a material point of mass M = 1kg with a speed VA parallel to the track (see figure 1). The contact between the cube and the part BC of the track is characterized by a coefficient dynamic friction $\mu_d = 0, 4.$
We give g = 10m/s2.

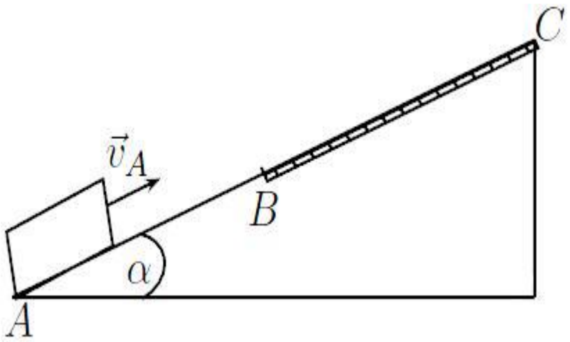


Figure 1

1- Represent the forces applied to the cube on parts AB and BC of the track.

2- What must be the speed VA1 so that the speed VB at point B is zero.

3- We launch the cube with another speed VA2. It stops at a point D between B and C.-

Derive the expression for the distance BD as a function of VA2, AB, g, α μ_d = 0, 4 .

- Calculate this distance knowing that VA2= 14m/s.

4. What will happen to the cube, after reaching point D, if the cube-track contact is

characterized by a coefficient of static friction equal to: a) μ_s = 0, 5 b) μ_s = 0, 65

We give g=10 m .s-²

Exercise 3:

A material point of mass m, slides along the path ABC shown in Figure 2

- The path AB is circular with center O, and radius R. Friction is negligible on along AB. -

The path BC is horizontal, characterized by a slip coefficient μ.

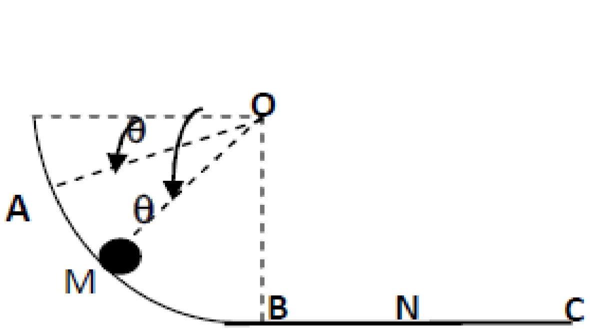


Figure 2

1-Represent the forces exerted on the mass at point M.

2-a) demonstrate that the speed acquired at the point M defined by the angle θ is given by the expression:

$$v=\sqrt{2Rg(\sin\theta-\frac{1}{2})}$$

2.b) Express the reaction force N at point M, Va=0?

3.b) Deduce the values of the speed and the reaction force N at point B?

4- Express the speed at point N; knowing that the friction force f = μ. mg, and BC=R?

5-Calculate the slip coefficient μ, so that the material point stops at point C?

Chapter IV

Work, power and energy

In this chapter we will introduce the important concepts of work of a force and mechanical energy. We will see that these two quantities and the relationships that connect them can be used to simply solve mechanical problems.

IV.1- The work of a force:

IV.1.1- Definition:

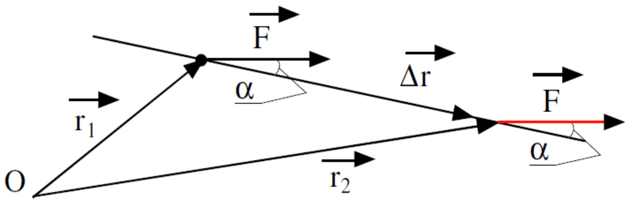
The work of a force measures the effort required to move an object along a path which may or may not be horizontal, rectilinear or not. A work can be positive in which case, we will speak of motor work, because a motor can very well carry out this movement effort. On the other hand, work can be negative, we speak of resistant work because it opposes the movement, this is the case of friction forces.

1-Case of a constant force and a rectilinear movement.

A material point, subjected to a force F constant in module and direction, Its displacement is therefore:

$$\Delta \vec{r} = \vec{r}_2 - \vec{r}_1$$

During this displacement, the force F exerts work



$$W_{12} = \vec{F} \cdot \Delta \vec{r} = |\vec{F}|\,|\Delta \vec{r}|\, \cos\alpha$$

This work can be positive, negative or zero, it depends on the sign of cos α:

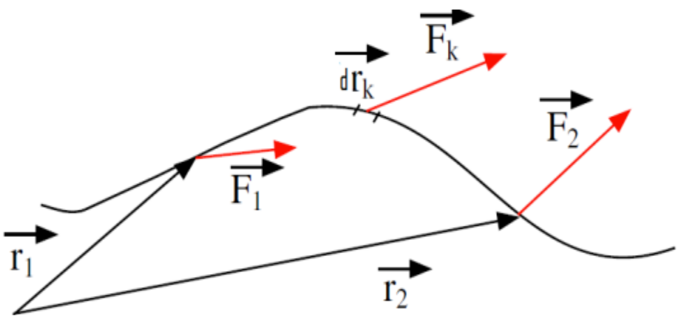
- cos α >0 it is motor work

- cos α <0 resistant work

- cos α =0 zero work

IV.1.2-Elementary work:

In the case where the force F is variable and any displacement. We calculate the work of this force for an infinitesimal rectilinear displacement dr, we speak of the elementary work dW defined by:



$$dW = \vec{F} \cdot d\vec{r}$$

To find the total work between the first point and the second, we integrate this last equation:

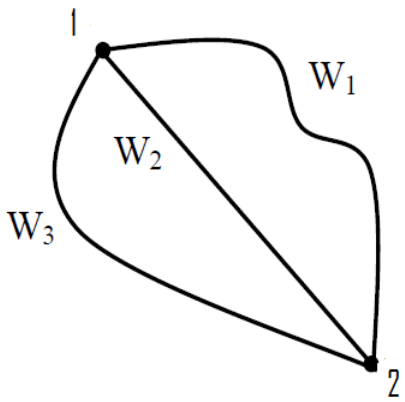
$$W_{1-2} = \int_{\vec{r_1}}^{\vec{r_2}} \vec{F}.\vec{dr}$$

If we want to use Cartesian coordinates, we express the force and the displacement in this system, for this we will have:

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$w_{1-2} = \int_1^2 F_x dx + F_y dy + F_z dz$$



Calculating the work requires knowing the path followed between the two points. For each path there is a job. In general the work depends on the path followed.

-Constant force on a rectilinear movement:

Consider a force $\vec{F}$ constant (in norm, sense and direction) applied to a material point M moving on a straight segment AB.

$$W_{\overrightarrow{AB}}(\vec{F}) = \vec{F}.\overrightarrow{AB}$$

$\vec{F}$ = const on $\overrightarrow{AB}$ $\quad W_{AB}(\vec{F}) = F.AB.\cos\alpha$ si AB= $\ell \Rightarrow W_{AB} = F.\ell.\cos\alpha$ (IV.1)

The angle α is the angle made by the force vector $\vec{F}$ with the displacement vector $\overrightarrow{AB}$ (see figure IV.1).

The work is expressed in N.m, that is to say in Joule (J). (1J corresponds to the work of a force of 1 N over a distance of 1 m).

There is non-zero work when the force acting on the object has a component in the direction of motion.

The work is either positive, zero or negative depending on the direction of the force relative to the displacement.

When the force opposes the movement the force is resistant and the work is negative. $W_{AB}(\vec{F}) < 0$.

When the force is driving the work is positive. $W_{AB}(\vec{F}) > 0$.

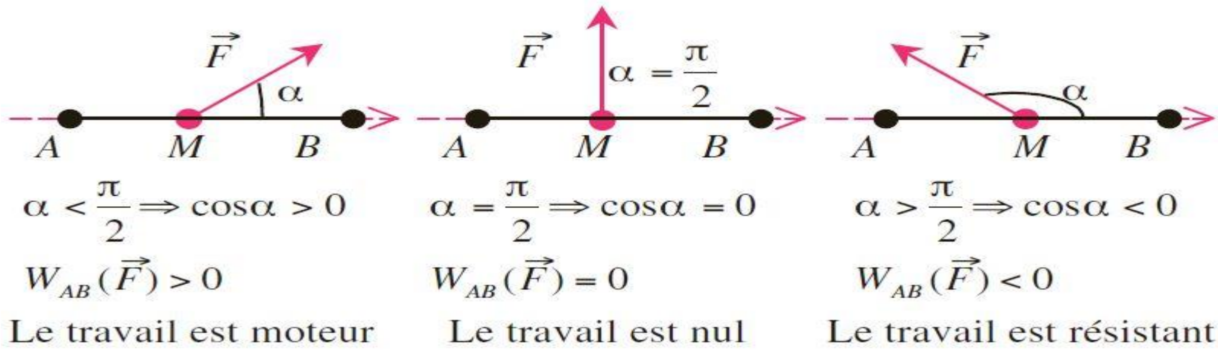


Figure IV.2 motor work, zero and resistant to a force.

IV.1.3-Definition of elementary work:

We now consider the more general case where the force $\vec{F} = \vec{F}(M)$ is a function of the position of the point of application M which moves from a point A to a point B on any curve (figure IV.3)

In this case, to calculate the corresponding work of the force, it is no longer possible to use the expression (IV.1). The calculation method then consists of breaking down the path into a succession of infinitely small and therefore rectilinear elementary movements (see figure IV.4). These elementary paths are sufficiently small to be able to consider the force vector as constant over the displacement and the definition (see relation (IV.1)) can be applied.

The expression of the elementary work on such an elementary displacement can therefore be written:

$\delta W = \vec{F}(M).\vec{d\ell}$ (IV.2)

The definition of elementary work of force corresponds to the more general definition of "elementary circulation of the force vector"

IV.1.4-Work of a variable force on any displacement:

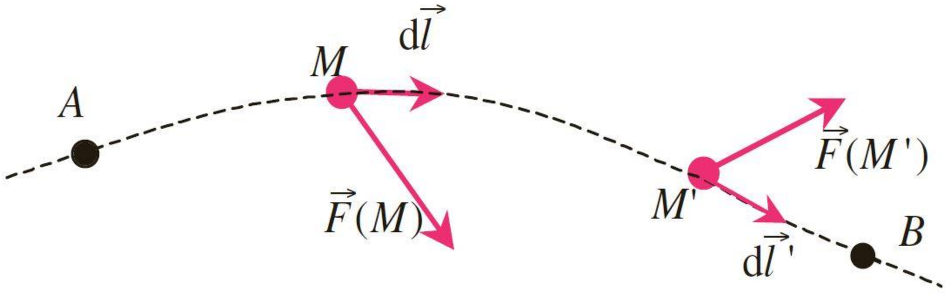


Figure IV.3 Variable force on any displacement AB

To get the work $W_{AB}(\vec{F})$ of the force $\vec{F}(M)$ on the move from A to B, simply add up all the elementary work between the starting point and the arrival point. This summation containing an infinity of terms all infinitely small corresponds to an integration between points A and B:

$$W_{AB}(\vec{F}) = \int_A^B \delta W(\vec{F}) = \int_A^B \vec{F}(M).d\vec{\ell} \quad \text{(IV.3)}$$

IV.1.5-Examples of calculating the work of a force on a path AB:
IV.1.5.1-Case of a constant force on any path AB:

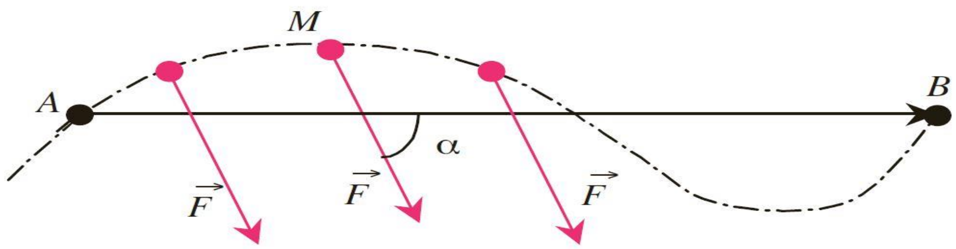


Figure IV.4. Case of a constant force

In the particular case of a constant force, the expression for work is simplified because we can leave the integral. We therefore obtain in this case:

$$W_{AB}(\overrightarrow{F}) = \int_A^B \delta W(\overrightarrow{F}) = \int_A^B \overrightarrow{F}(M).d\vec{\ell} = \overrightarrow{F}..\int_A^B d\vec{\ell} \quad \text{(IV.4)}$$

The addition of all elementary vectors $d\vec{\ell}$, placed end to end, starting from point A to reach point B gives, using the Chasles relation, the vector $\overrightarrow{AB}$:

$$\int_A^B d\vec{\ell} = \overrightarrow{AB} \quad \text{(IV.5)}$$

So the expression of the work will be:

$$W_{\overline{AB}}(\overrightarrow{F}) = \vec{F}.\overrightarrow{AB} = F.AB.\cos\alpha \quad \text{(IV.6)}$$

Noticed:

The work of a constant force does not depend on the path followed but only on the direct distance AB and the angle that the force makes with the segment AB.

Calculating the work on the direct path AB would have given the same result.

IV.1.5.2-Working with the weight of a body on any AB path:

Consider a point M of mass m moving from point A to point B and calculate the work of the weight of this body during this movement (see figure IV.5). The movement from A to B is assumed to be random, that is to say that the path which leads from A to B can take different trajectories.

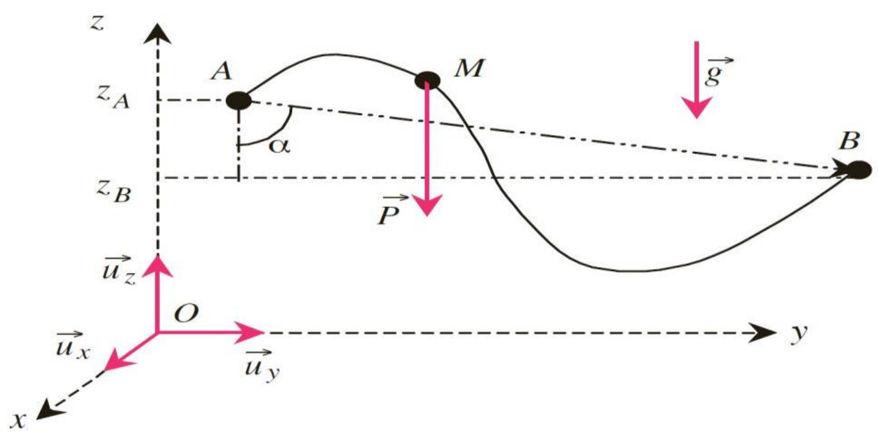


Figure IV.5 Weight work $\vec{P}$ of a body

According to the relation (IV.6), we can write:

$$W_{AB}(\vec{P}) = \vec{P}.\overrightarrow{AB} \quad \text{(IV.7)}$$

The calculation of the scalar product can be done in two ways:

- Using the cosine of the angle between $\vec{P}$ et $\overrightarrow{AB}$:

$$W_{AB}(\vec{P}) = mg.AB.\cos\alpha. \quad \text{(IV.8)}$$

- Using the components of the vectors and in the basis $(\vec{i}, \vec{j}, \vec{k})$ Cartesian coordinate system (O, x, y, z) (see figure IV.5).

The Oz axis is chosen vertical upwards. With (x_A, y_A, z_A) the coordinates of the starting point A and (xB, yB, zB) those of the arrival point B we have: the coordinates of the starting point A and (xB, yB, zB) those of the arrival point B we have:

$$\overrightarrow{AB} = \begin{cases} x_B - x_A = 0 \\ y_B - y_A \text{ et } \vec{P} = m\vec{g} = 0 \quad \text{(IV.9)} \\ z_B - z_A \quad \text{et } p_z = mg \end{cases}$$

$$W_{AB}(\vec{P}) = \vec{P}.\overrightarrow{AB} = -mg.(z_B - z_A) \qquad \text{(IV.10)}$$

The two results are identical. Indeed we can notice that:

$$AB.\cos\alpha = (z_A - z_B) \qquad \text{(IV.11)}$$

The expression of the work of the weight as a function of the altitude difference is therefore:

$$W_{AB}(\vec{P}) = -mg.\Delta h \qquad \text{(IV.12)}$$

IV.1.5.3-Case of a variable force: work of the tension of a spring

Consider a spring of stiffness k, of length at rest, at the end of which a mass m is attached (see figure IV.6). The spring and mass are on a horizontal plane and we are only interested in the tension of the spring.

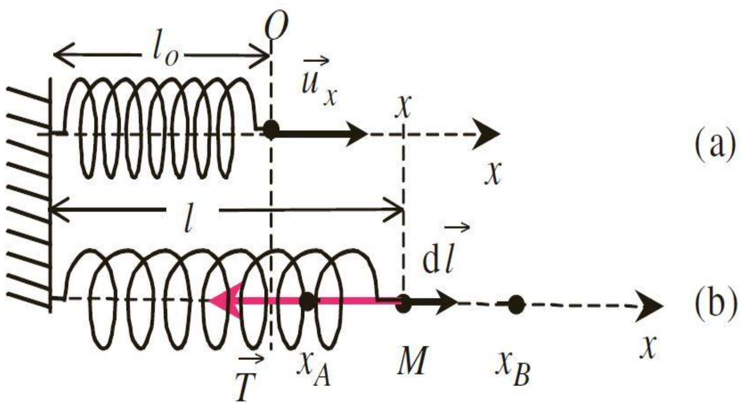


Figure IV.6 Case of a variable force $\vec{T}$

The elastic force, that is to say the tension of the spring, is a force which varies with the state of stretching of the spring of stiffness k. We locate the free end M of the spring on an axis Ox in the same direction as the spring, the origin point O corresponding to the position of the point M when the spring is at rest (neither stretched nor compressed). With the conventions of figure IV.6 we can write:

$$\vec{T} = -k.\Delta\vec{\ell} = -k.\Delta\ell\vec{i} = -k.(\ell - \ell_0)\vec{i} = -k.x\vec{i} \qquad \text{(IV.13)}$$

Consider an elementary displacement $d\vec{\ell} = dx.\vec{i}$ from the end M of the spring (see figure IV.6). The elementary work is written:

$$\delta W = \vec{T}.d\vec{\ell} = -k.x\vec{i}.dx\vec{i} = -k.x.dx \qquad \text{(IV.14)}$$

The work of the spring tension when the point M passes from the abscissa x_A at the abscissa x_B is given by the following expression:

$$W_{AB}(\vec{T}) = -k\left[\frac{x^2}{2}\right]_{x_A}^{x_B} = \frac{1}{2}k.x_A^2 - \frac{1}{2}k.x_B^2 \qquad \text{(IV.15)}$$

The work of this force does not depend on the path followed but only on the initial and final position of the spring.

IV.2-Power of a force:

The same work can be completed more or less quickly. The power of a force corresponds to the work done by this force per unit of time and provides information on the speed with which the work (energy transfer) is carried out.

If W is the work carried out during the duration Δt, the average power Pm of the force is defined by:

$$P_m = \frac{W}{\Delta t} \qquad \text{(IV.16)}$$

The unit of power is the Watt (symbol W) corresponding to 1J of work carried out in 1 s.

IV.3-Mechanical energy:

IV.3.1-Introduction:

Two types of energy will be introduced in this chapter: kinetic energy (Ec) linked to the movement of the object and potential energy (Ep) linked to its position. The mechanical energy (E) of a system is then defined by the sum of the kinetic and potential energies.

IV.3.2-Kinetic energy:

Let us consider a material point G, of mass m, moving, in a Galilean frame of reference (R), under the action of a set of external forces.

The application of the fundamental principle of dynamics gives:

$$\sum \vec{F}_{ex} = m.\vec{a}_G = m.\frac{d\vec{v}_G}{dt} \qquad \text{(IV.17)}$$

During an elementary move $d\vec{\ell}$, the sum of the elementary works of the external forces is given by:

$$\sum \delta W = \sum(\vec{F}_{ex}.d\vec{\ell}) = (\sum \vec{F}_{ex}).d\vec{\ell} \qquad \text{(IV.18)}$$

Using the relation (IV.17) the relation (IV.18) is written:

$$\sum \delta W = m.\frac{d\vec{v}_G}{dt}.d\vec{\ell} = m.d\vec{v}_G.\frac{d\vec{\ell}}{dt} = m.\vec{v}_{G.}d\vec{v}_G \qquad \text{(IV.19)}$$

Let us now consider a path AB carried out by the point G. Noting $\vec{v}_A$ et $\vec{v}_B$ the velocity vectors of G respectively at point A and point B, the integration of the second member of the relation (IV.19) gives:

$$m\int_{v_A}^{v_B} \vec{v}_G d\vec{v}_G = m.\left[\frac{1}{2}v_G^2\right]_{v_A}^{v_B} = \frac{1}{2}m.v_B^2 - \frac{1}{2}m.v_A^2 \qquad \text{(IV.20)}$$

The integration of the first member of the relation (IV.18) is written as :

$$\int_A^B (\sum \delta W) = \sum \int_A^B (\vec{F}_{ex}.d\vec{\ell}) = \sum W_{AB}(\vec{F}_{ex}) \qquad \text{(IV.21)}$$

Equality of relationships (IV.20) et (IV.21) leads to :

$$\frac{1}{2}m.v_B^2 - \frac{1}{2}m.v_A^2 = \sum W_{A\to B}(\vec{F}_{ex}) \qquad \text{(IV.22)}$$

Thus the total work of all the forces applied to point G between two positions A and B can be expressed solely as a function of the mass and the speeds at the starting point A and arrival point B. It therefore appears interesting to define a function of state depending only on the mass and speed of the point. This new quantity homogeneous with work is called kinetic energy.

IV.3.3-Definition of kinetic energy:

We define the kinetic energy Ec for a material point of mass m moving at speed in a Galilean frame of reference, by:

$$E_C = \frac{1}{2}m.v^2 \qquad \text{(IV.23)}$$

IV.3.4-Theorem of kinetic energy:

In a Galilean frame of reference, the variation in kinetic energy of a material point, subjected to a set of external forces, between a position A and a position B, is equal to the sum of the work of these forces between these two points:

$$\frac{1}{2}m.v_B^2 - \frac{1}{2}m.v_A^2 = E_C(B) - E_C = \Delta E_C = \sum W_{A \to B}(\vec{F}_{ex}) \quad \text{(VI.24)}$$

This theorem is very useful for determining the speed of a point mass m at a particular point (B) knowing its speed at another point (A) without going through the time equations of motion.

IV.3.5-Potential energy:

Potential energy is a form of energy linked to the position of the system. By changing position, this energy can increase (the system stores energy) or decrease (the system releases energy outside).

We can give the simple example of a mass that is released from a certain altitude: as it falls (reduction in altitude) its speed increases. The mass therefore potentially had energy which was returned in the form of kinetic energy during the fall. The only force then exerted on the mass is its weight which provides positive work independent of the path followed and linked only to the reduction in altitude. Just as for the kinetic energy defined from its variation linked to the work of all the forces, the Potential energy will be defined from its reduction linked to the work of certain forces: those whose work does not depend on the path followed.

IV.3.6-Conservative forces:

IV.3.6.1-Definition:

These are the strengths (noted $\vec{F}_{ex}^{C}$) whose work does not depend on the path followed but only on the initial (starting point) and final (arrival point) positions.

We can cite as examples encountered at the beginning of this chapter:

• weight work.

• work of spring tension.

• work with a constant force (in norm and direction).

IV.3.7-Non-conservative forces:

IV.3.7.1-Definition:

These are all the other forces (noted $\vec{F}_{ex}^{NC}$) whose work depends on the path followed.

Friction forces can be cited as an example. The work of these forces is always resistant (negative work W < 0). Let us take the case of a solid friction force. This force continually opposes the displacement (see chapter III on forces) and its norm F is constant. The force vector will be vector in the same direction but opposite in direction to the elementary displacement vector. The calculation of the work of the solid friction force gives:

$$\delta W = \vec{F}.d\vec{\ell} = -F.d\ell \Rightarrow W_{AB} = \int_A^B - F.d\ell = -F\int_A^B d\ell = -F.L_{AB} \qquad \text{(IV.25)}$$

The length LAB is the distance actually traveled between A and B. This distance obviously depends on the path followed.

IV.3.8-Definition of potential energy:

The work of a conservative force does not depend on the path followed but only on the initial state (state A) and final state (B). This work can be expressed from a state function EP (function depending only on the state of the system) called potential energy.

By definition: for a conservative force $\vec{F}_{ex}^{C}$ there exists a state function E_P such that:

$$W_{AB}(\vec{F}_{ex}^{C}) = E_P(A) - E_P(B) = -\Delta E_P \qquad \text{(IV.26)}$$

The change in potential energy between two points A and B is equal to the opposite of the work of the conservative force between these two points.

IV.3.9-Integral definition of potential energy:

From the integral expression, we can deduce the differential expression of the potential energy by revealing the elementary work of the conservative force. We will then have:

$$\delta W(\vec{F}_{ex}^{C}) = \vec{F}_{ex}^{C}.d\vec{\ell} = -dE_P \qquad \text{(IV.27)}$$

IV.3.10-Local definition of potential energy:

The differential of the potential energy function can be written as a function of the gradient of this function.

$$dE_P = \overrightarrow{grad}E_P.d\vec{\ell} = -\vec{F}_{ex}^{C}.d\vec{\ell}$$

$$\vec{F}_{ex}^{C} = -\overrightarrow{grad}E_P \text{ (IV.28)}$$

$$\vec{F}_{ex}^{C} = -\frac{\partial E_P}{\partial x}\vec{i} - \frac{\partial E_P}{\partial y}\vec{j} - \frac{\partial E_P}{\partial z}\vec{k} \qquad \text{(IV.29)}$$

In the case where the potential energy only depends, for example, on a variable x
we have :

$$\vec{F}_{ex}^{C} = -\frac{dE_P}{dx}\vec{i} \qquad \text{(IV.30)}$$

The force is directed towards decreasing potential energies.

Potential energy is defined from a comparison with a difference (integral definition) or an integration (differential or local definition). This means that this function is defined up to a constant. This constant does not matter because potential energy always occurs through differences.

IV.4-Mechanical energy:

IV.4.1-Work and energy:

Consider a material point M of mass m subject to a set of external forces $\vec{F}_{ex}$. These can be decomposed into conservative forces $\vec{F}_{ex}^{C}$ and not conservative $\vec{F}_{ex}^{NC}$.
Let us now consider a displacement of the point M of one position
A to a position B. The total work of all the forces for this displacement is related to the variation of kinetic energy by the kinetic energy theorem:

$$\sum W_{A\to B}(\vec{F}_{ex}) = \Delta E_C \qquad \text{(IV.31)}$$

$$\Rightarrow \sum W_{A\to B}(\vec{F}_{ex}^{\,C}) + \sum W_{A\to B}(\vec{F}_{ex}^{\,NC}) = E_C(B) - E_C(A) \qquad \text{(IV.32)}$$

The work of conservative forces can be expressed as a function of the potential energy from which they derive. Let Ep denote the total potential energy, sum of the potential energies from which each conservative force derives, we can write:

$$\sum W_{A\to B}(\vec{F}_{ex}^{\,C}) = E_P(A) - E_P(B) \qquad \text{(IV.33)}$$

By reporting this relation (IV.33) in the equality (IV.32) we have:

$$E_C(B) - E_C(A) = \sum W_{A\to B}(\vec{F}_{ex}^{\,NC}) + \left[E_P(A) - E_P(B)\right] \qquad \text{(IV.34)}$$

$$\left[E_C(B) - E_C(A)\right] + \left[E_P(B) - E_P(A)\right] = \sum W_{A\to B}(\vec{F}_{ex}^{\,NC}) \ \text{(IV.35)}$$

$$\left[E_C(B) + E_P(B)\right] - \left[E_C(A) + E_P(A)\right] = \sum W_{A\to B}(\vec{F}_{ex}^{\,NC}) \ \text{(IV.36)}$$

A new homogeneous state function appears at an energy and whose variation is expressed as a function only of the work of the non-conservative forces. This new function corresponds to mechanical energy.

IV.4.2-Definition of mechanical energy:

The mechanical energy of a system is equal to the sum of the kinetic and potential energies. It is a state function: $E = E_c + E_P$.

IV.4.3-Theorem of mechanical energy:

Based on the results of the relation (III.36) we deduce the following theorem:

The variation in mechanical energy of a system between two points A and B is equal to the sum of the work of the non-conservative forces applied to the system between these two points.

$$\Delta E = E(B) - E(A) = \sum W_{A\to B}(\vec{F}_{ex}^{\,NC}) \qquad \text{(IV.37)}$$

Noticed :

Non-conservative forces being resistant forces (their work being negative; W <0) the mechanical energy of a system can only decrease over time.

IV.4.4-Conservative system and conservation of mechanical energy:

A system is said to be conservative if this system only experiences conservative external forces.

As the system does not experience any non-conservative force, the friction forces are zero or negligible. The system is also called "mechanically isolated". By applying the mechanical energy theorem (IV.37) we obtain:

$$\Delta E = E(B) - E(A) = \sum W_{A\to B}(\vec{F}_{ex}^{\,NC}) = 0 \qquad \text{(IV.38)}$$

$$\Delta E = 0 \Rightarrow E = cons\tan te.$$

IV.4.5-Theorem of mechanical energy for a conservative system:

The mechanical energy of a conservative (or mechanically isolated) system is conserved over time.

Conservative system ⇒E = Ec + Ep= Eo = constant.

$$\Rightarrow \frac{dE}{dt} = 0.$$

⇒This constitutes the principle of conservation of mechanical energy.

IV.5-Applications :

IV.5.1-Use of energy to solve a problem:

Consider the case of a ball M of mass m located at altitude zA= het which is released without initial speed.

-Find the speed of this ball when it hits the ground at B altitude zB= 0.

-Determine its speed v as a function of its altitude z.

System ball M of mass m

-Galilean terrestrial reference frame. Choice of a reference: vertical axis ascending with point B on the ground (altitude zB= 0) and point A at altitude zA= h (see figure IV.7).

- Balance of forces: only the weight of the ball: (we neglect friction.

Weight is a conservative force that derives from potential energy. With an ascending vertical axis we have: EPP = mgz with zero potential energy on the ground (EPP(B) = EPP(zB= 0) = 0)).

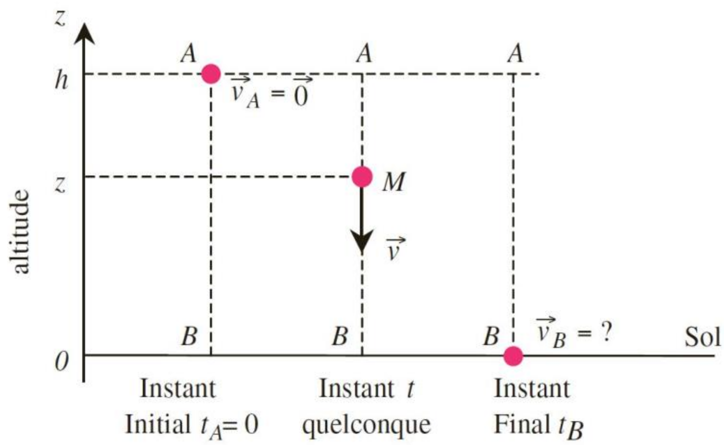


Figure IV.7 Free fall of a ball identified by its altitude z.

IV.5.2-Calcul en utilisant le théorème de l'énergie cinétique :

Let us apply the kinetic energy theorem between positions A at the start and B at the end.

The weight work is written:

$$W_{A\to B}(\vec{P}) = m.g.(Z_A - Z_B) = m.g.h\,.$$

The variation in kinetic energy (final kinetic energy – initial kinetic energy) is written:

$$\Delta E_C = E_C(B) - E_C(A) = \frac{1}{2}.m.v_B^2 - \frac{1}{2}.m.v_A^2 = \frac{1}{2}.m.v_B^2$$

The kinetic energy theorem:

$$W_{A\to B}(\vec{P}) = \Delta E_C = E_C(B) - E_A(A)$$

$$\frac{1}{2}.m.v_B^2 = m.g.h \Rightarrow v_B = \sqrt{2.g.h}$$

The application of the theorem between position A and any intermediate position in M gives:

The weight work is written:

$$W_{AB}(\vec{P}) = m.g.(z_A - z) = m.g.(h - z).$$

La variation d'énergie cinétique s'écrit :

$$\Delta E_C = E_C(M) - E_C(A) = \frac{1}{2}.m.v^2$$

Kinetic energy theorem gives:

$$\frac{1}{2}mv^2 = mg(h - z) \Rightarrow v = \sqrt{2g(h - z)}$$

IV.5.3-Calculation using energy conservation:

The system is conservative so there is conservation of mechanical energy. Let us express this energy at the different points A, M and B:

Point A : $E(A) = E_C(A) + E_{PP}(A) = \frac{1}{2}mv_A^2 + mgz_A = mgh$

Point M : $E(M) = E_C(M) + E_{PP}(M) = \frac{1}{2}mv^2 + mgz$

Point B : $E(B) = E_C(B) + E_{PP}(B) = \frac{1}{2}mv_B^2 + mgz_B = \frac{1}{2}mv_B^2$

The conservation of mechanical energy is written as:

$$E(A) = E(M) = E(B) = E_0 = mgh = \frac{1}{2}mv^2 + mgz = \frac{1}{2}mv_B^2$$

We can deduce : $\frac{1}{2}mv_B^2 = mgh \Rightarrow v_B = \sqrt{2gh}$

Likewise : $mgh = \frac{1}{2}mv^2 + mgz \Rightarrow v = \sqrt{2g(h-z)}$

IV.6-Linked states and stability of an isolated mechanical system:

IV.6.1-related states:

When a system is conservative, its mechanical energy is conserved. We therefore have for such a system:

$$E = E_C + Ep = \text{constant} \qquad \text{(IV.39)}$$

By definition, kinetic energy being a necessarily positive quantity, we obtain a condition restricting the possible energy states of the system. This restriction condition defines the so-called linked states of the system.

These states are defined by:

$$E_C = \frac{1}{2}mv^2 > 0 \Rightarrow E - E_P > 0 \qquad \text{(IV.40)}$$

Let us take the example of a mass attached to a spring and whose elongation corresponds to the variable x (figure IV.6). The linked states of the system are then defined by:

$$E-\frac{1}{2}kx^2 \geq 0 \Rightarrow -\sqrt{\frac{2E}{k}} = -x_M \leq x \leq \sqrt{\frac{2E}{k}} = x_M$$

Values of x outside this interval are inaccessible to the system which is said to be enclosed in a potential well because of the form taken by the potential energy function.

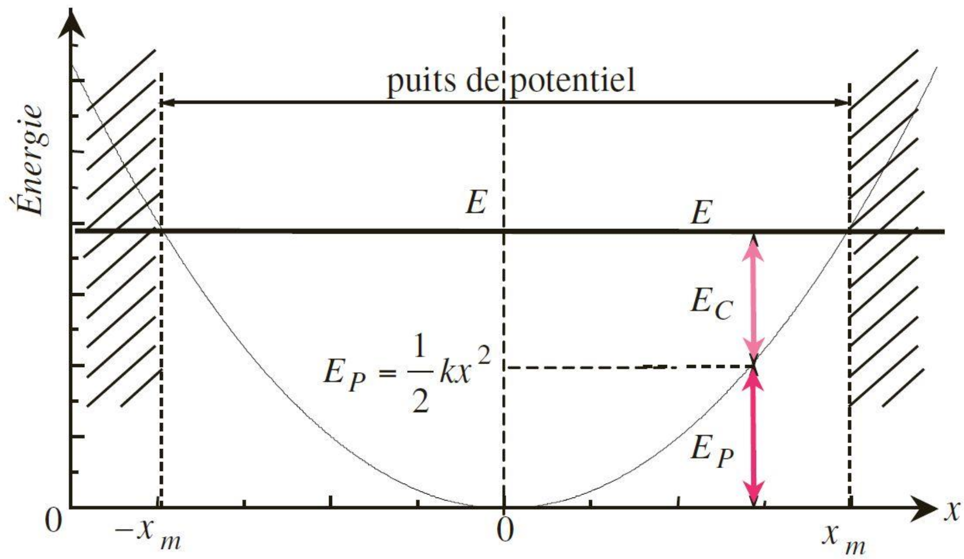


Figure IV.8 Graph of energies as a function of the elongation x of a spring.

IV.6.2-Stability of a system subjected to a conservative force:

For a system subjected only to a conservative resultant force, the local form of the potential energy allows us to write that:

$$\vec{F} = -\overrightarrow{grad}E_P$$

In the case where the potential energy only depends on a variable x, this amounts to saying that:

$$\vec{F} = -\frac{dE_p}{dx}\vec{i} \Rightarrow F = -\frac{dE_P}{dx}$$

The equilibrium condition resulting in $\vec{F} = \vec{0}$ can therefore also be written:

$$\frac{dE_P}{dx} = 0.$$

A position of equilibrium therefore results in an extremum of the potential energy function.

An equilibrium is stable if, following a disturbance which has moved the system away from this position, it returns there spontaneously. Otherwise the equilibrium is unstable.

• If there exists a stable equilibrium for x = xo, then Ep(x) is minimal for x = xo. So we have :

Stable equilibrium for x = $x_o \Leftrightarrow \frac{dE_P}{dx}(x_0) = 0$ and $\frac{d^2E_P}{dx^2}(x_0) > 0$

• If there exists an unstable equilibrium for x = xo, then Ep(x) is maximum for x =xo. So we have :

Stable equilibrium for x = $x_o \Leftrightarrow \frac{dE_P}{dx}(x_0) = 0$ and $\frac{d^2E_P}{dx^2}(x_0) < 0$

A system, left to its own devices, therefore evolves spontaneously towards a state of equilibrium which corresponds to a position for which the potential energy is minimal.

-Example of a marble on a bowl-shaped ground:

We consider the case of a ball that can roll on a ground with a bowl-shaped profile (see figure IV.9 (a)). The weight derives from the potential energy Ep = mgz with an axis Oz directed upwards and taking the origin of the potential energies at the bottom of the bowl enz = 0 (see figure IV.9 (b)).

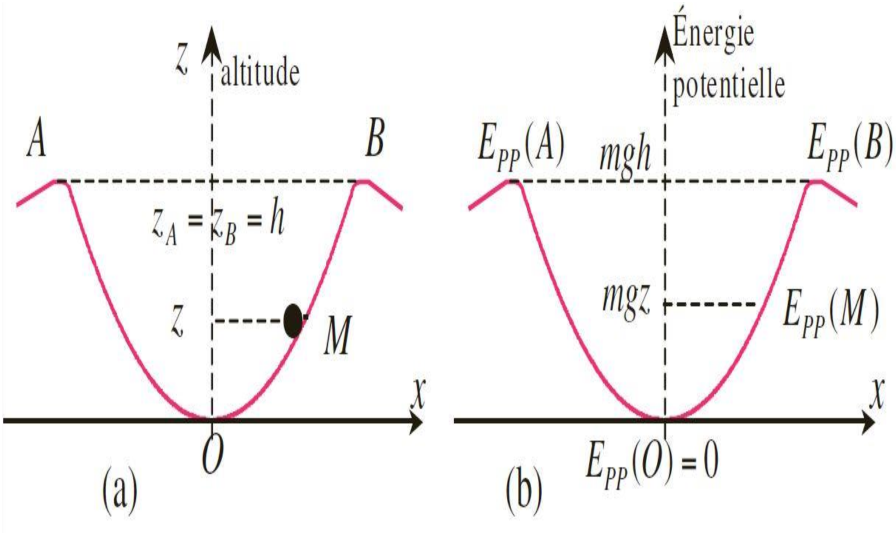
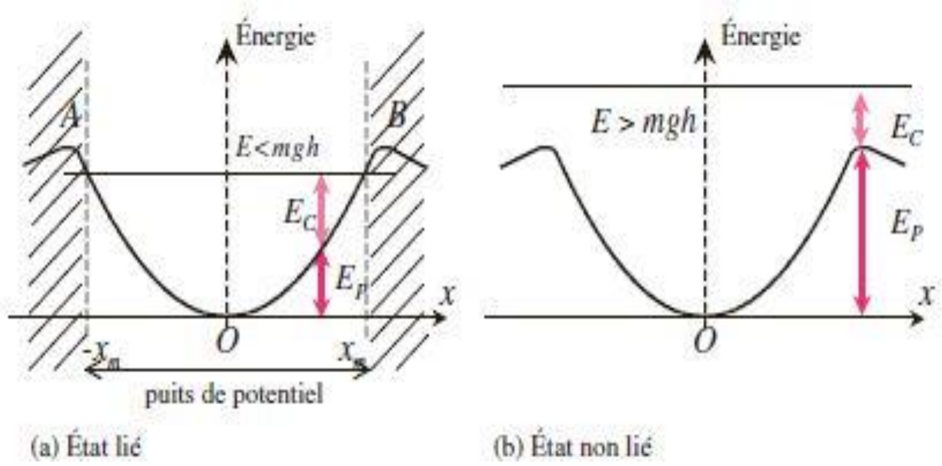


Figure IV.9 graph (a) represents the ground profile (altitude z) as a function of an abscissa x. Graph (b) represents the potential energy as a function of position.

The potential energy is minimal at the bottom of the bowl (lowest possible altitude). This position corresponds to the stable equilibrium position of the ball M. Positions A and B corresponding to maximum altitudes are unstable equilibrium positions. At the slightest disturbance directed towards the bowl, the ball placed in A or B will leave this position to move towards the stable equilibrium position O.



FigureIV.10 (a) Bound state (E <mhg) and (b) Unbound state (E >mhg).

If the mechanical energy of the ball is less than the maximum potential energy mgh (see figure IV.10 (a)), the ball is caught in a potential well. It can only evolve between –xmetxm values. If its mechanical energy is greater than the potential energy in A or B, it will be able to leave the bowl: the state is unbound (figure IV.10 (b)).

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