Ministry of Higher Education and Scientific Research

University of RELIZANE

Faculty of Science and Technology

Department of Physics

Mathematics

Analysis & Algebra

Courses and Corrected Exercises

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Intended for first year LMD-SM students

- First semester: 2024/2025 -

بسم الله الرحمن الرحيم

Foreword

This Mathematics I book is intended for first-year students in material sciences and science technologie. It has been designed to provide a rigorous and progressive introduction to the fundamental concepts

of analysis and algebra, emphasizing both theoretical understanding and the practical application of concepts through solved exercises.

The first part of the book is dedicated to analysis, where we introduce the concepts of ***set theory****,* ***the structure of the real number field****, and* ***real functions of a real variable****. We then cover essential notions such as limits, continuity, and reciprocal functions, which form the foundation of mathematical analysis.*

The second part focuses on algebra, exploring common algebraic structures, vector spaces, and linear applications. These notions are fundamental to many branches of mathematics and have direct applications in physics, engineering, and computer science.

To facilitate learning, each chapter includes a detailed explanation of concepts, followed by illustrative examples and a series of solved exercises. This approach allows students to test their understanding and progressively master the topics covered.

We hope that this book will be a valuable tool for students, helping them develop a solid mathematical intuition and providing them with the necessary foundations for the continuation of their scientific studies.

Course Content

Chapter 4: Limits of functions

Chapter 5: Continuous functions

Chapter 6: Reciprocal functions

Algebra section

Chapter 1: Common Algebraic Structures

Chapter 2: Vector spaces

Chapter 3: Linear Applications

ANALYSIS SECTION

Chapter 1:
Set theory

I. Definition

set theory is a branch of mathematics that deals with the properties of well-defined collections of objects, which may or may not be of a mathematical nature, such as numbers or functions. For example, a group of players in a football team is a set and the players in the team are its objects. Let's say we have a set of Natural Numbers then it will have all the natural numbers as its members and the collection of the numbers is well defined that they are natural numbers.

II. Notation

II.1. *Natural Whole Numbers*

A natural whole number is a whole number that is positive. The set of natural whole numbers is denoted by $\mathbb{N}$.

$\mathbb{N}= \{0 ;1;2 ;3;4...\}$.

Examples :

$4 \in \mathbb{N}$

$-2 \notin \mathbb{N}$

II.2. *Integers*

An integer is a whole number that is either positive or negative. The set of integers is denoted by $\mathbb{Z}$.

$\mathbb{Z} =\{...-3;-2 ;-1; 0 ;1; 2 ; 3...\}$.

Examples:

$-2 \in \mathbb{Z}$

$5 \in \mathbb{Z}$

$0,33 \notin \mathbb{Z}$

II.3. *Decimal Numbers*

A decimal number can be written with a finite number of digits after the decimal point. The set of decimal numbers is denoted by $\mathbb{D}$.

Examples:

$0,56 \in \mathbb{D}$

$3 \in \mathbb{D}$

$¼ \notin \mathbb{D}$ *mais* $3/4 \in \mathbb{D}$

II.4. Rational Numbers

A rational number can be written as a quotient a/b, where a is an integer and b is a non-zero integer. The set of rational numbers is denoted by ℚ.

Examples:

$1/3 \in \mathbb{Q}$

$4 \in \mathbb{Q}$

$-4{,}8 \in \mathbb{Q}$

$\sqrt{2} \notin \mathbb{Q}$

II.5. Real Numbers

The set of real numbers is denoted by ℝ. It is the set of all numbers that we will use in the second-year class.

Example :

2, 0, -5, 0.67,1/3, $\sqrt{3}$ ou π appartiennent à ℝ.

II.6. Empty Set

A set that contains no numbers is called the empty set and is denoted by Ø.

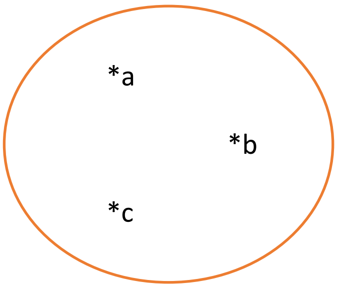
II.7. Exclusion Symbol

*The symbol * excludes the number 0 from a set. For example, ℝ*is the set of real numbers without 0.*

III. Representation of a Set

a) ***Graphical Representation:*** *The elements of the set are placed within a boundary enclosed by a closed curve.*

Example: This set A contains three elements: a, b, and c.



b) ***Enumeration:*** *All the elements of the finite set are listed within curly brackets {....}.*

Example: A = {a, b, c}

c) ***Set-builder Notation:*** *The set is defined based on the elements of another set E that satisfy a certain property P.*

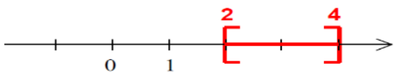
General form: A = {x ∈ E / P(x)}.

A contains all the elements x of E that satisfy the property P.

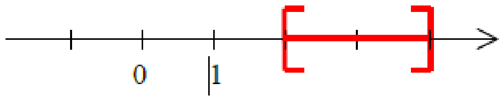
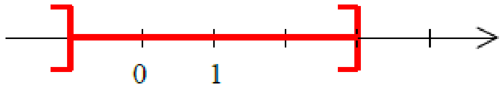
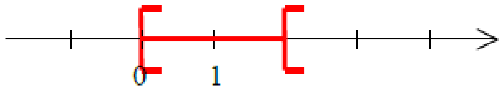
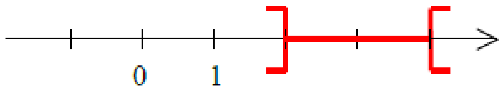
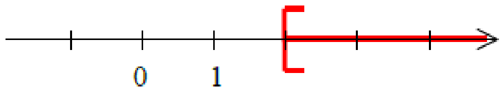
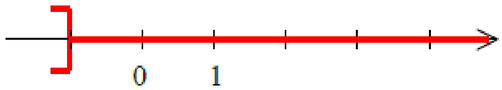
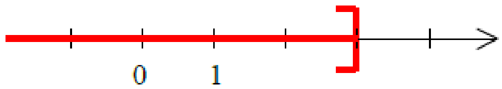
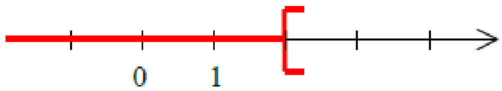
d) ***Intervals of a Set:***

The set of all real numbers x such that $2 \le x \le 4$ can be represented on a number line.

This set is called an interval and is denoted by : ***[2 ; 4]***



Example :

Nombres réels x	Notation	Représentation
$2 \le x \le 4$	[2 ; 4]	
$-1 < x \le 3$	] -1 ; 3]	
$0 \le x < 2$	[0 ; 2 [	
$2 < x < 4$	] 2 ; 4 [	
$x \ge 2$	[2 ; +∞ [∞ désigne l'infini	
$x > -1$	] -1 ; +∞ [	
$x \le 3$	] -∞ ; 3]	
$x < 2$	] -∞ ; 2 [	

IV. Operations on Sets

a) ***Equality:*** *Two sets A and B are equal if and only if they are composed of the same elements. In other words, if all the elements of A are also found in B and vice versa. We write: A= B* ***if and only if*** $\forall\, x, x \in A \leftrightarrow x \in B$

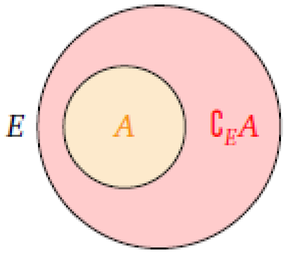
Example : if $A = \{0,1,2,3,4,5,6,7\}; B = \{x \in N / 0 \le x \le 7\}$ *so A=B*

b) ***Inclusion:*** E ⊂ F if every element of E is also an element of F. In other words: $\forall\, x, x \in E\ (x \in F)$, It is then said that E is a subset of F or a part of F..

c) *Set of Subsets of E: We denote P(E) as the set of subsets of E. For example, if* E = {1,2,3} : P({1,2,3}) = $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

d) ***Complement:*** *If A ⊂E,*

$$C_E A = \{x \in E / x \notin A\}$$

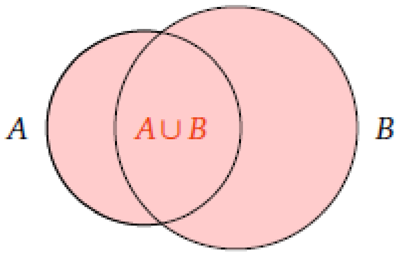


It is also denoted as E / A and simply as C_A if there is no ambiguity (and sometimes also as A^c or A').

e) **Union:** *For A, B⊂ E,*

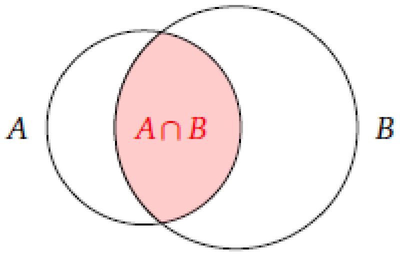
$$A \cup B = \{x \in E / x \in A \text{ or } x \in B\}$$

"The 'or' is not exclusive: x can belong to both A and B at the same time."



f) *Intersection :*

$$A \cap B = \{x \in E / x \in A \text{et } x \in B\}$$



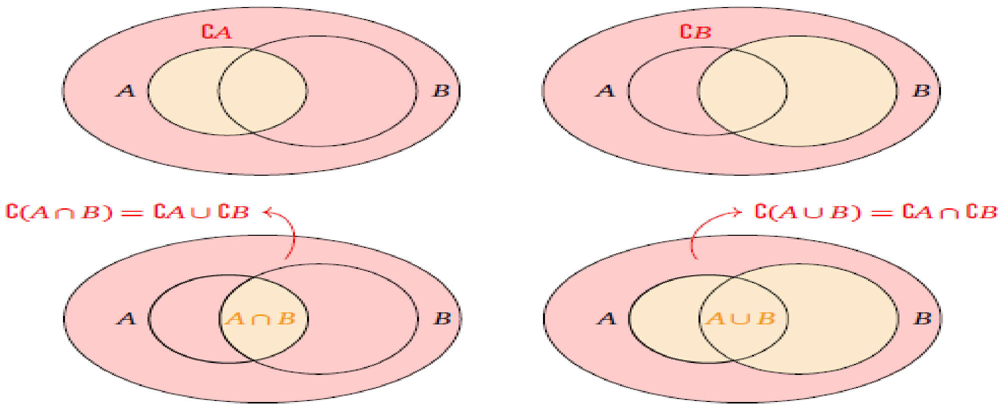
V. <u>*Calculations rules*</u>

Let A, B, and C be subsets of a set E

- $A \cap B = B \cap A$
- $A \cap (B \cap C) = (A \cap B) \cap C$ (We can therefore write $A \cap B \cap C$ without ambiguity)
- $A \cap \emptyset = \emptyset$, $A \cap A = A$, $A \subset B \leftrightarrow A \cap B = A$
- $A \cup B = B \cup A$
- $A \cup (B \cup C) = (A \cup B) \cup C$ (We can therefore write $A \cup B \cup C$ without ambiguity.)

- $A\cup\emptyset = A$, $A\cup A= A$, $A\subset B \leftrightarrow A\cup B = B$
- $A\cap (B\cup C) = (A\cap B)\cup(A\cap C)$
- $A\cup (B\cap C) = (A\cup B)\cap(A\cup C)$
- $C(C_A)$=A et donc $A\subset B\leftrightarrow C_B\subset C_A$
- $C_{(A\cap B)} = C_A\cup C_B$
- $C_{(A\cup B)} = C_A\cap C_B$

Here are the diagrams for the last two statements



VI. <u>Cartesian Product:</u>

Let E and F be two sets. The Cartesian product, denoted $E \times F$, is the set of pairs (x, y) where $x\in E$ and $y\in F$.

$$E\times F = \{ (x,y) \; ; \; x\in E \text{ and } y\in F\}$$

if *Card E = m and Card B =n than*

Card $E\times F$ = m.n

<u>Example</u>

Let A={ 1,2,3} et B ={0,1,2,3}

Write the sets $A\cap B$, $A\cup B$, $A\times B$.

<u>Note :</u> *$A\cap B$ = {1,2,3} ; ,$A\cup B$ = {0,1,2,3}*

<u>Noticed :</u>

As $A\subset B$ we have $A\cap B$ =A and $A\cup B$=B

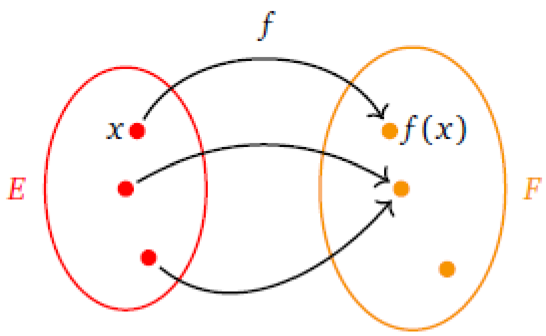
$A\times B$= {(1,0), (1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(2,3) ,(3,0),(3,1),(3,2),(3,3)}

Noticed Card ($A\times B$) = Card (A) × Card (B) = 3×4= 12

VII. Applications

VII.1. Definition :

*A function f: E→F is the assignment, for each element x∈E, of a unique element in F, denoted as **f(x)**. We will represent functions using two types of illustrations: the domain (and the codomain) will be schematized by an oval, and its elements by points. The mapping x→f(x) will be represented by an arrow.*



The other representation is for continuous functions from R to R (or subsets of R). The domain R is represented by the horizontal axis, and the codomain by the vertical axis. The mapping x→f(x) is represented by the point (x, f(x))

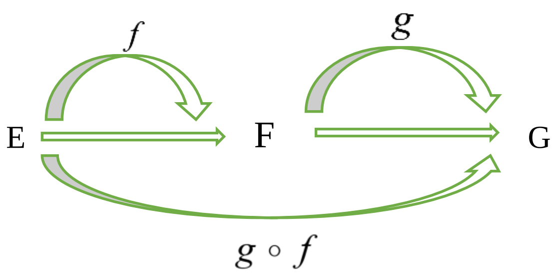
***Notation**: The set of functions from E to F is denoted by $\mathcal{F}$(E,F) ou F^E.*

- ***Equality of two functions**: Two functions f and g to be equal if:*

1. *f and g have the same domain E,*
2. *f and g have the same codomain F,*
3. *if and only if, for all x ∈ E , f(x) = g(x). We then write f = g.*

- ***Composition of two functions**: Let E, F, and G be three sets, f a function from E to F, and g a function from F to G. The composition of f by g, denoted by g ∘ f, is defined as follows:*

  f: E → F and g : F→G than g ∘ f : E →G** is the function defined by **g ∘ f(x)= g(f(x))



- ***Identity:***

Let E be a set. We call the identity map of E, and denote it by Id_E*, the application from E to E (**E →E**) defined by **x →x.***

For any application f from E to E, we have $f \circ Id_E = f$ ***and*** $Id_E \circ f = f$

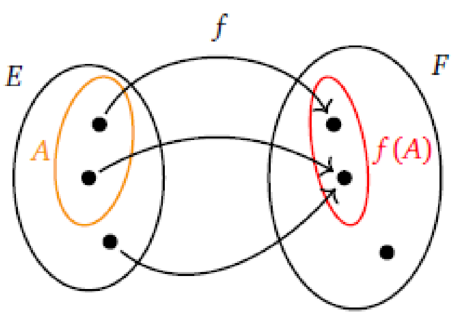
- ***Extensions of an application:*** *let E and F be two sets, f an application from E to F and G a set containing E. We call extension of f to G any application g from G to F such that, for all x ∈E, g(x) = f(x)*

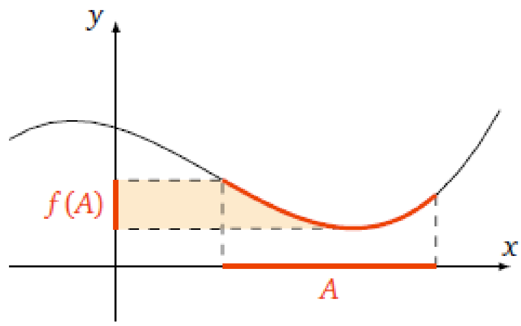
VII.2. Direct image and reciprocal image

1) Direct image

Let E and F be two sets, f an application of E in F and A a part of E. We call the direct image of A by f and we denote f(A) the subset of F defined by:

$$f(A) = \{ f(x), x \in A\}$$

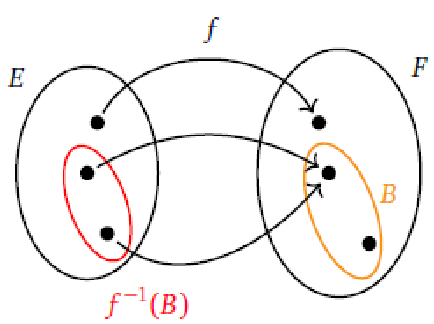


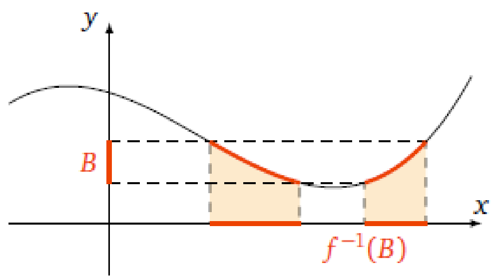


2) Reciprocal image

*Let E and F be two sets, f an application of E in F and B a subset of F. We call the **reciprocal image** of B by f and we denote by* $f^{-1}(B)$ *the subset of E defined by*

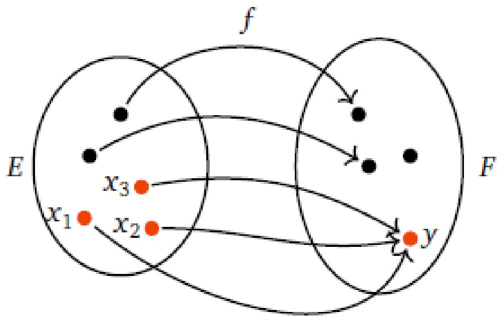
$$f^{-1}(B) = \{ x \in E , f(x) \in B\}$$

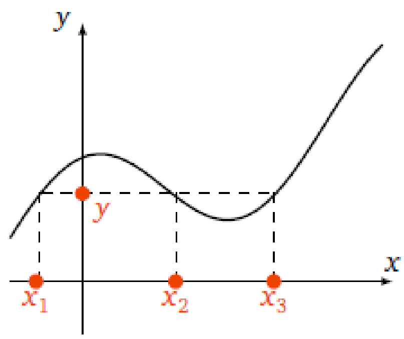




3) <u>Antecedent Image</u>

*Let $y \in F$. Any element $x \in E$ such that $f(x) = y$ is an **antecedent** of y. In terms of the inverse image the set of antecedents of y is $f^{-1}(\{y\})$.*

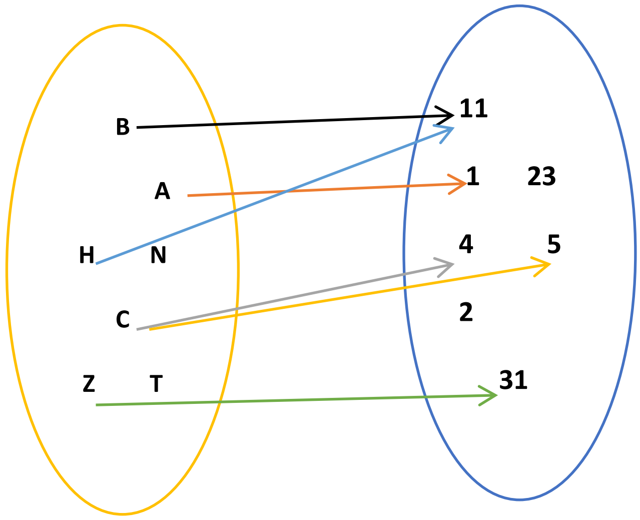




<u>Exercise :</u>

Find the image and antecedent of each element?

<u>Solution :</u>



<u>Image of an element</u>

- *A, B, H, Z each have one image*
- *C two images*
- *D, N, T have no image*

<u>Antecedent of an element</u>

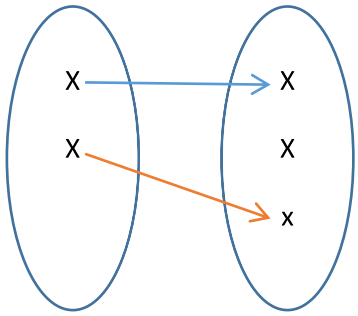
- *1, 4, 5, 31 each have one antecedent*
- *11 two antecedents*
- *23, 2 have no antecedent*

VIII. <u>Injective, surjective and bijective applications</u>

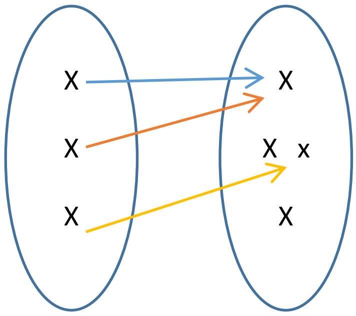
VIII.1. <u>Injective application:</u>

*Let f be an application of E into F. We say that f is injective when every element of F **has one or more** antecedents by f;*

$$\forall (x, x') \in E \times E, f(x) = f(x') \rightarrow x = x'$$



f is injective

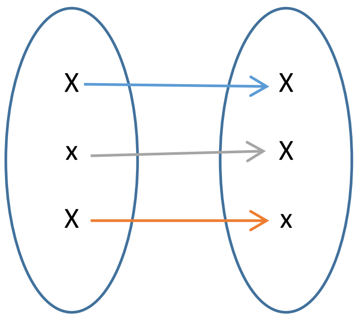


f is not injective

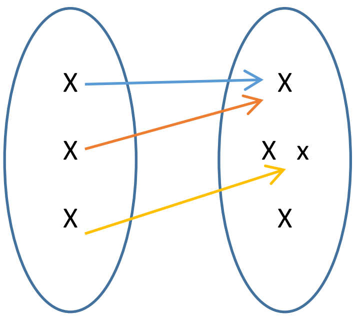
- ***Compound of two injections :*** *the compound of two injective applications is injective*

VIII.2. Surjective application

Let f be an application of E into F. We say that f is surjective when every element of F has at ***least one*** *antecedent by f; i.e.:*$\forall \boldsymbol{y} \in \boldsymbol{F}, \exists\, \boldsymbol{x} \in \boldsymbol{E}, \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$



f is surjective

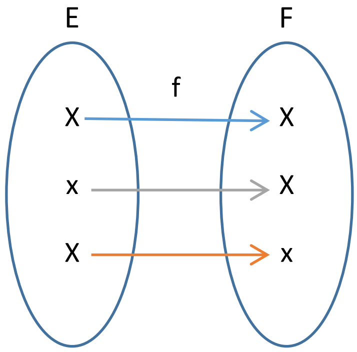


f is not surjective

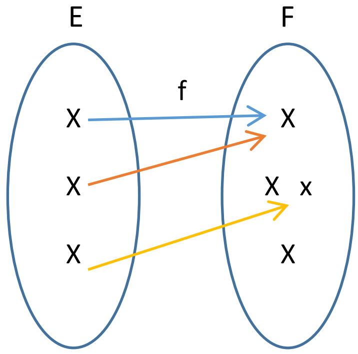
- ***Composed of two surjectives:*** *the composite of two surjective applications is surjective.*

VIII.3. Bijective application

Let f be an application of E into F. We say that f is ***bijective*** *when f is both injective and surjective. In other words, f is bijective when every element of F has* ***a unique*** *antecedent by f.*



f is bijective



f is not bijective

- ***Reciprocal application of a bijection:*** *let f be a bijection from E to F. We call the reciprocal application of f and we denote by* $\boldsymbol{f^{-1}}$ *the application of F to E which, to any element y of F, associates its unique antecedent by f. By definition, we have:*

$$\forall\ (\boldsymbol{x},\boldsymbol{y}) \in \boldsymbol{E}\ \times \boldsymbol{F}, \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})\ \leftrightarrow\ \boldsymbol{x} = \ \boldsymbol{f}^{-1}(\boldsymbol{y})$$

- ***Proposition:*** *let f be a bijection from E into F and* $\boldsymbol{f^{-1}}$ *: F →E its reciprocal application then :* $\boldsymbol{f \circ f^{-1} = Id_F}$ ***et*** $\boldsymbol{f^{-1} \circ f = Id_E}$

- ***Characterization of the reciprocal bijection:*** *an application f : E →F is a bijection if and only if there exists an application g : F →E verifying* $f \circ f^{-1} = Id_F$ *and et* $f^{-1} \circ f = Id_E$*. In this case, g is the application of f : g =* f^{-1}

- ***Composed of two bijections****: let f be an application from E to F and g an application from F to G. If f and g are bijective, then* $g \circ f$ *is a bijective from E to G and we have:* $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

<u>Example 1</u>

L'application f : R/ {- 2} → R defined by f(x) = $\frac{x-1}{x+2}$ *is it injective, surjective? What restriction must be made on the arrival set so that f becomes a bijection? In this case, explain the reciprocal application.*

<u>Solution :</u>

I. Injectivity:

A function is injective if different inputs give different outputs. In other words, if $f(x_1)=f(x_2)$*, then* $x_1=x_2$ *.*

To check injectivity, suppose $f(x_1)=f(x_2)$:

$$\frac{x_1-1}{x_1+2}=\frac{x_2-1}{x_2+2}$$

Cross-multiply to eliminate the denominators:

$$(x_1-1)(x_2+2)=(x_1+2)(x_2-1)$$

Expand both sides: $x_1x_2+2x_1-x_1-2=x_1x_2+2x_1-x_1-2$

Thus, $x_1=x_2$. Therefore, f(x) is injective.

II. Surjectivity

A function is surjective if every element of the codomain (here $\mathbb{R}$) has a preimage in the domain.

We want to check if for any $y\in\mathbb{R}$, $x\in\mathbb{R}\{-2\}$ such that f(x)=y:

$$y=\frac{x-1}{x+2}$$

Solve for x in terms of y:

$$y(x+2)=x-1$$

$$yx+2y=x-1$$

$$yx-x=-1-2y$$

$$x(y-1)=-1-2y$$

$$x=\frac{-1-2y}{y-1}, y\neq 1$$

So for any $y\neq 1$, there exists an $x\in R/\{-2\}$. However, y=1 does not have a preimage, because solving f(x)=1gives:

$$1=\frac{x-1}{x+2}$$

$$x+2=x-1$$

This leads to a contradiction, so f(x) is not surjective onto $\mathbb{R}$ because y=1 is not in the range.

III. Bijection:

To make f(x) bijective, we must restrict the codomain. From the surjectivity analysis, the only value that does not have a preimage is 1. Therefore, the function will be bijective if we restrict the codomain to $R\setminus\{1\}$.

Thus, the function f: $R\setminus\{-2\}\rightarrow R\setminus\{1\}$ is a bijection.

Inverse Function

Now, to find the inverse function $f^{-1}(y)$, *we solve for x in terms of y, as done previously:*

$$f(x) = y = \frac{x-1}{x+2}$$

Solving this equation, we found: $x = \dfrac{-1-2y}{y-1}$

Therefore, the inverse function is: $f^{-1}(y) = \dfrac{-1-2y}{y-1}, y \neq 1$

<u>Example 2 :</u>

Show that the application f is bijective and determine its reciprocal application.

$$f : \mathbb{R} \setminus \{3\} \to \mathbb{R} \setminus \{3\}$$
$$x \mapsto \frac{3x+1}{x-3}$$

<u>Solution :</u>

For f to be bijective we use the law directly:

$$\forall\, (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R} \setminus \{3\} \times \mathbb{R} \setminus \{3\}, \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}) \leftrightarrow \boldsymbol{x} = \boldsymbol{f}^{-1}(\boldsymbol{y})$$

$$y = f(x) \Rightarrow y = \frac{3x+1}{x-3}$$

$$\Rightarrow y(x-3) = 3x+1$$

$$\Rightarrow yx - 3y = 3x+1$$

$$\Rightarrow x(y-3) = 3y+1$$

$$\Rightarrow x = \frac{3y+1}{y-3}$$

The unique antecedent of y is $\dfrac{3y+1}{y-3}$ *, which is real distinct from 3.*

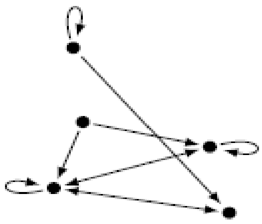
- *F is bijective if*

$$f^{-1}:\mathbb{R}\setminus\{3\}\rightarrow\mathbb{R}\setminus\{3\}$$
$$x\mapsto\frac{3x+1}{x-3}$$

IX. Equivalence relation

1. Definition

A relation on a set E is the data for any pair (x, y) 2 E × E of "True" (if they are related), or of "False" otherwise. We schematize a relation as follows: the elements of E are points, an arrow from x to y means that x is related to y, that is to say that we associate "True" with the pair (x,



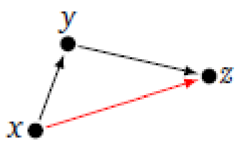
y).

let E be a set and R a relation, it is an equivalence relation if:

- $\forall x\in E, x\Re x$, (***reflexivity***)

- $\forall x,y\in E, x\Re y\Rightarrow y\Re x$ (***Symmetry***)

- $\forall (x,y)\in E^2, x\Re y\ ou\ y\Re x\Rightarrow x\Re z$ (***Transitivity***)



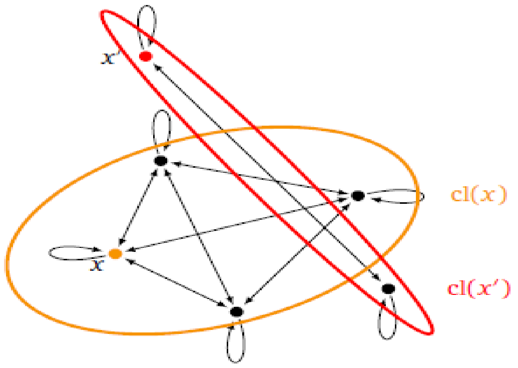
- $\forall x,y\in E, x\Re y\wedge y\Re x\Rightarrow x=y$ ***(Anti-symmetric)***

Definition: *An equivalence relation is a relation that is reflexive, symmetric and transitive.*

2. Equivalence classes

Let $\Re$ an equivalence relation on a set E. Let $x\in E$ the equivalence class of x is

$$cl(x) = \{x \in E \,/\, a\Re x\}$$



Remark:

- x *is a representation of the equivalence class* $\dot{x}$
- *We call the set quotient of E by* $\Re$ *the set of equivalence classes all the elements of E it is noted* $E/\Re$
- *Application* $E \to E/\Re$ *is called "canonical surjection"*

$$x \mapsto \dot{x}$$

Example:

In $\mathbb{R}$ *we define the relation* $\Re$ *by:* $\forall x, y \in E, x\Re y \Rightarrow x^2 - 1 = y^2 - 1$

- *Show that* $\Re$ *is an equivalence relation and give the quotient set* $\mathbb{R}/\Re$.

Solution :

1.To have an equivalence relation $\Re$, *it must satisfy:*

- ***Reflexivity***

$\forall x, y \in \mathbb{R}, x\Re x \Rightarrow x^2 - 1 = x^2 - 1$

which shows that $\Re$ *is reflexive.*

- ***Symmetry***

$$\begin{aligned}\forall x, y \in \mathbb{R}, x\Re y &\Rightarrow x^2 - 1 = y^2 - 1 \\ &\Rightarrow y^2 - 1 = x^2 - 1 \\ &\Rightarrow y\Re x\end{aligned}$$

which shows that $\Re$ *is symmetric.*

- ***Transitivity***

$$\forall(x,y)\in\mathbb{R}^2, x\Re y \text{ ou } y\Re x \Rightarrow (x^2-1=y^2-1)\wedge(y^2-1=z^2-1)$$
$$\Rightarrow \begin{cases} x^2-1=y^2-1 \\ y^2-1=z^2-1 \end{cases}$$
$$\Rightarrow x^2-1=z^2-1$$
$$\Rightarrow x\Re z$$

Therefore $\Re$ *is transitive.*

we deduce that $\Re$ *is an equivalence relation*

2. let $a\in\mathbb{R}$

$$cl(x)=\dot{a}=\{x\in\mathbb{R} / a\Re x\},$$
$$\dot{a}=\{x\in\mathbb{R} / a^2-1=x^2-1\},$$
$$=\{x\in\mathbb{R} / a^2-x^2=0\},$$
$$=\{x\in\mathbb{R} / (a+x)(a-x)=0\},$$
$$=\{x\in\mathbb{R} / x=a \ \text{ or } \ x=-a\},$$
$$=\{a,-a\},$$

So $\mathbb{R}/_{\Re}=\{a\in\mathbb{R} / \{a,-a\},$

X. Order relation:

A binary relation $\Re$ *on E is said to be an order relation if it is reflexive, transitive and antisymmetric.*

Remark :

- *The order is said to be **total** if it allows any two elements to be compared*
  $\forall(x,y)\in E^2, x\Re y \text{ ou } y\Re x$
- *The order is said to be **partial** otherwise*

Example *:*

Let $\Re$ *the relation be defined on* $\mathbb{N}^*$ *by the relation "x divided y" let us verify that it is antisymmetric.*

$$\left.\begin{array}{l} x\Re y \Rightarrow \exists k\in\mathbb{N}^*, y=kx \\ y\Re x \Rightarrow \exists k'\in\mathbb{N}^*, x=k'y \end{array}\right\} \Rightarrow kk'=1$$

as $k,k'\in\mathbb{N}^*$*, then*

$k=k'=1$ *i.e. x=y*

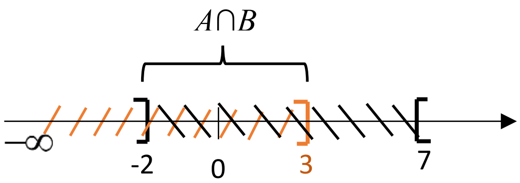
Applications

Exercise n°1

Let A =]-∞; 3], B =] - 2; 7[and C =] - 5;+ ∞ [three subsets of R.

- *Determine* $A\cap B$, $A\cup B$, $B\cap C$, $B\cup C$, A^c, $A\setminus B$, $A^c\cap B^c$, $(A\cup B)^c$, $(A\cap B)\cup(A\cap C)$ *and* $A\cap(B\cup C)$.

Solution



- $A\cap B =]-2,3]$
- $A\bigcup B =]-\infty, 7[$
- $B\bigcap C =]-2, 7[$
- $B\bigcup C =]-5, +\infty[$
- $A^C =]3, +\infty[$
- $A\setminus B =]-\infty, -2[$
- $A^C \bigcap B^C =]7, +\infty[$
- $(A\bigcup B)^C =]7, +\infty[$
- $(A\bigcap B)\bigcup(A\bigcap C) =]-5, 3]$
- $A\bigcap(B\bigcup C) =]-5, 3]$

Exercise n°2

Let f, g and h three function difined by : $f(x) = \dfrac{1}{x^2-4}$, $g(x) = x^3 + 3x^2 + 5$, $h(x) = \sqrt{4-2x}$.

- *Determine the sets* $D_f, D_g, D_h, C_{D_f}, C_{D_g}, C_{D_h}, D_f \bigcup D_h, D_g \setminus D_h, D_g \bigcap D_h$.

Solution

- $D_f : x^2 - 1 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$ *so* $D_f =]-\infty, -2[\bigcup]-2, 2[\bigcup]2, +\infty[$
- $D_g = \mathbb{R} =]-\infty, +\infty[$
- $D_h : 4 - 2x \geq 0 \Rightarrow -2x \geq -4 \Rightarrow x \leq 2$ *so* $D_h =]-\infty, 2]$

- $C_{D_f} = \{-2, 2\}$
- $C_{D_g} = \varnothing$
- $C_{D_h} =]2, +\infty[$
- $D_f \bigcup D_h = \mathbb{R}$
- $D_g \setminus D_h =]2, +\infty[$
- $D_g \bigcap D_h =]-\infty, 2]$

Exercise n°3

True or False? For every subset A, B, and C of E, we have

i) $[(A \cap B) \cup C] \cap B = B \cap (A \cup C)$,

ii) $C \cap [(A \cap B \cap C^c) \cup (A^c \cap B)] = A^c \cap B \cap C$,

iii) $A \cup (B \cap C) = (A \cup B) \cap C$.

2) if $A \cap B = A \cup B$ *What can we say about sets A and B?*

Solution

i) $[(A \bigcap B) \bigcup C] \bigcap B = (A \bigcap B \bigcap B) \bigcup (C \bigcap B)$

$$= (B \bigcap A) \bigcup (B \bigcap C)$$
$$= B \bigcap (A \bigcup C)$$ ***true***

ii) $C \cap [(A \cap B \cap C^c) \bigcup (A^c \cap B)] = (C \cap A \cap B \cap C^c) \bigcup (C \bigcap A^c \bigcap B)$

Therefore: $C \cap A \cap B \cap C^c = \varnothing$

So: $C \cap [(A \cap B \cap C^c) \bigcup (A^c \cap B)] = C \bigcap A^c \bigcap B = A^c \bigcap B \bigcap C$ ***true***

iii) $A \cup (B \cap C) = (A \cup B) \cap C$. ***False***

we can take for example the following subsets of $\mathbb{N}$ **:** $A = \{0\}, B = \{0\}$ *and* $C = \{1\}$

so $A \bigcup (B \bigcap C) = \{0\}$ *and* $(A \bigcup B) \bigcap C = \varnothing$

2) if $A \cap B = A \cup B$ *we say that* $A = B$

Exercise n°3

Determine the complement in $\mathbb{R}$ of the following parts:

$A_1=]-\infty,0]$; $A_2=]-\infty,0[$; $A_3=]0,+\infty[$; $A_4=[0,+\infty[$; $A_5=]1,2[$; $A_6=[1,2[$.

2. Let $A=]-\infty,1[\cup]2,+\infty[$, $B=]-\infty,1[$ and $C=]2,+\infty[$ .Compare the following sets:

$$C_{\mathbb{R}}A \text{ et } C_{\mathbb{R}}B\cap C_{\mathbb{R}}C$$

Solution

- $C_{\mathbb{R}}A_1=]0,+\infty[$
- $C_{\mathbb{R}}A_2=[0,+\infty[$
- $C_{\mathbb{R}}A_3=]0,+\infty]$
- $C_{\mathbb{R}}A_4=]-\infty,0[$
- $C_{\mathbb{R}}A_5=]-\infty,1]\cup[2,+\infty[$
- $C_{\mathbb{R}}A_6=]-\infty,1[\cup[2,+\infty[$

2. $C_{\mathbb{R}}A=[1,2]$ and $C_{\mathbb{R}}B\cap C_{\mathbb{R}}C=[1,+\infty[\cap]2,+\infty[=[1,2]$

So :

$C_{\mathbb{R}}B\cap C_{\mathbb{R}}C=C_{\mathbb{R}}A$

Exercise n°4

Let $f : R\to R$ and $g : R\to R$ be such that $f(x) = 3x+1$ and $g(x) = x^2-1$.

Do we have $f \circ g = g \circ f$?

Solution

We have $f(x) = 3x+1$ and $g(x) = x^2-1$

- $f\circ g(x)=f(g(x))=f(x^2-1)=3(x^2-1)+1=3x^2-2$
- $g\circ f(x)=g(f(x))=g(3x+1)=(3x+1)^2-9=9x^2+6x$

So : $f\circ g(x)\neq g\circ f(x)$

Exercise n°5

Let f : [0;1] → [0;1] such as:

$$F(x) = \begin{cases} x & si\ x \in [0,1] \cap \mathbb{Q} \\ 1 - x & Otherwise \end{cases}$$

Demonstrate that: $f \circ f = id$.

Solution

- *If* $x \in \mathbb{Q}\ ,\ f(x) = x \Rightarrow f(f(x)) = x$
- *If* $x \notin \mathbb{Q},\ f(x) = 1 - x \Rightarrow f(f(x)) = f(1-x)$
  $= 1 - (1 - x) = x$

So : $\forall x \in [0,1],\ f \circ f = id_f$

Exercice n°6

Let $f : [1;+\infty[\to [0;+\infty[$ *such as* $f(x) = x^2 - 1$. *f is it bijective?*

Solution

$$f : [1;+\infty[\to [0;+\infty[$$
$$x \mapsto x^2 - 1$$

f is bijective $\Rightarrow y = f(x)$

$$y = f(x) \Leftrightarrow y = x^2 - 1$$
$$\Leftrightarrow x^2 = y + 1$$
$$\Leftrightarrow x = \pm\sqrt{y+1} \qquad (x \text{ is positive } x \in [1;+\infty[)$$

$$f^{-1} : [0,+\infty[\to [1,+\infty[$$
$$y \mapsto \sqrt{y+1}$$

T verify:

$$\forall y \in [0,+\infty[: f \circ f^{-1} = y \qquad (f \circ g = id)$$
$$\forall x \in [1,+\infty[: f^{-1} \circ f = x \qquad (g \circ f = id)$$

So f is bijective

Exercise n°7

Are the following applications injective, surjective, bijective?

1. $f : \mathbb{N} \to \mathbb{N}$; $n \mapsto n+1$
2. $g : \mathbb{Z} \to \mathbb{Z}$; $n \mapsto n+1$
3. $h : \mathbb{R}^2 \to \mathbb{R}^2$; $(x,y) \mapsto (x+y, x-y)$
4. $k : \mathbb{R} \setminus \{1\} \to \mathbb{R}$; $x \mapsto \frac{x+1}{x-1}$

Solution

1. $\boldsymbol{f : \mathbb{N} \to \mathbb{N};}$ $\boldsymbol{n \mapsto n+1}$

- ***injective*** $\Leftrightarrow \forall n, n' \in \mathbb{N},\ f(n) = f(n') \Rightarrow n = n'$

$$\Rightarrow n+1 = n'+1$$
$$\Rightarrow n = n'$$
$$\Rightarrow f \text{ is surjective}$$

- *f is not surjective because 0 has no antecedent,* $\exists n \in \mathbb{N},\ f(n) = 0 \Rightarrow n = -1 \notin \mathbb{N}$.
- *f is not bijective because is not surjective*

2. $\boldsymbol{g : \mathbb{Z} \to \mathbb{Z};}$ $\boldsymbol{n \mapsto n+1}$

- ***g is injective, surjective and bijective because a change in domain and codomain***

3. $\boldsymbol{h : \mathbb{R}^2 \to \mathbb{R}^2;}$ $\boldsymbol{(x,y) \mapsto (x+y, x-y)}$

- ***injective*** $\Leftrightarrow \forall (x,y),(x',y') \in \mathbb{R}^2,\ f(x,y) = f(x',y') \Rightarrow \begin{cases} x = x' \\ y = y' \end{cases}$

$$f(x,y) = f(x',y') \Rightarrow (x+y, x-y) = (x'+y', x'-y')$$

$$\Rightarrow \begin{cases} x+y = x'+y' \\ x-y = x'-y' \end{cases}$$

$$\Rightarrow \begin{cases} 2x = 2x' \\ 2y = 2y' \end{cases} \Rightarrow (x,y) = (x',y')$$

f is injective

- ***f surjective*** $\Rightarrow \forall (X,Y) \in \mathbb{R}^2, h(x,y) = (X,Y)$

$$h(x,y)=(X,Y)\Rightarrow(x+y,x-y)=(X,Y)$$
$$\Rightarrow\begin{cases}x+y=X\\x-y=Y\end{cases}$$
$$\Rightarrow\begin{cases}2x=X+Y\\2y=X-Y\end{cases}\Rightarrow\begin{cases}x=\dfrac{X+Y}{2}\\y=\dfrac{X-Y}{2}\end{cases}$$

h has a unique antecedent

so h is surjective and id bijective

4. $\boldsymbol{k:\mathbb{R}\setminus\{1\}\to\mathbb{R};\quad x\mapsto\frac{x+1}{x-1}}$

- ***injective*** $\Leftrightarrow\forall(x,x'\in\mathbb{R}-\{1\},k(x)=k(x')\Rightarrow x=x'$

$$k(x)=k(x')\Rightarrow\frac{x+1}{x-1}=\frac{x'+1}{x'-1}$$
$$\Rightarrow(x+1)(x'-1)=(x'+1)(x-1)$$
$$\Rightarrow xx'+x'-x=xx'-x'+x$$
$$\Rightarrow 2x=2x'$$
$$\Rightarrow x=x'$$

K is injective

- ***surjective*** $\Rightarrow\forall x\in\mathbb{R}-\{1\},k(x)=y$

$$k(x)=y\Rightarrow\frac{x+1}{x-1}=y$$
$$\Rightarrow x+1=y(x-1)$$
$$\Rightarrow x=\frac{y+1}{y-1}$$

But if y=1 we cannot find an antecedent x

So, k is not surjective

Exercise n°8

We consider the application f of $\mathbb{R}^2$*in* $\mathbb{R}^2$ *defined by :*

f (x,y) = (x-4y, 2x+3y)

- *Show that f is bijective.*

Solution

For f to be bijective we use the law directly:

$\forall (x,y),(x',y') \in \mathbb{R}^2, f(x,y)=(x',y')$

$f(x,y)=(x',y') \Rightarrow (x-4y,\ 2x+3y)=(x',y')$

$$\Rightarrow \begin{cases} x-4y=x' \\ 2x+3y=y' \end{cases}$$

$$\Rightarrow \begin{cases} x=\dfrac{3x'+4y'}{11} \\ y=\dfrac{y'-2x}{11} \end{cases}$$

F is bijective because he have unique antecedent

$$f^{-1}:\mathbb{R}^2 \to \mathbb{R}^2$$
$$\text{x} \mapsto (\frac{3x+4y}{11},\frac{y-2x}{11})$$

Exercise n°9

We consider the following application:

$$f:\mathbb{R} \to \mathbb{R}$$
$$x \mapsto f(x)=\frac{2x}{x^2+1}$$

1. *Let's calculate* $f(\{2\}),\ f(\left\{\frac{1}{2}\right\}),\ f^{-1}(\{2\})$
2. *Let's show that f is not injective and not surjective.*
3. *Determine the reciprocal function in [-1,1] of f.*

Solution

1.

$$f(\{2\})=\left\{\frac{4}{5}\right\},\ f(\left\{\frac{1}{2}\right\})=\left\{\frac{4}{5}\right\}$$

$$\begin{aligned} f^{-1}(\{2\})=?\Rightarrow\ & f(x)=y \\ \Rightarrow\ & f(x)=2 \\ \Rightarrow\ & \frac{2x}{x^2+1}=2 \\ \Rightarrow\ & x^2-x+1=0 \quad \textit{has no solution} \end{aligned}$$

$$so: f^{-1}(\{2\})=\varnothing$$

2. *f is not injective because* $f(\{2\})\,f(\left\{\frac{1}{2}\right\})=\frac{4}{5}$

f is not surjective because y=2 not have solution

3.*To find the inverse, we set y=f(x) and solve for x:*

$$y=\frac{2x}{x^2+1}\Rightarrow y(x^2+1)=2x$$
$$\Rightarrow yx^2-2x+y=0$$

so :

$$\Delta=4-4y^2$$

$$x=\pm\frac{1+\sqrt{1-y^2}}{y}$$

On [−1,1], f(x) is one-to-one and symmetric about x=0. Therefore, select the positive branch:

$$[-1,1]\mapsto[-1,1]$$
$$y\mapsto\frac{1+\sqrt{1-y^2}}{y}$$

Exercise n°10

$$f:\mathbb{R}^+\to[1,+\infty[$$

$$x\to f_m(m)=(m-1)^2x^2+1$$

1) *Discuss, based on the values of m, the injectivity of* f_m.
2) *Discuss, based on the values of m, the surjectivity of* f_m.
3) *Deduce the values of m for which fm is bijective and determine, in this case, the inverse application* f_m^{-1}.

Solution

$$f:\mathbb{R}^+ \to [1,+\infty[$$
$$x \to f_m(m)=(m-1)^2x^2+1$$

1°/ f_m *inj* $\Leftrightarrow \forall x, x' \in \mathbb{R}^+ \qquad f_m(x)=f_m(x') \Rightarrow x=x'$

Let $x, x' \in \mathbb{R}^+,\ f_m(x)=f_m(x') \Rightarrow (m-1)^2+1=(m-1)^2x'^2+1$

$$\Rightarrow (m-1)^2(x^2-x'^2)=0$$
$$\Rightarrow (m-1)(x-x')(x+x')=0$$

i) $m=1 \quad :^* \Leftrightarrow 0=0$ *Then f is not injective*

(Or Remark: For $m=1,\ f_1(x)=1$ *(a constant function):not injective* $\forall x, x'\ f(x)=f(x')=1$

ii) $m\neq 1 :^* \Rightarrow (x-x')(x+x')=0 \Rightarrow x=x'$ *ou* $\underbrace{x=-x'}_{impossible}\ x, x' \in \mathbb{R}$

$$\Rightarrow x=x' \quad f \text{ is injective}$$

$m=1 \quad f$ *is not injective*

$m\neq 1 \quad f$ *is injective*

2°/Déf : f_m *is surjictive* $\Leftrightarrow \forall y \in [1,+\infty[, \exists x \in \mathbb{R}^+, f_m(x)=y$

let $y \in [1,+\infty[$ *équation* $y=f_m(x)$ *(The unknown variable x)*

$$y=f_m(x) \Leftrightarrow y=(m-1)^2x^2+1 \Leftrightarrow y-1=(m-1)^2x^2$$

a) if $(m-1)^2 \neq 0$ *i.e if* $m\neq 1$ *then :* $(y-1)=(m-1)x^2 \Leftrightarrow x^2=\dfrac{y-1}{(m-1)^2} \quad (y \succ 0)$

i.e $x=\pm\sqrt{\dfrac{y-1}{(m-1)^2}}$, $x \in \mathbb{R}^+$ *we have*

$$y=(m-1)x^2+1 \Leftrightarrow x=\sqrt{\frac{y-1}{(m-1)^2}} \geq 0 \quad , \left(x=\frac{\sqrt{y-1}}{(m-1)} \right)$$

Case where $m \neq 1$; $\forall y \in [1,+\infty[, \exists x=\sqrt{\frac{y-1}{(m-1)^2}} \in \mathbb{R}, y=f_m(x)$

i.e if $m \neq 1$ *thene* f_m *is surjective*

b)if $m=1$, $\forall y \in]1,+\infty[, \forall x \in \mathbb{R}^+, y \neq f_1(x)$ *(because* $f_1(x)=1$*)*

if $m=1$ *therfore* f_1 *is not surjective*

Summary of the second question :

If $m=1$ *:* f *is not injective ,not surjective And thus,* f *is not bijective.*

If $m \neq 1$ *:* f *is injective and surjective , thus* f *is bijective.*

$m \neq 1$: f_m^{-1}? $\qquad f_m^{-1}$: $\begin{array}{l} [1,+\infty[\rightarrow \mathbb{R}^+ \\ y \rightarrow x \end{array}$

$$f_m^{-1}(y)=x \Leftrightarrow y=f_m(x)$$

i.e ;

$$f_m^{-1}:[1,+\infty[\rightarrow \mathbb{R}^+$$
$$y \rightarrow f_m^{-1}(y)=\sqrt{\frac{y-1}{(m-1)^2}}$$

$$f_m^{-1}:[1,+\infty[\rightarrow \mathbb{R}^+$$
$$x \rightarrow f_m^{-1}(x)=\sqrt{\frac{x-1}{(m-1)^2}}$$

Exercise n°11

Let $f : \mathbb{R}-\left\{-\frac{3}{2}\right\} \rightarrow \mathbb{R}$ be defined by $f(x)=2-\frac{8-x}{4x+6}$.

⋆ *Is f injective? Surjective? Justify your answer.*

⋆ *What restriction must be made on the codomain for* f *to become a bijection? In that case, give the inverse function of* f *.*

Solution

f *is injective* $\Leftrightarrow \forall x_1, x_2 \in \mathbb{R}-\left\{-\frac{3}{2}\right\},\ f(x_1)=f(x_2) \Rightarrow x_1=x_2$?

$$f(x_1)=f(x_2) \Rightarrow 2-\frac{8-x_1}{4x_1+6}=2-\frac{8-x_2}{4x_2+6}$$
$$\Rightarrow \frac{8-x_1}{4x_1+6}=\frac{8-x_2}{4x_2+6}$$
$$\Rightarrow (8-x_1)(4x_2+6)=(8-x_2)(4x_1+6)$$
$$\Rightarrow 32x_2+48-4x_1x_2-6x_1=32x_1+48-4x_1x_2-6x_2$$
$$\Rightarrow 32x_1+6x_1=32x_2+6x_2$$
$$\Rightarrow 38x_1=38x_2 \Rightarrow x_1=x_2$$

Thus f *is injective.*

f *is surjective* $\Leftrightarrow \forall y \in \mathbb{R}, \exists x \in \mathbb{R}-\left\{-\frac{3}{2}\right\}, y=f(x)$.

Let $y \in \mathbb{R}$ *:*

$$y=f(x) \Rightarrow y=2-\frac{8-x}{4x+6} \Rightarrow y-2=-\frac{8-x}{4x+6}$$
$$\Rightarrow 8-x=(2-y)(4x+6)$$
$$\Rightarrow 8-x=8x+12-4xy-6y$$
$$\Rightarrow 8-9x+4xy=12-6y$$
$$\Rightarrow 4xy-9x=4-6y$$
$$\Rightarrow x=\frac{4-6y}{4y-9} \quad \text{,if} \quad 4y-9 \neq 0$$

$y=\frac{9}{4}$ *has no preimage ,so* f *is not surjective.*

$x=\frac{4-6y}{4y-9} \in \mathbb{R}-\left\{-\frac{3}{2}\right\}$, *because* $\frac{4-6y}{4y-9}=-\frac{3}{2} \Leftrightarrow 27=8$ *wichi s impossible.*

**for* f *to be bijective,it is necessary that the codomain be* $\mathbb{R}-\left\{\frac{9}{4}\right\}$

**the inverse application of* f *is :*

$$f^{-1}:\mathbb{R}-\left\{\frac{9}{4}\right\}\rightarrow\mathbb{R}-\left\{-\frac{3}{2}\right\}$$

$$x\rightarrow f^{-1}(x)=\frac{4-6x}{4x-9}$$

Exercise n°12

Say whether the following relations are reflexive, symmetric and transitive :

1. $E=\mathbb{Z}$ and $x\Re y \Leftrightarrow x=-y$
2. $E=\mathbb{R}$ and $x\Re y \Leftrightarrow \cos^2 x+\sin^2 y=1$
3. $E=\mathbb{R}$ and $x\Re y \Leftrightarrow xe^y=ye^x$
4. $\forall x,y\in\mathbb{R},\ x\Re y \Leftrightarrow x-y=x^2-y^2$

Which of the above examples are the order relations and the equivalence relations?

Solution

2. *We show that* $\Re$ *is reflexive, symmetric and transitive.*

- ***Reflexivity***

This means the condition must hold when y=x :

$\forall x,y\in\mathbb{R}, x\Re x\Rightarrow \cos^2 x+\sin^2 x=1$

This is true for all x, as it follows from the trigonometric identity.

which shows that $\Re$ *is reflexive.*

- ***Symmetry***

If $\cos^2 x+\sin^2 y=1$*,does* $\cos^2 y+\sin^2 x=1$ *also hold?*

Since the equation involves separate functions of x and y, there is no inherent guarantee that $\cos^2 y+\sin^2 x=1$ *will hold just because* $\cos^2 x+\sin^2 y=1$

Counterexample:

Take x=0,y=$\frac{\pi}{2}$ *:*

$$\cos^2(0)+\sin^2(\frac{\pi}{2})=1$$

but

$$\cos^2(\frac{\pi}{2})+\sin^2(0)=0\neq 1$$

which shows that $\Re$ *is not symmetric.*

- ***Transitivity***

$$\forall (x,y)\in \mathbb{R}^2, x\Re y \text{ ou } y\Re z \Rightarrow (\cos^2 x+\sin^2 y=1)\wedge(\cos^2 y+\sin^2 z=1)$$

$$\Rightarrow \begin{cases} \cos^2 x+\sin^2 y=1...............(1) \\ \cos^2 y+\sin^2 z=1...............(2) \end{cases}$$

there is no guarantee that $\cos^2 x+\sin^2 z=1$

Counterexample

Take $x=0, y=\dfrac{\pi}{4}, z=\dfrac{\pi}{2}$.

$$\Rightarrow \begin{cases} \cos^2 0+\sin^2(\dfrac{\pi}{4})=1 \\ \cos^2(\dfrac{\pi}{4})+\sin^2(\dfrac{\pi}{2})=1 \end{cases}$$

But

$\cos^2(0)+\sin^2(\dfrac{\pi}{2})=1$ $\quad$ *(fails due to intermediate relationship not guaranteed)*

Therefore $\Re$ *is not transitive.*

$\Re$ *is reflexive, no symmetric and no transitive then it is not an equivalence relation.*

3. We show that $\Re$ *is reflexive, symmetric and transitive.*

- ***Reflexivity***

$$\forall x,y\in\mathbb{R}, x\Re x \Rightarrow xe^x = xe^x$$

which shows that $\Re$ *is reflexive.*

- ***Symmetry***

$$\forall x,y\in\mathbb{R}, x\Re y \Rightarrow xe^y = ye^x$$
$$\Rightarrow ye^x = xe^y$$
$$\Rightarrow y\Re x$$

which shows that $\Re$ *is symmetric.*

- ***Transitivity***

$$\forall (x,y)\in \mathbb{R}^2, x\Re y \text{ ou } y\Re z \Rightarrow (xe^y = ye^x)\wedge(ye^z = ze^y)$$

$$\Rightarrow \begin{cases} xe^y = ye^x...............(1) \\ ye^z = ze^y...............(2) \end{cases}$$

$$(1) \Rightarrow y = \frac{xe^y}{e^x}, \quad e^y = \frac{ye^x}{x}$$

$$(2) \Rightarrow \frac{xe^y}{e^x} e^z = z\frac{ye^x}{x}, \text{since } e^x \neq 0 \text{ Thus}$$

$$\Rightarrow xe^z(\frac{e^y}{e^x}) = ze^x(\frac{y}{x})$$

$$\Rightarrow xe^z(xe^y) = ze^x(ye^x)$$

$$\Rightarrow xe^z = ze^x$$

$$\Rightarrow x\Re z$$

Therefore $\Re$ *is transitive.*

$\Re$ *is reflexive, symmetric and transitive then it is an equivalence relation.*

4.We show that $\Re$ *is reflexive, symmetric and transitive.*

- ***Reflexivity***

$\forall x, y \in \mathbb{R}, x\Re x \Rightarrow x - x = x^2 - x^2$

which shows that $\Re$ *is reflexive.*

- ***Symmetry***

$$\forall x, y \in \mathbb{R}, x\Re y \Rightarrow x - y = x^2 - y^2$$
$$\Rightarrow y - x = y^2 - x^2$$
$$\Rightarrow y\Re x$$

which shows that $\Re$ *is symmetric.*

- ***Transitivity***

$$\forall (x, y) \in \mathbb{R}^2, x\Re y \text{ ou } y\Re z \Rightarrow (x - y = x^2 - y^2) \wedge (y - z = y^2 - z^2)$$

$$\Rightarrow \begin{cases} x - y = x^2 - y^2 \text{...............}(1) \\ y - z = y^2 - z^2 \text{...............}(2) \end{cases}$$

$$(1)\text{-}(2) \Rightarrow x - z)x^2 - z^2$$
$$\Rightarrow x\Re z$$

Therefore $\Re$ *is transitive.*

$\Re$ *is reflexive, symmetric and transitive then it is an equivalence relation.*

Exercise n°13

Let $\alpha \in \mathbb{R}$ *and* $\Re$ *be a binary relation defined on* $\mathbb{R}$ *by :*

$$\forall x, y \in \mathbb{R} \quad x\Re y \Leftrightarrow x^3 - y^3 = \alpha(x^2 - y^2)$$

1. Show that $\Re$ *is an equivalence relation on* $\mathbb{R}$.

2. Let $\alpha = 7$. *Determine the equivalence class of 6.*

II· Let Φ *be the relation defined on* $\mathbb{N}^*$ $a\Phi b \Leftrightarrow \exists n \in \mathbb{N}$ *such that* $a^n = b$.

- Show that Φ *is a order relation on* $\mathbb{N}^*$.

− Is the order total?

Solution

$$\forall x, y \in \mathbb{R} \quad x\Re y \Leftrightarrow x^3 - y^3 = \alpha(x^2 - y^2)$$

- $\Re$ *is Reflexive* $\Leftrightarrow \forall x \in \mathbb{R}, x\Re x$?

a) Let $x \in \mathbb{R}$ *we have*

$$x^3 - x^3 = \alpha\left(x^2 - x^2\right) \Rightarrow x\Re x$$

Then $\Re$ *is Reflexive*

- $\Re$ *is symmetrique* $\forall x, y \in \mathbb{R}, x\Re y \Rightarrow y\Re x$?

b)Let $x, y \in \mathbb{R}$

$$\begin{aligned} x\Re y &\Leftrightarrow x^3 - y^3 = \alpha\left(x^2 - y^2\right) \\ &\Leftrightarrow y^3 - x^3 = \alpha\left(y^2 - x^2\right) \\ &\Leftrightarrow y\Re x \end{aligned}$$

Then $\Re$ *is symmetric.*

- $\Re$ *is Transitive* $\Leftrightarrow \forall x, y, z \in \mathbb{R}, (x\Re y) \text{ and } (y\Re z) \Rightarrow x\Re z$?

c)Let $x, y, z \in \mathbb{R}$ *such that* $\begin{cases} x\Re y \\ and \\ y\Re z \end{cases} \Leftrightarrow \begin{cases} x^3 - y^3 = \alpha\left(x^2 - y^2\right) \\ y^3 - z^3 = \alpha\left(y^2 - z^2\right) \end{cases} \Rightarrow x^3 - z^3 = \alpha\left(x^2 - z^2\right)$

$\Rightarrow x\Re z$ *Therefore* $\Re$ *is transitive.*

Conclusion *:* $\Re$ *is an equivalence relation in* $\mathbb{R}$.

2*.the equivalence class of 6*

$$\begin{aligned}\dot{6} &= \{x \in \mathbb{R} : x\Re 6\} \\ &= \{x \in \mathbb{R} : x^3 - 6^3 = 7(x^2 - 6)\} \\ &= \{x \in \mathbb{R} : x^3 - 7x^2 + 36 = 0\} \\ &= \{x \in \mathbb{R} : (x-6)(x^2 - x - 6) = 0\} \\ &= \{6, 3, -2\}\end{aligned}$$

II. $a\Phi b \Leftrightarrow \exists n \in \mathbb{N}$ *such that* $a^n = b$

(1) Show that Φ *is an ordre relation in* $\mathbb{N}^*$

a) Φ *is it reflexive ?*

Φ *is reflexive* $\Leftrightarrow \forall a \in \mathbb{N}^*, a\Phi a$ *?*

$\forall a \in \mathbb{N}^* \Rightarrow \exists n = 1 \in \mathbb{N}$ *such that :* $a^1 = a \Rightarrow a\Phi a \Rightarrow \Phi$ *is reflexive*

b) Φ *is it antisymetric* $\Leftrightarrow \forall a, b \in \mathbb{N}^*, a\Phi b$ *and* $b\Phi a \Rightarrow a = b$ *?*

Let $a, b \in \mathbb{N}^*$, *if* $a\Phi b$ *and* $b\Phi a \Rightarrow \exists n_1 \in \mathbb{N}$ *such that :* $a^{n_1} = b$

Et $\exists n_2 \in \mathbb{N}$ *such that :*

$$\begin{aligned}& b^{n_2} = a \Rightarrow \left(b^{n_2}\right)^{n_1} = a^{n_1} = b \\ & \Rightarrow n_1 n_2 = 1 \Rightarrow n_1 = n_2 = 1 \\ & \Rightarrow a = b \Rightarrow \Phi \quad is \text{ antisymmetric.}\end{aligned}$$

c) Φ *is transitive ?*

Φ *is transitive* $\Leftrightarrow \forall a, b, c \in \mathbb{N}^*, a\Phi b$ *and* $b\Phi c \Rightarrow a\Phi c$

Let $a, b, c \in \mathbb{N}^*$,

$a\Phi b$ *and* $b\Phi c \Rightarrow \exists n_1 \in \mathbb{N}$ *such that :* $a^{n_1} = b$

And $\exists n_2 \in \mathbb{N}$ *such that :* $b^{n_2} = c \Rightarrow (a^{n_1})^{n_2} = c$

$\exists n = n_1 n_2 \in \mathbb{N}$ such that $: a^n = c$

$\Rightarrow a\Phi c$

$\Rightarrow \Phi$ *is transitive*

(2) *Is this ordere total ?*

The order is not total because for the two natural integers $\{2,3\}$ *we have neither* $2\Phi 3$ *neither* $3\Phi 2$

Exercise n°14

find the domain of definition for the following inequalities in $\mathbb{R}$ *:*

1. $(2x-3)(5-7x)<0$ $\quad D_f = \left]-\infty, \frac{5}{7}\right[\cup \left]\frac{3}{2}, +\infty\right[$
2. $(8x-2)^2 > 9x^2$ $\quad D_f = \left]-\infty, \frac{2}{11}\right[\cup \left]\frac{2}{5}, +\infty\right[$
3. $\sqrt{5-7x}$ $\quad D_f = \left[\frac{5}{7}, +\infty\right[$
4. $9x^2 - 6x > -1$ $\quad D_f = \mathbb{R} \setminus \left\{\frac{1}{3}\right\}$
5. $\frac{9}{4(x-1)(x+2)} \geq -1$ $\quad D_f = \mathbb{R} \setminus \{1, -2\}$
6. $\frac{5x^2 - x + 2}{\sqrt{-2x^2 + 5x - 3}}$ $\quad D_f = \left]1, \frac{3}{2}\right[$

Chapter 2:

Structure of the real number field in $\mathbb{R}$

I. Field of real numbers

There is an ensemble, called field of real numbers and denoted, many of 2 internal operations (operations) internes

+ and **×**

$"+" : \mathbb{R} \to \mathbb{R}$ $\quad$ $"\times" : \mathbb{R} \to \mathbb{R}$

$(x, y) \mapsto x + y$ $\quad$ $(x, y) \mapsto x \times y$

1. Addition

The addition of real numbers has the following properties :

- *Commutativity: for all real numbers x and y ;* $x + y = y + x$.
- *Associativity: for all real numbers x,y and z :* $(x + y) + z = x + (y + z)$
- *0 is the neutral element: for any real number x :* $x + 0 = 0 + x$
- *Every real number has an opposite :* $x + (-x) = (-x) + x = 0$

By stating that ($\mathbb{R}$ *,+)is a* commutative group.

2. Multiplication

Multiplication of real numbers has the following properties:

- *Commutativity: for any real number x and y ;* $x \times y = y \times x$.
- *Associativity: for any real number x, y and z :* $(x \times y) \times z = x \times (y \times z)$
- *1 is the neutral element: for any real number x :* $x \times 1 = 1 \times x = x$
- *Every real number has an opposite :* $x \times \frac{1}{x} = \frac{1}{x} \times x = 1$

By stating that ($\mathbb{R}$ *,+,×) is a* commutative group.

3. Order relation

Comparing real numbers has the following properties:

- *Reflexivity : for any real number x :* $x \leq x$
- *Antisymmetry : for any real number x and y :* $(x \leq y \text{ and } y \leq x) \Rightarrow x = y$
- *Transitivity : for any real number x, y and z :* $(x \leq y \text{ and } y \leq z) \Rightarrow x \leq z$

By stating that ≤ that is an order relation

- *On a* $\forall x, y \in \mathbb{R}, x \leq y \text{ or } y \leq x \Rightarrow y \leq x$ *: we say that the order is* total

Moreover the order is compatible *with :*

- *The addition :* $\forall x, y, z \in \mathbb{R}^3, x \leq y \Rightarrow x+z \leq y+z$
- *Multiplication by positive real numbers:*
  $\forall (x, y) \in \mathbb{R}^2, \forall z \in \mathbb{R}_+ : x \leq y \Rightarrow xz \leq yz$

By stating that $(\mathbb{R}, +, \times, \leq)$ *is a totally ordered commutative body*

V. The upper bound property

a. Majorant and minorant

Let A a part of $\mathbb{R}$ *:*

- *A real number M is an upper bound of A when:* $\forall x \in A, x \leqslant M$.
- *A real number m* $\in R$ *is a lower bound of A when:* $\forall x \in A, m \leqslant x$.
- *A is an increased part when there is an increased part of A*
- *A is a minor part when there exists a lower bound of A*
- *A is a bounded part when it is both upper and lower bound.*

b. Largest and smallest element

Let A a part of E.

- *A has a greatest element if there exists an element of A which is also an upper bound of A.*
- *Such an element is necessarily unique and is called the largest element of A. We denote it max(A).*
- *A has a smallest element if there exists an element of A which is also a lower bound of A. Such an element is necessarily unique and is called the smallest element of A. We denote it by min(A).*

c. Upper and lower bound

Let A a part of $\mathbb{R}$ *:*

- *We say that A has an upper bound when the set of all majorants of A has a smallest element. In this case, we call the smallest element of the set of all majorants of A the upper bound of A, which we denote by sup(A).*
- *We say that A has a lower bound when the set of lower bounds of A has a greatest element. In this case, we call the lower bound of A, which we denote by inf(A), the greatest element of the set of lower bounds of A.*

VI. Absolute value

Definition: Given a real x, we call the absolute value of x, which we denote by $|x|$, the positive real defined by:

$$|x| = \begin{cases} x & si\ x \geq 0 \\ -x & si\ x \prec 0 \end{cases}$$

Proposal :

- $\forall x \in \mathbb{R}, |x| = 0 \Rightarrow x = 0$
- $\forall (x,y) \in \mathbb{R}^2, |x \times y| = |x| \times |y|$
- $\forall (x,y) \in \mathbb{R}^2, |x + y| \leq |x| + |y|$ *(First triangular inequality)*
- $\forall (x,y) \in \mathbb{R}^2, \left||x| - |y|\right| \leq |x - y| \leq |x| + |y|$ *(Second triangular inequality)*
- $\forall (x,y) \in \mathbb{R}^2, \left|\frac{x}{y}\right| = \frac{|x|}{|y|}, (y \neq 0)$
- $\forall x \in \mathbb{R}, \sqrt{x^2} = |x|$

VII. Intervals of $\mathbb{R}$

A part I of is an interval if as soon as it contains two real numbers, it contains all the intermediate real numbers: $\forall (c,d) \in I^2, \forall x \in \mathbb{R}, (c \leq x \leq d \Rightarrow x \in I)$

We can have:

- $I = \{x \in \mathbb{R}, a \leq x \leq b\} = [a,b]$ *Bounded closed interval or segment*
- $I = \{x \in \mathbb{R}, a \leq x \prec b\} = [a,b[$ *Right-hand half-open bounded interval*
- $I = \{x \in \mathbb{R}, a \prec x \leq b\} =]a,b]$ *Left-half-open bounded interval*
- $I = \{x \in \mathbb{R}, a < x < b\} =]a,b[$ *Open bounded interval*
- $I = \{x \in \mathbb{R}, x \geq a\} = [a,+\infty[$ *Closed interval not increased*
- $I = \{x \in \mathbb{R}, x \succ a\} =]a,+\infty[$ *Open interval not increased*
- $I = \{x \in \mathbb{R}, x \leq b\} =]-\infty,b]$ *Closed interval not reduced*
- $I = \{x \in \mathbb{R}, x < b\} =]-\infty,b[$ *Open interval not reduced*
- $I = \mathbb{R} =]-\infty,+\infty[$ *Interval neither reduced nor increased*

***Theorem** (density of rationals): any non-empty interval not reduced to a singleton contains at least one rational, i.e.* $\forall (a,b) \in \mathbb{R}^2, (a < b), \exists r \in / a \prec r \prec b$

VIII. Principle of recurrence

Let $P(n)$ *be a property depending on* $n \in \mathbb{N}$ *, and* $n_0 \in \mathbb{N}$ *.we assume that:*

- *The property* $P(n_0)$ *is true*
- *For any integer* $n \geq n_0$ *,* $P(n)$ *implies* $P(n+1)$ *.*
- *Pour tout entier, implique.*

So the property $P(n)$ *is true for any integer* $n \geq n_0$

IX. Principle of strong recurrence (or recurrence with predecessors)

Let $P(n)$ *be a property depending on* $n \in \mathbb{N}$ *, and* $n_0 \in \mathbb{N}$ *.we assume that:*

- *The property* $P(n_0)$ *is true*
- *For any integer* $n \geq n_0, (P(n_0) \text{ and } P(n_0+1) \text{ and } P(n))$ *implies* $P(n+1)$

So, the property $P(n)$ *is true for any integer* $n \geq n_0$

APPLICATIONS

Exercise 1

Find upper bound, lower bound, min, max, inf and sup of $\mathbb{R}$ *parts:*

$$A_1=[0,1],\ A_2=[0.2[,\ A_3=]0,1[,\ A_4=[1,+\infty[,\ A_5=]-\infty,+\infty[,\ A_6=\left\{\frac{1}{n}-1, n\in\mathbb{N}^*\right\}$$

Solution

We note that M : upper bound

m : lower bound

1. $A_1=[0,1]$

$M_1=[1,+\infty[$, $\quad SupA_1=1$

$m_1=]-\infty,0]$, $\quad \inf A_1=0$

$maxA_1=1$, $\quad \min A_1=0$

2. $A_2=[0.2[$

$M_2=[2,+\infty[$, $\quad SupA_1=2$

$m_2=]-\infty,0]$, $\quad \inf A_2=0$

$maxA_2=\nexists$, $\quad \min A_2=0$

3. $A_3=]0,1[$

$M_3=[1,+\infty[$, $\quad SupA_3=1$

$m_3=]-\infty,0]$, $\quad \inf A_3=0$

$maxA_3=\nexists$, $\quad \min A_3=\nexists$

4. $A_4=[1,+\infty[$

$M_4=\nexists$, $\quad SupA_4=\nexists$

$m_4=]-\infty,1]$, $\quad \inf A_4=1$

$maxA_4=\nexists$, $\quad \min A_4=1$

5. $A_5=]-\infty,+\infty[$

$M_5=\nexists$, $\quad SupA_5=\nexists$

$m_5=\nexists$, $\quad \inf A_5=\nexists$

$maxA_5=\nexists$, $\quad \min A_5=\nexists$

6. $A_6=\left\{\frac{1}{n}-1, n\in\mathbb{N}^*\right\}$

$M_6 = [0, +\infty[$, $SupA_6 = 0$

$m_6 =]-\infty, -1]$, $\inf A_6 = -1$

$maxA_6 = 0$, $\min A_6 = -1$

Exercise 2

Are the following functions bounded below? bounded above? If so, determine a lower bound and/or an upper bound.

1. $f : \mathbb{R} \to \mathbb{R}$ *such as* $f(x) = -2\sin(x^2 + 2x - 1)$
2. $g : \mathbb{R} \to \mathbb{R}$ *such as* $g(x) = \dfrac{1}{1+x^2}$
3. $h : \mathbb{R} \to \mathbb{R}$ *such as* $h(x) = \dfrac{2}{3-\cos x}$
4. $f :]0,5] \to \mathbb{R}$ *such as* $k(x) = \sqrt{x} - \dfrac{1}{1+x}$

Solution

1. *We know that the sine function is bounded on* $\mathbb{R}$ *by:* $-1 \leq \sin(x) \leq 1$

   By multiplying by -2, we obtain: $-2 \leq \sin(x^2 + 2x - 1) \leq 2$

   Thus, f is bounded above by : $M = 2$

   bounded below by : $m = -2$.

2. $x^2 \succ 0$, *So* $1 + x^2 \geq 1$ *which implies:*

$$0 < \frac{1}{1+x^2} \leq 1$$

The maximum value is reached in $x = 0$, *where* $g(0) = 1$

When $x \to \pm\infty$, *we have* $g(x) \to 0$

Thus, g is bounded above by : $M = 1$

bounded below by : $m = 0$.

3. *We know that the sine function is bounded on* $\mathbb{R}$ *by:* $-1 \leq \cos(x) \leq 1$

$$3 - \cos x \in [2, 4]$$

Thus, h(x) is bounded by: $$\frac{2}{4} \leq h(x) \leq \frac{2}{2}$$

$$\frac{1}{2} \leq h(x) \leq 1$$

Thus, g is bounded above by : $M = 1$

bounded below by : $m = \frac{1}{2}$.

4. *Let's study the limits at the endpoints of the interval* $]0,5]$.

- *On the right in x=5:* $k(5) = \frac{\sqrt{5}-1}{6} \approx 0.206$
- *On the left as x→0⁺:* $k(x) = \frac{\sqrt{x}-1}{1+x} \approx \frac{-1}{1} \approx -1$

Thus, g is bounded above by : $M = 0.206$

bounded below by : $m = -1$.

Chapter 3 :

Real Functions of a Real Variable

I. The domain of definition

A real function of a real variable is any application f defined on a subset D of $\mathbb{R}$ with values in $\mathbb{R}$. D is called the domain of definition of f and is denoted by D_f .

$$D_f = \{x \in \mathbb{R} \ : f(x) \text{ exists } \}$$

Example

$$f(x) = \frac{1}{x}, D_f = \{x \in \mathbb{R}, x \neq 0\} = \mathbb{R}^*$$

$$g(x) = \sqrt{x}, D_g = \{x \in \mathbb{R}, x \geq 0\} = [a, +\infty[$$

$$h(x) = \frac{x+1}{1-e^{\frac{1}{x}}}, D_h = \mathbb{R} \setminus \{0\}$$

$$k(x) = e^{\frac{1}{1-x}}\sqrt{x^2-1}, D_k =]-\infty, -1] \cup]1, +\infty[$$

$$l(x) = (1+\ln(x))^{\frac{1}{x}}, D_l = \left]\frac{1}{e}, +\infty\right[$$

II. Definition (Graph of a function)

In a plane associated with a coordinate system (O;i,j) (usually orthonormal), the points M(x;f(x)) with $x \in D_f$ form the representative curve of f, denoted C_f.

$$C_f = \{M(x; f(x)) : x \in D_f\}$$

We call the graph of f the set of pairs (x; f(x)) where $x \in D_f$

III. Composition of functions

Three generic cases: Let P(x) and Q(x) be two functions

1st case: *function of the type $f(x) = \frac{P}{Q}$* — *f is defined for all $Q \neq 0$*

2nd case: *function of the type $f(x) = \sqrt{Q}$* — *f is defined for all $Q \geq 0$*

3rd case: *function of the type $f(x) = \frac{P}{\sqrt{Q}}$* — *f is defined for all $Q > 0$*

IV. Periodic functions

A function f is said periodic and p is its period if T is the smallest positive real number that satisfies if:

$$\exists p \in \mathbb{R}^* : f(x+T) = f(x), \forall x \in D_f$$

Example: For all $x \in \mathbb{R}$ *and all* $k \in \mathbb{Z}$ *we have:* $\cos(x+2k\pi)=\cos x$

Whene : $T=2\pi$ *is the period of the function cos(x) defined on* $\mathbb{R}$

Remark :

Si f est de période T alors ∀ x ∈ N : (x + nT))∈ Df ; f(x + nT) = f(x)

V. ***Even functions***

A function f is called an even function if :

$$\forall x \in D_f : f(-x)=f(x)$$

Example : $f(x)=\cos x$ *is even because we have* $f(-x)=\cos(-x)=\cos x = f(x)$

$f(x)=\dfrac{x^2-1}{x^2+1}$ *is even because we have* $f(-x)=\dfrac{(-x)^2-1}{(x)^2+1}=\dfrac{x^2-1}{x^2+1}=f(x)$

VI. ***Odd functions***

the function f is odd If: $\forall x \in D_f , f(-x)=-f(x)$

Example : $g(x)=\sin x$ *is odd because we have:* $\sin(-x)=\sin x$

VII. ***Bounded functions***

The function is said to be bounded if:

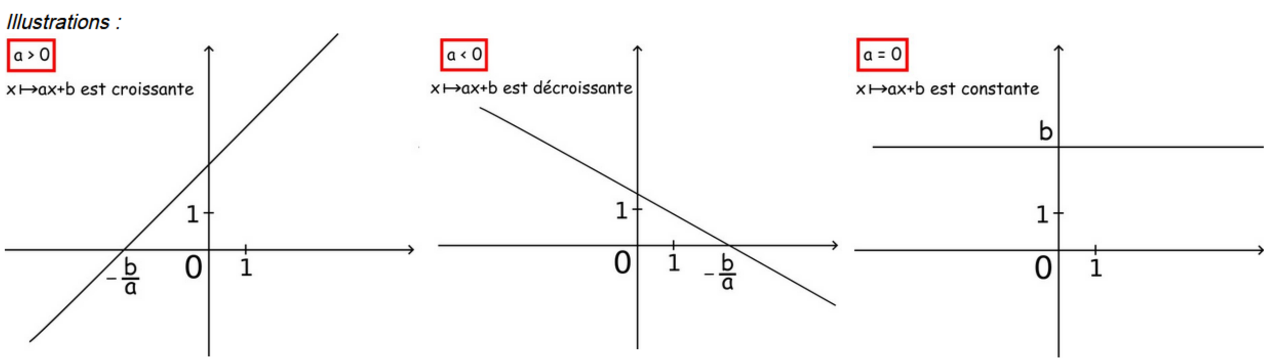
$$\exists a, A \in \mathbb{R} : a \le f(x) \le A \forall x \in D_f$$

As: $-1 \le \cos x \le 1$ *then the function cos(x) is bounded.*

VIII. ***Direction of variation of functions***

Let f defined by : ***f(x) = ax + b****.*

- *if* ***a>0****, then f is increasing on* $\mathbb{R}$*.*
- *if* ***a<0****, then f is decreasing on* $\mathbb{R}$*.*
- *if* ***a=0****, then f is constant on* $\mathbb{R}$*.*



In general:

The function f defined on D is said to be:

- ***Increasing if****:* $\forall (x,x') \in D^2, (x \le x' \Rightarrow f(x) \le f(x'))$

- ***Decreasing if***: $\forall (x,x') \in D^2, (x \le x' \Rightarrow f(x) \ge f(x'))$
- ***Strictly increasing if:*** $\forall (x,x') \in D^2, (x < x' \Rightarrow f(x) < f(x'))$
- ***Strictly decreasing if***: $\forall (x,x') \in D^2, (x < x' \Rightarrow f(x) > f(x'))$
- ***Monotone if:*** *it is increasing or decreasing*
- ***Strictly monotonic if:*** *it is strictly increasing or strictly decreasing*
- ***Increased if***: $\exists M \in \mathbb{R}, \forall x \in D, f(x) \le M$
- ***Reduced if***: $\exists m \in \mathbb{R}, \forall x \in D, f(x) \ge m$.

IX. Operation on functions

Let f, g be two functions defined on D *:*

- ***An addition:*** $\forall x \in D : (f+g)(x) = f(x) + g(x)$
- ***A multiplication***: $\forall x \in D : (f.g)(x) = f(x).g(x)$
- ***An external multiplication by real numbers***: $\forall x \in D : (\alpha.f)(x) = \alpha.f(x)$
- ***An order relationship:*** $f \le g \Leftrightarrow \forall x \in D; f(x) \le g(x)$

Applications

Exercise n°1

Given the functions: $f(x)=\sqrt{x-1}, g(x)=\dfrac{2}{x-3}$

Find the domain of the function $(f\circ g)(x)$.

Solution

We need to determine when g(x)is in the domain of f(x).

1. **Domain of g(x):** $g(x)=\dfrac{2}{x-3}$ *is undefined at x=3.*

   $D_g=\mathbb{R}\setminus\{3\}$

2. **Domain of** $(f\circ g)(x)$ **:** $f(g(x))=\sqrt{g(x)-1}=\sqrt{\dfrac{2}{x-3}-1}$

We require:

$$\frac{2}{x-3}-1\geq 0\Rightarrow \frac{x-5}{x-3}\geq 0$$
$$\Rightarrow x\leq 5$$

Solve the inequality:

- *The expression* $\dfrac{x-5}{x-3}\Rightarrow x\leq 5$
- $D_g=\mathbb{R}\setminus\{3\}$

Solution: $3<x\leq 5$

So, the domain of $(f\circ g)(x)$ *is* $]3,5]$

Exercise n°2

Let $f(x)=\sqrt{x}$ *and* $g(x)=\ln(x)$. *Find the domain of* $(f\circ g)(x)$ *and* $(g\circ f)(x)$

Solution

1. $(f\circ g)(x)=\sqrt{\ln(x)}$

   Domain: $\ln(x)\geq 0\Rightarrow x\geq 1$

2. $(g\circ f)(x)=\ln(\sqrt{x})=\dfrac{1}{2}\ln(x)$

   Domain: $x>0$

Exercise n°3

Find the period of the function : $f(x)=\sin(3x)+\cos(5x)$

Solution

1. *The period of sin(3x) is:* $T_1 = \frac{2\pi}{3}$
2. *The period of cos(5x) is:* $T_2 = \frac{2\pi}{5}$
3. *The period of f(x) is the* ***least common multiple (LCM)*** *of* T_1 *and* T_2*:*

$$T = lcm(\frac{2\pi}{3}, \frac{2\pi}{5})$$

4. *The LCM of denomination (3 and 5) is 15.Thus, the LCM of the period is :*

$$T = \frac{2\pi \times 15}{15} = 2\pi$$

Chapter 4:
Limits of functions

I. Limits

Let f a function y = f (x) defined on an interval I containing the point x_0. We say that f has the limit at this point x_0 of the real number L if:

$$\forall \varepsilon > 0, \forall n > 0 \text{ such as } \forall x \in I : 0 < |x - x0| < n \rightarrow |f(x) - l| < \varepsilon$$

We note: $\lim_{x \to x0} f(x) = l$

Note: *If a function has a limit, that limit is unique.*

II. Right limit, left limit

- *We say that f has a right limit at if the restriction of f to* $]x_0, +\infty[\cap I$ *admits a limit in* x_0

Notation : $\lim_{x \to x_0^+} f(x)$ *or* $\lim_{x \xrightarrow{>} x_0} f(x)$

- *We say that f has a left limit if the restriction of f to* $]-\infty, x_0[\cap I$ *admits a limit in* x_0

Notation : $\lim_{x \to x_0^-} f(x)$ *ou* $\lim_{x \xrightarrow{<} x_0} f(x)$

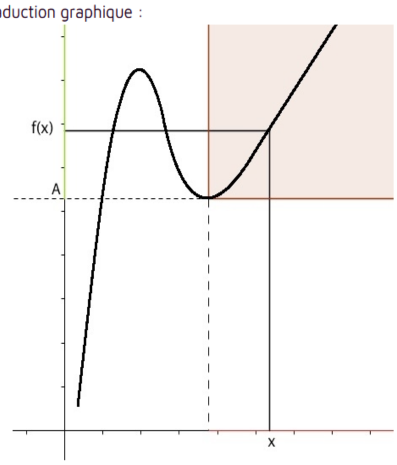
Exemple : $\lim_{x \to 0} \frac{|x|}{x} = \begin{cases} \lim_{x \to 0-} \frac{-x}{x} = -1 \\ \lim_{x \to 0+} \frac{x}{x} = 1 \end{cases}$

III. Infinite limits and limit at infinity

a. Plus infinity

« f(x) tends to +∞ as x tends to +∞" or "f has limit +∞ at +∞" means that any interval]A; +∞[, with A > 0, contains all values of f(x) for x large enough.

$$\lim_{x \to \infty} f(x) = +\infty$$



b. In less infinity

« f(x) tends to -∞ as x tends to +∞" or "f has limit -∞ at +∞" means that any interval]-∞ ;B[, with B > 0, contains all values of f(x) for x large enough.

$$\lim_{x \to -\infty} f(x) = -\infty$$

c. Limits of elementary functions

c.1. Limits in infinity

$f(x)$	x^n	$\frac{1}{x^n}$	$\sqrt{x}$	$\frac{1}{\sqrt{x}}$
$\lim_{x\to+\infty} f(x)$	$+\infty$	*0*	$+\infty$	*0*
$\lim_{x\to-\infty} f(x)$	$+\infty$ *if n even* $-\infty$ *if n odd*	*0*	*Not defined*	*Not defined*

c.2. Limits in 0

$f(x)$	$\frac{1}{x^n}$	$\frac{1}{\sqrt{x}}$
$\lim_{\substack{x\to0\\x>0}} f(x)$	$+\infty$	$+\infty$
$\lim_{\substack{x\to0\\x<0}} f(x)$	$+\infty$ *if n even* $-\infty$ *if n odd*	*Not defined*

IV. Indeterminate forms

There are indeterminate forms such as: $\frac{0}{0}, \frac{\infty}{\infty}, (+\infty-\infty), 0\times\infty$

V. Algebraic operations on limits

5.1. Sum of functions

if ***f*** *has a limit*	*L*	*L*	*L*	$+\infty$	$-\infty$	$+\infty$
if ***g*** *has a limit*	*L'*	$+\infty$	$-\infty$	$+\infty$	$-\infty$	$-\infty$
so ***f + g*** *has a limit*	*L+L'*	$+\infty$	$-\infty$	$+\infty$	$-\infty$	*Ind. F*

Example

1. *limit as x approaches* $+\infty$ *of the function defined on* $\mathbb{R}^*$ *by:* $f(x) = x+3+\frac{1}{x}$

5.2. Produit de fonctions :

if **f** has a limit	L	$L \neq 0$	0	∞
Si **g** has a limit	L'	∞	∞	∞
So $\boldsymbol{f} \times \boldsymbol{g}$ has a limit	$L \times L'$	∞^*	Ind. F	∞^*

* Apply the rule of signs

Example

1. *Limit as x approaches $-\infty$ of the previous function.:* $f(x) = x^2 + x$

   $\lim_{x \to -\infty} f(x) = x^2 + x = +\infty$

2. *limit as x approaches $+\infty$ of the function defined on $\mathbb{R}^+$ by:* $f(x) = x - \sqrt{x}$

   $\lim_{x \to +\infty} f(x) = x - \sqrt{x} = +\infty$

5.3. Quotient de fonctions

Si **f** a pour limite	L	$L \neq 0$	0	L	∞	∞
Si **g** a pour limite	$L' \neq 0$	$0^{(1)}$	0	∞	$L'^{(1)}$	∞
Alors $\frac{f}{g}$ a pour limite	$\frac{L}{L'}$	∞^*	F.Ind	0	∞^*	F.Ind

Example

1. *Limit in -2 of fonction* $\mathbf{R} - \{-2\}$ *by :* $f(x) = \frac{2x-1}{x+2}$

Solution :

By substituting -2 into the rational function, we obtain "$\frac{-5}{0}$".

Therefore, $\lim_{x \to -2} \frac{2x-1}{x+2} = \infty$

For the sign analysis, we distinguish between the left-hand and right-hand limits.

The numerator is always positive.

- *if,* $x < -2, x+2$ *the denominator is strictly negative*
- *if* $x > -2, x+2$ *, the denominator is strictly positive. Thus:*

$$\lim_{x \xrightarrow{<} -2} \frac{2x-1}{x+2} = -\infty$$

$$\lim_{x \xrightarrow{>} -2} \frac{2x-1}{x+2} = +\infty$$

VI. *Limit of a Composite Function*

Theorem: *Let f and g be two functions. Let a, b, and c be real numbers or* $+\infty$ *or* $-\infty$*. If:*

$$\lim_{x\to a} f(x) = b \quad and \quad \lim_{x\to b} g(x) = c \quad so \quad \lim_{x\to a} g\left[f(x)\right] = c$$

Example :

1. $\lim_{x\to+\infty} h(x)$ *avec* $h(x) = \sqrt{2 + \frac{1}{x^2}}$

2. $\lim_{x\to+\infty} k(x)$ *avec* $k(x) = \cos(\frac{1}{x^2+1})$

Applications

Exercise 1 with solution

1. $\lim\limits_{x\to+\infty} \dfrac{3x^2+4}{x^2+x+1}=3$

2. $\lim\limits_{x\to-\infty} \dfrac{-8x^2+1}{4x+16}=-\infty$

3. $\lim\limits_{x\to2} \dfrac{x^2-x-2}{x-2}=3$

4. $\lim\limits_{x\to1} \dfrac{x^2+2x-3}{2x^2-x-1}=\dfrac{4}{3}$

5. $\lim\limits_{x\to9} \dfrac{\sqrt{x}-3}{x-9}=\dfrac{1}{6}$

6. $\lim\limits_{x\to+\infty} \sqrt{\dfrac{4x+5}{x-2}}=$

7. $\lim\limits_{x\to0^+} \dfrac{e^x}{x}=+\infty$

8. $\lim\limits_{x\to\frac{\pi}{2}} (\sin x-\cos x)=1$

9. $\lim\limits_{x\to+\infty} \dfrac{x^2-3x+2}{x^4+x}=0$

10. $\lim\limits_{x\to\frac{\pi}{2}} \ln(\sin x)=0$

11. $\lim\limits_{x\to+\infty} \sqrt{x^2+4x+3}-(x+2)=0$

12. $\lim\limits_{x\to+\infty} \dfrac{x^2}{-2+\dfrac{3}{x}}=-\infty$

13. $\lim\limits_{x\to+\infty} (x^2+e^x)=+\infty$

14. $\lim\limits_{x\to+\infty} (\dfrac{1}{x}-3e^x)=+\infty$

15. $\lim\limits_{x\to2^+} \dfrac{x^2-4}{\sqrt{2}-\sqrt{x}}=-8\sqrt{2}$

16. $\lim\limits_{x\to1} \dfrac{-2x^2-x+3}{x-1}=-5$

17. $\lim\limits_{x\to1} \dfrac{\sqrt{x^2-1}+\sqrt{x-1}}{\sqrt{x-1}}=\sqrt{2}$

18. $\lim\limits_{x\to0} x\sin(\dfrac{1}{x})=0$

19. $\lim\limits_{x\to0} \dfrac{x-\sin(2x)}{x+\sin(3x)}=\dfrac{-1}{4}$

20. $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\sin x-\cos x}{1-\tan x}=-\dfrac{\sqrt{2}}{2}$

21. $\lim\limits_{x\to+\infty} (\sin\sqrt{x+1}-\sin\sqrt{x})=0$

22. $\lim\limits_{x\to1} (1-x)\tan(\dfrac{\pi x}{2})=\dfrac{2}{\pi}$

23. $\lim\limits_{x\to+\infty} (1+\dfrac{1}{x})^x=e$

24. $\lim\limits_{x\to+\infty} x\sin(\dfrac{1}{x})=1$

Exercise n°2: *we consider the numerical function f defined on $\mathbb{R}$ by:*

$$f(x)=\frac{e^x}{e^x+1}$$

1. *Determine the limit of f(x) when x tends to $-\infty$*

2. *Show that* $f(x)=\dfrac{1}{1+e^{-x}}$ *, and calculate the limit of f(x) when x tends to* $+\infty$

Solution

1. $\lim\limits_{x\to-\infty} f(x)=0$

2. $f(x)=\dfrac{e^{x}}{1+e^{x}}=\dfrac{1}{1+e^{x}}$

Since $\lim\limits_{x\to+\infty} e^{x}=0, \lim\limits_{x\to+\infty} f(x)=0$

Exercise n°3

Determine the domain of definition of the following functions:

$$f(x)=\sqrt{3x-x^{3}} \quad ,g(x)=\ln(x-2)+\ln(x+2),\ h(x)=\sqrt{\frac{2+3x}{5-2x}}$$

Using the definition of the limit, show that: $\lim\limits_{x\longrightarrow 2}(3x+1)=7$

Solution

1) *f is defined if and only if,*

$$\begin{aligned} 3x-x^{3}\geq 0 &\Leftrightarrow x\left(3-x^{2}\right)\geq 0\\ &\Leftrightarrow x\left(\sqrt{3}-x\right)\left(\sqrt{3}+x\right)\geq 0\\ &\Leftrightarrow x\in\left]-\infty,-\sqrt{3}\right]\cup\left[0,\sqrt{3}\right] \end{aligned}$$

Therfore

$$D_{f}=\left]-\infty,-\sqrt{3}\right]\cup\left[0,\sqrt{3}\right]$$

2) *g is defined if and only if ,*

$$x-2\succ 0 \ \ and\ \ x+2\succ 0\Leftrightarrow\begin{cases} x\succ 2\\ and\\ x\succ -2\end{cases}\Leftrightarrow x\succ 2$$

Therefore

$$D_{g}=\left]2,+\infty\right[$$

2) *h is defined if and only*

$$\frac{2+3x}{5-2x} \geq 0 \ \ and \ \ 5-2x \neq 0 \Leftrightarrow (2+3x \geq 0 \ \ and \ \ 5-2x \succ 0) \ \ \text{or} \ \ (2+3x \leq 0 \ \ and \ \ 5-2x \prec 0)$$

$$\Leftrightarrow \left(x \geq -\frac{3}{2} and x \prec \frac{5}{2} \right) or \left(x \leq -\frac{2}{3} and x \succ \frac{5}{2} \right)$$

$$\Leftrightarrow x \in \left[-\frac{3}{2}, \frac{5}{2} \right[$$

Therefore

$$D_h = \left[-\frac{3}{2}, \frac{5}{2} \right[$$

x	$-\infty$ $-\frac{3}{2}$	$\frac{5}{2}$	$+\infty$
$2+3x$	-	+	+
$5-2x$	+	+	-
$(2+3x)(5-2x)$	-	+	-

**let's prove it using the definition of the limit :* $\lim_{x \to 2} (3x+1) = 7$

In general, we have :

$$\forall \varepsilon \succ 0, \exists \alpha \succ 0, \forall x \in D_f, |x - x_0| \prec \alpha \Rightarrow |f(x) - l| \prec \varepsilon$$

Then :

$$\forall \varepsilon \succ 0, \exists \alpha \succ 0, \forall x \in \mathbb{R}, |x-2| \prec \alpha \Rightarrow |f(x) - 7| \prec \varepsilon$$

$$|f(x) - 7| \prec \varepsilon \Rightarrow \ |(3x+1) - 7| \prec \varepsilon$$

$$\Rightarrow |3x - 6| \prec \varepsilon$$

$$\Rightarrow |3(x-6)| \prec \varepsilon$$

$$\Rightarrow |x-2| \prec \frac{\varepsilon}{2}$$

Therefore

$$\forall \varepsilon \succ 0, \exists \alpha \succ 0, \forall x \in \mathbb{R}, |x-2| \prec \alpha \Rightarrow |f(x) - 7| \prec \varepsilon$$

Suchthat $\alpha = \frac{\varepsilon}{3}$

Chapter 5: Continuous functions

I. Definition of continuity at a point

Let $f: I \to R$ a function and $a \in I$.

- *We say that f is continuous at a if f has the limit f(a) at a:*

$\forall \varepsilon > 0, \forall n > 0$ *such as* $\forall x \in I : |x - a| < n \to |f(x) - f(a)| < \varepsilon$

II. Continuity on the right, continuity on the left

1. *f is continuous on the right at* $x_0 \in I$, *when:* $\lim_{x \xrightarrow{>} x_0} f(x) = f(x_0)$
2. *f is continuous on the left at* $x_0 \in I$, *when :* $\lim_{x \xrightarrow{<} x_0} f(x) = f(x_0)$
3. *f is continuous in* $x_0 \in I$, *when:* $\lim_{x \xrightarrow{>} x_0} f(x) = \lim_{x \xrightarrow{<} x_0} f(x) = f(x_0)$

III. Extension by continuity

Let $f : I \to \mathbb{R}$, *a function defined on I, can be in* x_0 $f : I \to \mathbb{R}$ *and **l** a real number suppose that :* $\lim_{\substack{x \to x_0 \\ x \neq x_0}} f(x) = l$ *and define the function* $\tilde{f} : I \cup \{x_0\} \to \mathbb{R}$ *by posing*

$$\tilde{f}(x) = \begin{cases} f(x) & si\ x \in I (x \neq x_0) \\ l & si\ x = x_0 \end{cases}$$ *so the function* $\tilde{f}$ *is continuous in* x_0

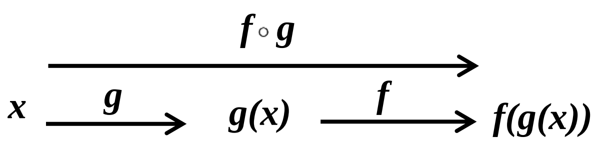
- *The function is called the extension by continuity of f in* x_0.

IV. Algebraic operations on continuous functions

Let f and g be functions of I in $\mathbb{R}$ *suppose that f and g are continuous in* $x_0 \in I$.

- *Functions* $f + g$, $f.g$ *et* λf *for everything* $\lambda \in \mathbb{R}$, *are continuities in* x_0.
- *if* $g(x_0) \neq 0$ *then the functions* $\frac{f}{g}$ *is continuous in* x_0.

V. Continuity of a composite function



- *If g is continuous at x_0 and if **f** is continuous at **g** (x_0) then $f \circ g$ is continuous at x_0*

VI. ***Continuity over an interval***

*If g is continuous on **I** and if **f** is continuous on **g(I)** then $f \circ g$ is continuous on **I***

VII. ***Continuous function on an interval***

Let I be an interval and $f : I \to \mathbb{R}$,a function. If f is continuous, then f(I) is an interval

Theorem: *let I be an interval and $f : I \to \mathbb{R}$, a continuous and strictly monotone function, then the reciprocal bijection f^{-1} of f is continuous, strictly monotone and of the same direction of variation as f.*

VIII. ***Intermediate Value Theorem***

Soit f:[a,b]→R une fonction continue. Soit γ∈R tel que γ est compris entre f(a) et f(b) ($f(a) < \gamma < f(b)$). Alors il existe c∈[a,b] tel que : f(c)=γ.

IX. ***Fonctions monotones continues***

- *We say that f is monotonic on I if its direction of variation does not change on I. In other words, if it is increasing on all I or decreasing on all I.*
- *f is said to be strictly monotone on I if it is strictly increasing on all I, or strictly decreasing on all I.*

APPLICATIONS

Exercise n°1

Determine whether the function f(x) is continuous at x=1 :

$$f(x) = \begin{cases} x^2 & x < 1 \\ 3 & x = 1 \\ x+1 & x > 1 \end{cases}$$

Solution

A function is continuous at x=1 if : $\lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) = f(1)$

1. ***Find f(1) :*** ***f(1)=3***

2. *Find* $\lim_{x\to 1-} f(x)$: $\lim_{x\to 1-} x^2 = 1$

3. *Find* $\lim_{x\to 1+} f(x)$: $\lim_{x\to 1+} (x+1) = 2$

Compare limits and function value:

Since $\lim_{x\to 1-} f(x)$ *and* $\lim_{x\to 1+} f(x)$ *the left and right limits are not equal.*

Therefore, $\lim_{x\to 1} f(x)$ ***does not exist**, so the function is **not continuous at x=1.***

Exercise n°2

Let $g:\mathbb{R}\to\mathbb{R}$ *be a function defined by:*

$$f(x) = \begin{cases} x^2+1 & x \le 2 \\ 3x-1 & x > 2 \end{cases}$$

Determine whether ff is continuous at x=2x=2. If not, classify the discontinuity.

Solution

A function is continuous at x=2 if : $\lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x) = f(2)$

1. *Find f(2) :* *f(2)=5*

2. *Find* $\lim_{x\to 2-} f(x)$: $\lim_{x\to 2-} x^2+1 = 2^2+1 = 5$

3. *Find* $\lim_{x\to 2+} f(x)$: $\lim_{x\to 2+} (3x-1) = 5$

Compare limits and function value:

Since $\lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x) = f(2)$

*Therefore, f is **continuous** at x=2.*

Exercise n°3

Let g: R→R be a function defined by:

$$g(x)=\begin{cases}\dfrac{x^2-4}{x-2} & x\neq 2\\ 5 & x=2\end{cases}$$

Determine whether g is continuous at x=2. If not, classify the discontinuity.

Solution

Simplify the expression: $\dfrac{x^2-4}{x-2}=\dfrac{(x-2)(x+2)}{x-2}$

For x≠2, we can cancel (x−2): *g(x)=x+2.*

A function is continuous at x=2 if : $\lim\limits_{x\to 2} g(x)=\lim\limits_{x\to 2}(x+2)=4$

1. From the definition of g, at x=2: *g(2)=5.*

2. Find $\lim\limits_{x\to 2} g(x)$: $\lim\limits_{x\to 2-} x^2-4=4$

***Compare limits and function value**:*

Since $\lim\limits_{x\to 2} g(x) \neq g(2)$

Therefore, f is not ***continuous*** *at x=2.*

Exercise n°4

Let f *be the function defined on* $\mathbb{R}$ *by :*

$$f(x)=\begin{cases}\dfrac{\sin x}{|x|} & if \quad x\neq 0\\ 1 & if \quad x=0\end{cases}$$

- *-calculate* $\lim\limits_{x\to 0} f(x)$.
- *-Is continous at 0 ?*
- *Is it differentiable at 0 ?Justify your answers.*

Solution

Let f *be the function defined on* $\mathbb{R}$ *by :*

$$f(x)=\begin{cases}\dfrac{\sin x}{|x|} & if \quad x\neq 0\\ 1 & if \quad x=0\end{cases}$$

1.We have

$$\lim_{x \xrightarrow{\succ} 0} f(x) = \lim_{x \xrightarrow{\succ} 0} \frac{\sin x}{x} = 1 \quad \text{(we have used a know limit)}$$

$$\lim_{x \xrightarrow{\prec} 0} f(x) = \lim_{x \xrightarrow{\prec} 0} (-\frac{\sin x}{x}) = -1$$

Therefore, the limit does not exist .

2. We have $\lim_{x \to 0} f(x)$ *does not exist, which means that the function is not continuous at 0 f is not differentiable at 0, since it is not continuous at this point, because any function differentiable is continuous, which is equivalent to saying that any function is discontinuous at a point cannot be differentiable at this point.*

Exercise n°5

Let $f : \mathbb{R} \to \mathbb{R}$ *be a function defined by :*

$$f(x) = \begin{cases} x^3 \cos\frac{1}{x} & if \quad x \succ 0 \\ 0 & if \quad x = 0 \\ x^3 \sin\frac{1}{x} & if \quad x \prec 0 \end{cases}$$

a-Is f *continous at* $x = 0$ *?*

b-Calculate $f^{'}(x)$ *for* $x \neq 0$*.Deduce the equation of the tangent line to* f *at* $x = \frac{1}{\pi}$.

c-Is f *differentiable at* $x = 0$ *?* **Is* f *of class* $C^1(\mathbb{R})$ *?*

Solution

1. *The function is clearly continous for* $x \neq 0$*.for* $x = 0$

$$\lim_{x \xrightarrow{\succ} 0} f(x) = \lim_{\xrightarrow{\succ} 0} x^3 \cos\frac{1}{x} = 0 \quad \text{(we have used the squeeze theorem)}$$

$$\lim_{x \xrightarrow{\prec} 0} f(x) = \lim_{\xrightarrow{\prec} 0} x^3 \sin\frac{1}{x} = 0 \quad \text{(we have used the squeeze theorem)}$$

Therefore $\lim_{x \to 0} f(x) = f(0)$ *:* f *is contiuos at 0.*

2. *for* $x \neq 0$ *the function i clearly differentiable and we have.*

$$f'(x)=\begin{cases} 3x^2\cos\frac{1}{x}+x\sin\frac{1}{x} & if \quad x\succ 0 \\ x^2\sin\frac{1}{x}\text{-}x\cos\frac{1}{x} & if \quad x\prec 0 \end{cases}$$

The tangent line to f *at* $x=x_0$ *is the aquation* $y=f'(x_0)(x-x_0)+f(x_0)$ *therefore,for* $x_0=\frac{1}{\pi}$

We have $y=-\frac{3}{\pi^2}x+\frac{2}{\pi^3}$.

3. *the function is differentiable at* $x=0$ *if and only if the limit* $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ *has a finite limit.*

We have

$$\lim_{x\overset{\succ}{\to}0}\frac{f(x)-f(0)}{x-0}=\lim_{x\overset{\succ}{\to}0}\frac{x^3\cos\frac{1}{x}-0}{x-0}=0$$

$$\lim_{x\overset{\prec}{\to}0}\frac{f(x)-f(0)}{x-0}=\lim_{x\overset{\prec}{\to}0}\frac{x^3\sin\frac{1}{x}-0}{x-0}=0$$

Therefore $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=0$: f *is differentiable at 0 and we have*

$$f'(x)=\begin{cases} 3x^2\cos\frac{1}{x}+x\sin\frac{1}{x} & if \quad x\succ 0 \\ 0 & if \quad x=0 \\ 3x^2\sin\frac{1}{x}\text{-}x\cos\frac{1}{x} & if \quad x\prec 0 \end{cases}$$

4. f' *is clearly continous for* $x\neq 0$.*for* $x=0$ *we have*

$$\lim_{x\overset{\succ}{\to}0}f'(x)=\lim_{x\overset{\succ}{\to}0}\left(3x^2\cos\frac{1}{x}+x\sin\frac{1}{x}\right)=0$$

$$\lim_{x\overset{\prec}{\to}0}f'(x)=\lim_{x\overset{\prec}{\to}0}\left(3x^2\sin\frac{1}{x}-x\cos\frac{1}{x}\right)=0$$

Then $\lim_{x\to 0}f'(x)=f'(0)$ *,therefore* f' *is continous at 0.therefore* f *is of class* $C^1(\mathbb{R})$.

Exercise n°6

Let f and g be two functions defined by:

$$f(x)=\begin{cases}\dfrac{x}{1+e^{\frac{1}{x}}} & x\neq 0\\ 0, & x=0\end{cases} \quad , \quad g(x)=\begin{cases}xe^{\frac{1}{x}} & x<0\\ 0, & x=0\\ x^2\ln(\dfrac{x+1}{x}) & x>0\end{cases}$$

Exercise n°7

Let f and g be two functions defined by:

$$f_1(x)=\begin{cases}2x-4 & x\leq 0,\\ \dfrac{1}{2}x-2 & x<0,\\ 2x^2-5x+4 & x\geq 4,\end{cases} \qquad f_2(x)=\begin{cases}ln(x^2+1) & x\leq 0,\\ \dfrac{e^x-1}{x^2+1} & x>0,\end{cases}$$

Solution

We need to check the continuity at ***x=0 and x=4.***

- *At x=0:*

  Left-hand limit: $\lim_{x\to 0^-} f_1(x)=2(0)-4=-4$

  Right-hand limit: $\lim_{x\to 0^+} f_1(x)=\frac{1}{2}(0)-2=-2$

Since −4≠−2, the left and right limits are not equal.

So, $f_1(x)$ *is* ***discontinuous at x=0****.*

- *At x=4:*

  Left-hand limit: $\lim_{x\to 4^-} f_1(x)=\frac{1}{2}(4)-2=0$

  Righ limit: $\lim_{x\to 4^+} f_1(x)=2(4)^2-5(4)+4=16$

Since 0≠16, the function has a jump.

$f_1(x)$ *is discontinuous at x=4.*

The function f2(x) is defined as: $f_2(x)=\begin{cases} ln(x^2+1) & x\le 0,\\ \dfrac{e^x-1}{x^2+1} & x>0,\end{cases}$

We need to check the continuity at ***x=0.***

- *At x=0:*

  Left-hand limit: $\lim_{x\to 0^-} f_2(x)=ln(0^2+1)=ln(1)=0$

Right-hand limit: $\lim_{x\to 0^+} f_2(x)=\lim_{x\to 0^+}\dfrac{e^x-1}{x^2+3}$

Using a first-order Taylor expansion at x=0 :

$$e^x\approx 1+x$$

Substituting this into $f_2(x)$: $\dfrac{(1+x)-1}{x^2+3}=\dfrac{x}{x^2+3}$

As $x\to 0$, $\lim_{x\to 0^+}\dfrac{x}{x^2+3}=\dfrac{0}{3}=0$

Since the left-hand and right-hand limits are equal,

$$\lim_{x\to 0^-} f_2(x)=\lim_{x\to 0^+} f_2(x)$$

The function's value at x=0is also f2(0)=ln(1)=0.

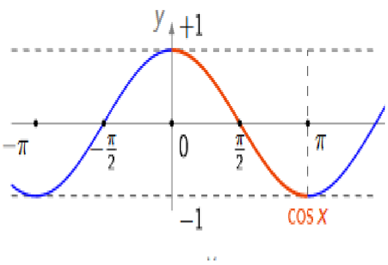
So, $f_2(x)$ *is* ***continuous at x=0****.*

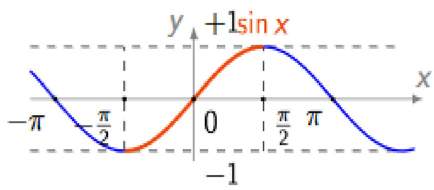
Chapter 6:
Reciprocal functions

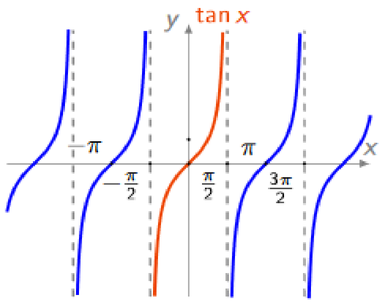
I. Definition

The sine, cosine functions defined for r in the interval [-1;1] are surjective applications by definition, that is to say:

$$\forall y \in [-1\,;1],\ \exists x \in \mathbb{R},\ \text{such as } \sin(x) = y \text{ and } \cos(x) = y\ .$$







The defined tangent function of $\mathbb{R}\,\{x \in \mathbb{R} / x = 2\pi + k\pi\ , k \in z\}$ *in* $\mathbb{R}$ *is a surjective application by definition*

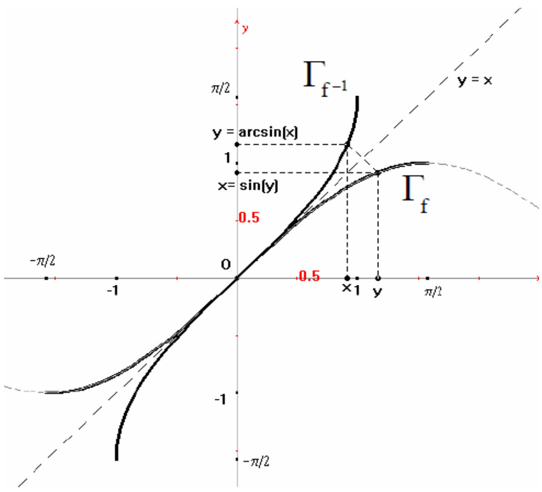
II. Reciprocal trigonometric functions

- *For the sine function, we restrict its domain of definition to the interval* $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ *and we have:*

$$\sin:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to[-1,1]$$
$$x \mapsto \sin x$$

So this function "sin" is bijective and we can define its reciprocal function called arcsine as follows:

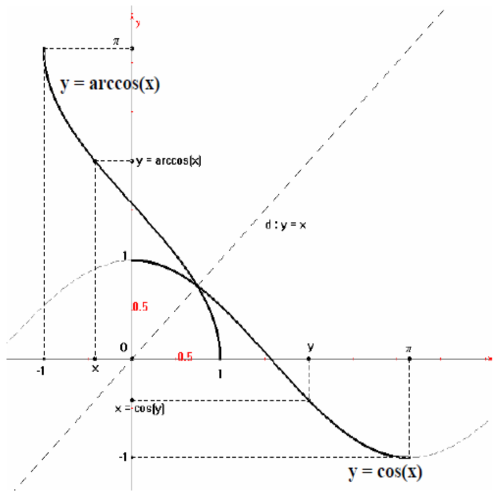
$$\arcsin:[-1,1]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$
$$x \mapsto \arcsin x$$



with equivalence: ***y = arcsin(x)*** $\Leftrightarrow$ ***x = sin(y)***

The graphic representation $\Gamma_{f^{-1}}$ *of a function* f^{-1}*, reciprocal of a bijective application f is always symmetric of* Γ_f *with respect to the bisector d of the first and third quadrant of equation d: y = x*

- *For the cosine function, we restrict its domain of definition to the interval [0;π] and we have:*



$$\cos : [0,\pi] \to [-1,1]$$
$$x \quad \mapsto \cos x$$

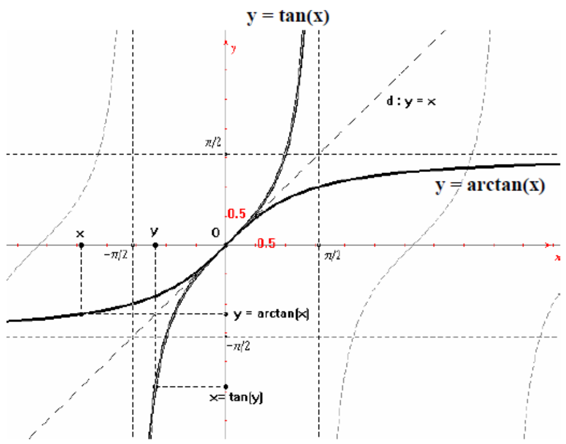
So this function "cos" is bijective and we can define its reciprocal function called arc cosine as follows:

$$\arccos : [-1,1] \to [0,\pi]$$
$$x \quad \mapsto arc\cos x$$

with equivalence: ***y = arccos(x)*** ⇔ ***x = cos(y)***

- *For the tangent function, we restrict its domain of*

definition to the interval $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ *and we have:*



$$\tan x : \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\to \mathbb{R}$$
$$x \quad \mapsto \tan x$$

So this function "tan" is bijective and we can define its reciprocal function called arctangent as follows:

$$arc\tan x : \mathbb{R} \to \left]-\frac{\pi}{2},\frac{\pi}{2}\right[$$
$$x \quad \mapsto arc\tan x$$

with equivalence: ***y = arctan(x)*** ⇔ ***x = tan(y)***

III. Direct hyperbolic functions

i. Hyperbolic sine and hyperbolic cosine

The hyperbolic sine function is called the function:

$$shx : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto sinh(x) = shx = \frac{e^x - e^{-x}}{2}$$

The hyperbolic cosine function is called the function:

$$chx : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto cosh(x) = chx = \frac{e^x + e^{-x}}{2}$$

Remarks

• *The **sh** function is odd.*

Indeed, it is defined on R and, for al $x \in \mathbb{R} l$ *, we have:*

$$sh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{-e^{-x} + e^{x}}{2} = -shx$$

- *The ch function is even.*

Indeed, it is defined on R and, for all $x \in \mathbb{R}$*, we have:*

$$ch(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^{x}}{2} = chx$$

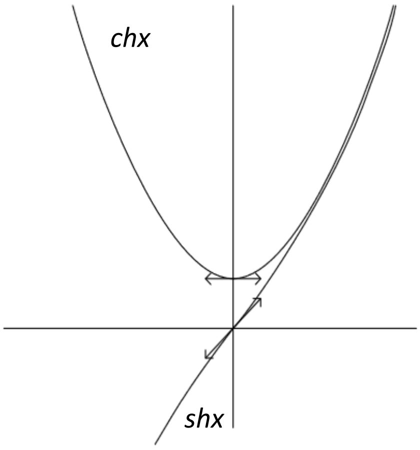
The graph of the ch function therefore takes the ordinate axis as its axis of symmetry.

- *For all* $x \in \mathbb{R}$*, on a* $ch^2 x - sh^2 x = 1$*.*

Proposition:

- The function sh is differentiable on R and its derivative is ch.

- The function ch is differentiable on R and its derivative is sh.



ii. Hyperbolic tangent

The hyperbolic tangent function is called the function:

$thx : \mathbb{R} \to \mathbb{R}$

$$x \mapsto tanh(x) = thx = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

Remarks

• *The sh function is odd.*

Indeed, it is defined on R and, for all $x \in \mathbb{R}$*, we have :* $th(-x) = \frac{sh(-x)}{ch(-x)} = \frac{-sh(x)}{ch(x)} = -thx$

- *For all* $x \in \mathbb{R}$*, we have :* $1 - th^2(x) = \frac{1}{ch^2(x)}$

iii. Identities for Hyperbolic Functions

Hyperbolic functions share many properties with trigonometric functions, but they are based on exponential functions. Below are the key identities for hyperbolic functions:

$$ch^2 a - sh^2 a = 1$$

IV. Reciprocal hyperbolic functions

1. Reciprocal of the hyperbolic sine function

- *The function **sh** is continuous and strictly increasing on R, it therefore performs a bijection of this interval on its image R and we can define its reciprocal application.*
- *We call the hyperbolic cosecant function* $\operatorname{Csch}(x)$ *of the hyperbolic sine function , and we note:*

$$Csch\ x : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto Csch\ x = Argsh(x) = \frac{1}{\sinh\ x} = \frac{2}{e^{x} - e^{-x}}$$

The reciprocal application of the hyperbolic sine function.

- *The* $Csch(x)$ *function is differentiable on R and For all* $x \in \mathbb{R}$, $Csch'(x) = \frac{1}{\sqrt{1+x^2}}$

2. Reciprocal of the hyperbolic cosine function

- *The function **ch** is continuous and strictly increasing on* $[0,+\infty[$ *, it therefore performs a bijection of this interval on its image* $[1,+\infty[$ *and we can define its reciprocal application.*
- *We call the hyperbolic Secant function, and we note:*

$$Sech : [1,+\infty[\to [0,+\infty[$$

$$x \mapsto Sechx = Argch(x) = \frac{1}{\cosh\ x} = \frac{2}{e^{x} + e^{-x}}$$

The reciprocal application the restriction of the function to the interval [0,+ ∞ [.

- *The hyperbolic Secant function is differentiable on]1,+ ∞ [and for all* $x \in \mathbb{R}$,

$$Sech'x = \frac{1}{\sqrt{x^2-1}}$$

3. Reciprocal of the hyperbolic tangent function

- *The function **th** is continuous and strictly increasing on R, it therefore performs a bijection of this interval on its image] − 1, 1[and we can define its reciprocal application.*
- *We call the hyperbolic Cotangent function, and we note:*

$$Coth :]-1,1[\to \mathbb{R}$$

$$x \mapsto Cothx = Arctanh(x) = \frac{1}{tanhx} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$

The reciprocal application of the hyperbolic tangent function.

- *The hyperbolic* ***cotangent*** *function is differentiable on] − 1, 1[and for all x∈] − 1, 1[:*

$$Coth'x = \frac{1}{th'(Cothx)} = \frac{1}{1 - th^2(Cothx)}$$

V. Identities and relationships

1. Some formulas of hyperbolic trigonometry

$$ch(a+b) = ch(a)\,ch(b) + sh(a)\,sh(b)$$

$$ch(a-b) = ch(a)\,ch(b) - sh(a)\,sh(b)$$

$$sh(a+b) = sh(a)\,ch(b) + ch(a)\,sh(b)$$

$$sh(a-b) = sh(a)\,ch(b) - ch(a)\,sh(b)$$

$$th(a+b) = \frac{th(a) + th(b)}{1 + th(a)th(b)}$$

$$th(a-b) = \frac{th(a) - th(b)}{1 - th(a)th(b)}$$

from which we deduce

$$ch(2a) = ch^2(a) + sh^2(a) = 1 + 2sh^2(a)$$

$$sh(2a) = 2sh(a)ch(a)$$

$$th(2a) = \frac{2th(a)}{1 + th^2(a)}$$

2. Expression of reciprocal hyperbolic functions with the natural logarithm

1. *For all* $x \in \mathbb{R}$, *on a* $sinh^{-1}x = \ln(x + \sqrt{x^2+1})$

2. *For all* $x \geq 1$, *on a :* $Cosh^{-1}x = \ln(x + \sqrt{x^2-1})$

3. *For all* $x \in \left]-1,1\right[$, *on a :* $Tanh^{-1}x = \frac{1}{2}\ln(\frac{1+x}{1-x})$

Applications

Exercise n°1

Solve the following equations

1. $5chx - 3shx = 4$
2. $3shx - chx = 1$
3. *ch(x)=2*
4. $ch(2x) = ch^2(x) + sh^2(x)$
5. $Coth^2 x - Csch^2 x = 1$
6. $Cschx = 2$

solution

1. $5chx - 3shx = 4$

$$ch(x) = \frac{e^x + e^{-x}}{2} \quad ; \quad sh(x) = \frac{e^x - e^{-x}}{2}$$

$$5chx - 3shx = 5(\frac{e^x + e^{-x}}{2}) - 3(\frac{e^x - e^{-x}}{2}) = e^x + 4e^{-x}$$

$$= e^x + 4\frac{1}{e^x} = \frac{e^{2x} - 4e^x + 4}{e^x}$$

which is also written $(e^x - 2)^2 = 0$ *or even* $e^x = 2$

Finally, the equation admits a unique solution $x = \ln 2$

2. $3shx - chx = 1$

$$ch(x) = \frac{e^x + e^{-x}}{2} \quad ; \quad sh(x) = \frac{e^x - e^{-x}}{2}$$

$$3shx - chx = 1 \Rightarrow 3(\frac{e^x - e^{-x}}{2}) - (\frac{e^x + e^{-x}}{2}) - 1 = 0$$

$$\Rightarrow e^x - 2e^{-x} - 1 = \frac{e^{2x} - e^x - 2}{e^x}$$

x is the solution if and only $e^{2x}-e^{x}-2=0$ *what is written* $X^2-X-2=0$ *by posing that* $X=e^{x}$ *, two roots -1 and 2 this shows that x is a solution if and only* $e^{x}=-1$ *(impossible) or* $e^{x}=2$ *the unique solution of the equation is* ***Ln2*** **.**

3. *ch(x)=2*

$$ch(x)=\frac{e^{x}+e^{-x}}{2}=2\Rightarrow e^{x}+e^{-x}=4$$

Let $y=e^{x}$ *, then* $e^{-x}=\frac{1}{y}$ *:*

$$y+\frac{1}{y}=4\Rightarrow y^2-4y-4=0$$

Solve the quadratic equation: $y=\frac{4\pm\sqrt{12}}{2}=2\pm\sqrt{3}$

Thus: $x=\ln(2+\sqrt{3})$ *or* $x=\ln(2-\sqrt{3})$

4. $ch(2x)=ch^2(x)+sh^2(x)$

$$ch(2x)=\frac{e^{2x}+e^{-2x}}{2};$$

$$ch^2(x)=(\frac{e^{2x}+e^{-2x}}{2})^2=\frac{e^{2x}+e^{-2x}+2}{4};$$

$$sh^2(x)=(\frac{e^{2x}-e^{-2x}}{2})^2=\frac{e^{2x}+e^{-2x}-2}{4};$$

$$ch^2(x)+sh^2(x)=\frac{e^{2x}+e^{-2x}+2}{4}+\frac{e^{2x}+e^{-2x}-2}{4}$$

$$=\frac{e^{2x}+e^{-2x}}{2}=ch(2x)$$

The identity is verified.

5. $Coth^2x-Csch^2x=1$

From the definitions of reciprocal hyperbolic functions:

$$Cothx=\frac{1}{tanhx}=\frac{Chx}{Shx}\quad ;\quad Csch\ x=\frac{1}{\sinh\ x}$$

Square both functions:

$$Coth^2x = \frac{1}{tanh^2x} = \frac{Ch^2x}{Sh^2x} \quad ; \quad Csch^2\,x = \frac{1}{\sinh^2 x}$$

Substitute into the left-hand side of the identity:

$$\frac{Cosh^2x}{\sinh^2 x} - \frac{1}{\sinh^2 x} = \frac{Cosh^2x - 1}{\sinh^2 x}$$

$$according\ to:\ Cosh^2x - \sinh^2 x = 1 \Rightarrow \sinh^2 x = Cosh^2x - 1$$

$so: Coth^2x - Csch^2x = \frac{\sinh^2 x}{\sinh^2 x} = 1$ *the identity is verified.*

Exercise n°3

$$we\ pose\ f(x) = Csch\left(\sqrt{\frac{chx+1}{2}}\right)$$

1. *Determine the definition set of the function f.*
2. *Calculate* $f'(x)$ *when possible. deduce a simple expression for f(x).*

Exercise n°4

Solve the following equations

1. $\arccos x = 2\arccos\frac{3}{4}$
2. $\arcsin x = \arcsin\frac{2}{5} + \arcsin\frac{3}{5}$

Solution

1. $\cos(\arccos x) = x$ *,So* $\cos(\arccos x) = \cos(2\arccos\frac{3}{4})$

Thus, $x = \cos(2\arccos\frac{3}{4}) = 2\cos(\arccos\frac{3}{4}) - 1 = 2.\frac{3}{4} - 1 = \frac{1}{2}$

2. $\cos(\arcsin x) = \sqrt{1-x^2}$ *and* $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$

$x=\sin(\arcsin x)$

$=\sin(\arcsin\frac{2}{5}+\arcsin\frac{3}{5})$

$=\sin(\arcsin\frac{2}{5}).\cos(\arcsin\frac{3}{5})+\sin(\arcsin\frac{3}{5}).\cos(\arcsin\frac{2}{5})$

$=\frac{2}{5}\cos(\arcsin\frac{3}{5})+\frac{3}{5}\cos(\arcsin\frac{2}{5})$

$=\frac{2}{5}\sqrt{1-(\frac{3}{5})^2}+\frac{2}{5}\sqrt{1-(\frac{2}{5})^2}=\frac{3\sqrt{21}+8}{25}$

Exercise n°5 : *Simplify the following expressions*

1. $ch(Argshx)$, 2. $th(Argshx)$, 3. $sh(2Argshx)$

4. $sh(Argchx)$, 5. $th(Argchx)$, 6. $ch(2Argchx)$

7. $\dfrac{2ch^2(x)-sh(2x)}{x-Ln(ch(x)-ln(2)}$

Solution

1.we have nverse Hyperbolic Sine Function : $Argshx=ln(x+\sqrt{x^2+1})$

And Hyperbolic Cosine Function : $ch(y)=\dfrac{e^y+e^{-y}}{2}$

- *Let* $y=Argshx$: $y=ln(x+\sqrt{x^2+1})$

Using the definition of y : $e^y=e^{ln(x+\sqrt{x^2+1})}=x+\sqrt{x^2+1}$

$$e^{-y}=\frac{1}{e^y}=\frac{1}{x+\sqrt{x^2+1}}$$

Thus: $$ch(y)=\frac{x+\sqrt{x^2+1}+\dfrac{1}{x+\sqrt{x^2+1}}}{2}$$

Multiplying the numerator and denominator of the second term by $x+\sqrt{x^2+1}$:

Which simplifies to:

$$ch(Argshx)=\sqrt{x^2+1}$$

2. *We use the definition:* $Argshx=ln(x+\sqrt{x^2+1})$

And the formulas: $sh(Argshx)=x,\quad ch(Argshx)=\sqrt{x^2+1}$

Thus: $$th(Argshx)=\frac{sh(Argshx)}{ch(Argshx)}=\frac{x}{\sqrt{x^2+1}}$$

3. *We use the double-angle formula:* $sh(2y)=2sh(y)ch(y)$

Since: $sh(Argshx) = x, \quad ch(Argshx) = \sqrt{x^2+1}$

we get: $sh(2Argshx) = 2x\sqrt{x^2+1}$

4. We use the definition: $ch^2(\alpha) - sh^2(\alpha) = 1$,with $\alpha = Argchx$

$$sh^2(Argchx) = ch^2(Argchx) - 1 = x^2 - 1$$

as, $Argchx \geq 0$, and $sh(Argchx) \geq 0$:

So $sh(Argchx) = \sqrt{x^2-1}$

5. We use the formulas: $th(Argchx) = \dfrac{sh(Argchx)}{ch(Argchx)}$

Since: $sh(Argchx) = \sqrt{x^2-1}, ch(Argchx) = x$

we obtain : $th(Argchx) = \dfrac{\sqrt{x^2-1}}{x}$

6. Using the double-angle identity: $ch(2y) = 2ch^2(y) - 1$

Since $ch(Argchx) = x$

We get : $ch(2Argchx) = 2x^2 - 1$

7. We use the identities: $ch^2(x) - sh^2(x) = 1$

$$sh(2y) = 2sh(y)ch(y)$$

which simplifies the numerator: $2ch^2(x) - sh(2x) = 2(1 + sh^2(x)) - 2sh(x)ch(x)$

$$= 2 + 2sh^2(x) - 2sh(x)ch(x)$$

For the denominator, we simplifly $ln(ch(x) + ln(2)$:

$$ln(ch(x) + ln(2) = ln(2ch(x))$$

So, $\quad x - ln(ch(x) - ln(2) = x - ln(2ch(x))$

Thus, the final simplified form depends on whether additional factorizations are possible, but the expression can be rewritten as: $\dfrac{2 + +2sh^2(x) - 2sh(x)ch(x)}{x - ln(2ch(x))}$

ALGEBRA

SECTION

Chapter 1:

Common Algebraic Structures

I. Internal composition law

1. Definition

Let E be a set. An internal composition law on E is an application of E×E In E.

If we note it $\begin{matrix} E\times E \to E \\ (a,b)\mapsto a*b \end{matrix}$, *we speak of the law* $*$ *and we say that* $a*b$ *is the composite of a and b for the law* $*$.

Example : *On E=Z, the addition defined by* $\begin{matrix} \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z} \\ (a,b)\mapsto a+b \end{matrix}$, *multiplication* $\begin{matrix} \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z} \\ (a,b)\mapsto a\times b \end{matrix}$ *and subtraction* $\begin{matrix} \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z} \\ (a,b)\mapsto a-b \end{matrix}$ *are internal composition laws. This is not the case for division because a/b is not defined for all pairs (a, b) of integers.*

2. Properties: *throughout this paragraph (E,*) designates a set provided with an I.C.L.*

a) **Associativity:** *we say that is associative when, for all x, y, z of E :* $x*(y*z)=(x*y)*z$

b) **Commutativity:** *we say that is commutative when, for all x, y of E:* $x*y=y*x$.

c) **Neutral element:** *or, we say that e is a neutral element for * when for all* $x\in E$, $x*e=e*x=x$ *(both equalities must be verified when * is not commutative)*

d) **Symmetrical :** *we assume here that E admits a neutral e for *.*
*let x, x' two elements of E. in says that x' is symmetrical to x (for *), x is invertible (symmetrizable) when* $x*x'=x'*x=e$.

e) **Distribution :** *we assume that E is equipped with a second I.C.L noted ♯, we say that is distributive on ♯ when for all x, y, z of E :*$x*(y♯z)=(x*y)♯(z*x)et$

f) $(y♯z)*x=(y*x)♯(z*x)$ *

g) **Monoid:** *let (E,*) be a monoid (thus* is associative then the set s of symmetrizable elements of E is stable by *.*

3. Stability

(E,) always denotes a set equipped with an ICL, let F be a part of E. We say that F is stable by * when for all x, y of F, is still in F. In this case, we can say that * defines by restriction an ICL on F.*

II. Groups

*A group is a non-empty set with an internal composition law (G, *) such that:*

- ** is associative;*
- ** admits a neutral e;*
- *Any element of G is symmetrizable (admits a symmetric) for *.*

*If * is commutative, we say that (G, *) is commutative, or abelian.*

1. Subgroups

Let (G,) be a group. We say that a subgroup H of G is a subgroup of G if:*

i) $H \neq \varnothing$ and $x * y \in H, \forall x, y \in H$

ii) (H,*) is a group

Proposal: *let (G,*) is group and $H \subset G$. Then H is a subgroup of*

$$G \Leftrightarrow \begin{cases} H \text{ is not empty} : (H \neq \varnothing) \\ x * y^{-1} \in H, \forall x, y \in H \end{cases}$$

Example : $\{e\}$ *is a subgroup of G.*

III. Ring structure

Definition : *we call ring, all together A provided with two internal composition laws + and such that:*

- *(A,+) is an abelian group (we will note 0 or 0_A the neutral element of +)*
- *$\times$ is associative and distribution relative to +*

If moreover $\times$ is commutative, we say that (A,+,) is a commutative ring.

IV. Body

A unit ring is said $(\mathrm{K}, +, \times)$ to be a body if every non-zero element of K is invertible for the law $\times$.

If moreover $\times$ is commutative, we say K that is a commutative body

Applications

Exercise n°1

1. we provide $\mathbb{R}$ *the internal composition law* $*$ *defined by:*

$\forall x, y \in \mathbb{R}, x * y = xy + (x^2-1)(y^2-1)$

*show that * is commutative, not associative and that 1 is a neutral element.*

2. we provide $\mathbb{R}^{+*}$ *the internal composition law* $*$ *defined by:* $\forall x, y \in \mathbb{R}, x * y = \sqrt{x^2+y^2}$

*show that * is commutative, associative and that 1 is a neutral element.*

Solution

1. *a. * is commutative if :* $x * y = y * x$

$x * y = xy + (x^2-1)(y^2-1) = yx + (y^2-1)(x^2-1) = y * x$

*So law * is commutative*

*b. * is associative:* $(x * y) * z = x * (y * z)$

$$(x * y) * z = (x * y)z + ((x * y)^2 - 1)(y^2-1)$$
$$= (xy + (x^2-1)(y^2-1))z + ((xy + (x^2-1)(y^2-1))^2 - 1)(y^2-1)$$

$$x * (y * z) = x(y * z) + (x^2-1)((y * z)^2 - 1)$$
$$= x(yz + (y^2-1)(z^2-1)) + (x^2-1)((yz + (y^2-1)(z^2-1))^2 - 1)$$

Is clear that $(x * y) * z \neq x * (y * z)$

*So law * is not associative*

*c. * has a neutral element if : x*e=e*x=x*

$$1 * x = 1 \times x + (1^2-1)(x^2-1) = x$$
$$x * 1 = x \times 1 + (x^2-1)(1^2-1) = x$$

*So 1 is a neutral element of **

*2. a. * is commutative if :* $x * y = y * x$

$$x*y=\sqrt{x^2+y^2}=\sqrt{y^2+x^2}=y*x$$

*So law * is commutative*

*b. * is associative:* $(x*y)*z=x*(y*z)$

$$(x*y)*z=(\sqrt{x^2+y^2})*z=\sqrt{(\sqrt{x^2+y^2})^2+z^2}=\sqrt{x^2+y^2+z^2}$$
$$x*(y*z)=x*(\sqrt{y^2+z^2})=\sqrt{x^2+\sqrt{y^2+z^2}}=\sqrt{x^2+y^2+z^2}$$

*So law * is associative.*

*c. * has a neutral element if : x*e=e*x=x*

$$x*e=\sqrt{x^2+e^2}\Rightarrow x*0=\sqrt{x^2+0^2}=|x|=x$$

*So 0 is a neutral element of **

Exercise n°2

let $A=\mathbb{R}x\mathbb{R}$ *of two laws defined by.*

$$\forall x,y\in\mathbb{R},(x,y)*(x',y')=(xx',xy'+x'y)$$

*a) Show that the law * is commutative.*

*b) Show that * is associative.*

*c) Determine the neutral element for the law *.*

Solution

*a. * is commutative if :* $(x,y)*(x',y')=(x',y')*(x,y)$

$$(x,y)*(x',y')=(xx',yy'+x'y)$$

$$(x',y')*(x,y)=(x'x,x'y+xy')$$

*So * is commutative*

*b. * is associative:* $(x*y)*z=x*(y*z)$

$$\left[(x,y)*(x',y')\right]*(x'',y'')=(x,y)*\left[(x',y')*(x'',y'')\right]$$

$$(x,y)*\left[(x',y')*(x'',y'')\right]=(xx'x'',xx'y''+x''(xy'+x'y))$$

$$=xx'x'',xx'y''+x''xy+x''x'y).$$

$$\left[(x,y)*(x',y')\right]*(x'',y'')=(xx',xy'+x'y)*(x'',y'')$$

$$=(xx'x'',xx'y''+x''xy+x''x'y)$$

Therefore : $\left[(x,y)*(x',y')\right]*(x'',y'')=(x,y)*\left[(x',y')*(x'',y'')\right]$

*the law * is associative*

c. ** has a neutral element if : (x,y)*(e,f)= (e,f)*(x,y)=(x,y)*

let (e,f) *such that for all* $(x,y)\in A,(x,y)*(e,f)=(x,y)$

$$\begin{cases} xe=x \\ xf+ye=y \end{cases} \Rightarrow \begin{cases} e=1 \\ xf+y=y \end{cases} \Rightarrow \begin{cases} e=1 \\ f=0 \end{cases}$$

$(1,0)\in \mathrm{A}$ *is the identity element of A for the law.*

Exercise n°3

*In the set R, we define the law * by:*

$$\forall yx\in\mathbb{R},\forall y\in\mathbb{R},x*y=|x|+|y|$$

*1) Is * associative? Commutative? Does it admit an identity element? Justify.*

2) Is $(\mathbb{R},*)$ *a group? Justify.*

Solution

1/ associative :let $x\in\mathbb{R},y\in\mathbb{R}$ et $z\in\mathbb{R}$

$$(x*y)*z=(|y|+|z|)*z=||x|+|y||+|z|=|x|+|y|+|z|$$

$$x*(y*z)=x*(|y|+|z|)=|x|+||y|+|z||=|x|+|y|+|z|$$

$$x*(y*z) = x*(|y|+|z|) = |x| + ||y|+|z|| = |x| + ||x|+|z|| = |x|+|y|+|z|.$$

Therefore : $(x*y)*z = x*(y*z)$

b/commutative : $\forall x \in \mathbb{R}, \forall y \in \mathbb{R} :\ x*y = y*x$

$$x*y = |x|+|y| = |y|+|x| = y*x$$

c/identity element : $\exists e \in \mathbb{R}, \forall x \in \mathbb{R}, x*e = e*x = x$

$$x*e = x \Leftrightarrow |x|+|e| = x \rightarrow \begin{cases} x+|e| = x & si\ x \geq 0 \\ -x+|e| = x & si\ x \leq 0 \end{cases}$$

$$\begin{cases} |e| = 0 \\ |e| = 2x \end{cases}$$

We can deduce from the law: e does not exist because e is unique $\Rightarrow$ *e is not an identity element.*

Exercise n°4

We define $\mathbb{Z}$ *a binary operation * as follows :*

$$\forall (x, y) \in \mathbb{Z}, x*y = xy(x+y)$$

*1) Show that the law * is commutative.*

*2) Calculate (1 * (-1)) * 2 and 1 * ((-1) * 2). Is the law * associative?*

3) Solve the equation in $\mathbb{Z}$*:* $x*x = 16$.

*4) Show that * does not admit an identity element.*

Solution

1. *The commutative law :* $\forall (x, y) \in \mathbb{Z} : x*y = y*x$

$x*y = xy(x+y) = yx(y+x) = y*x$

*The law * is commutative.*

2. ** associative :* $\forall x, y, z \in \mathbb{Z} : (x*y)*z = x*(y*z)$

$$\big(1*(-1)\big)*2 = \big(-1(1-1)\big)*2 = 0*2 = 0*2(2+0) = 0$$

$$1*\big((-1)*2\big) = 1*(-2*1) = 1*(-2) = -2*(-1) = 2$$

Therefore : $\big(1*(-1)\big)*2 \neq 1*\big((-1)*2\big)$

*Therefore, the law * is not associative.*

3. solve the equation : $x*x = 16$

We have $x*x = x^2*2x = 2x^3$

Therefore : $x*x = 16 \Leftrightarrow 2x^3 = 16$

$$\Leftrightarrow x^3 = 8$$

$$\Leftrightarrow \sqrt[3]{x^3} = \sqrt[3]{8}$$

$(x^3 \succ 0)$ $\quad \Leftrightarrow x = 2$

4.we have : $\forall x \in \mathbb{Z} : x*e = e*x = x$

$$xe(e+x) = x \Rightarrow xe(e+x) - x = 0$$

$$\Rightarrow x\big(e(e+x)-1\big) = 0$$

$$\Rightarrow \begin{cases} x = 0 \\ e(e+x) - 1 = 0 \end{cases}$$

$$e(e+x) = 1 \Rightarrow \begin{cases} x = -e \\ x = 0 \end{cases} \Rightarrow 0 \neq 1$$

*We take which is impossible, therefore * does not admit an identity element.*

Exercise n°5

*the internal composition law * defined by:*

$$\forall x, y \in \mathbb{R}, x * y = \frac{x+y}{1+xy}$$

*- show that * is commutative, associative, and has an identity element.*

Solution

1. ** is commutative if :* $x * y = y * x$

$$x * y = \frac{x+y}{1+xy} \quad , \; y * x = \frac{y+x}{1+yx}$$

*So * is commutative*

2 . is associative if :* $x*(y*z) = (x*y)*z$

$$y * z = \frac{y+z}{1+yz} \quad , \; x*(y*z) = \frac{x+\left(\frac{y+z}{1+yz}\right)}{1+x\left(\frac{y+z}{1+yz}\right)} = \frac{x+y+z+xyz}{1+yz+xy+xz}$$

$$(x*y)*z = \frac{\frac{x+y}{1+xy}+z}{1+\left(\frac{x+y}{1+xy}\right)z} = \frac{x+y+z+xyz}{1+xy+xz+yz}$$

*Therefore, * is associative*

3. has an identity element*

$$x*e = e*x = 1 \Rightarrow e = 1$$

*So 1 is a neutral element of **

Exercise n°6

*Let the internal composition law * defined by:*

$$\forall x, y \in \mathbb{R}, x * y = x + y + xy$$

- *Show that * is commutative, associative, and admits an neutral element*

Solution

1) ** is commutative :* $x*y=y*x$

$x*y=x+y+xy \quad /y*x=y+x+yx$

2/ ** is associative :* $(x*y)*z=x*(y*z)$

$(x*y)*z=(x+y+xy)*z=x+y+z+xy+xz+yz+xyz$

$x*(y*z)=x*(y+z+yz)=x+y+z+xy+xz+yz+xyz$

*Therefore * is associative*

3) *O is the neutral element :* $x*o=x+o+o=x \quad /o*x=x$

*Therefore, O is the neutral element of *.*

Chapter 2:
Vector spaces

I. Vector spaces

Définition : *A vector space (abbreviated as v.s.) over a field is a set K,E, equipped with two operations, + and.*

**The operation + is a mapping E×E→E, and the operation* (scalar multiplication) is a mapping K×E→E, such that the following properties are satisfied:*

1. $(E,+)$ *is a commutative group (with a neutral element denoted as* 0_E
2. $\forall \alpha \in \mathrm{K}, v_1, v_2 \in E, \alpha * (v_1 + v_2) = \alpha * v_1 + \alpha * v_2$
3. $\forall \alpha, \beta \in \mathrm{K}, v_1 \in E, v_1 * (\alpha + \beta) = \alpha * v_1 + \beta * v_1$
4. $\forall \alpha, \beta \in \mathrm{K}, v_1 \in E, v_1 * (\alpha \bullet \beta) = \alpha * (\beta \bullet v_1)$
5. $\forall v_1 \in E, 1 \bullet v_1 = v_1$ ***(****1_{K} .is the unit element of* K *)*

- *The elements of E are called: vectors, and those of K are called scalars*

Example :

II. Subvector Space:(s.e.v)

*If E is a vector space over K, F is said to be a **subvector space** of E*

1. $F \neq \varnothing$ $(0_E \in F)$
2. *Closed under addition, That is :* $\forall v_1, v_2 \in F, v_1 + v_2 \in F$
3. *Closed under scalar multiplication :* $\forall v \in F, \forall \alpha \in \mathrm{K}, \alpha v \in F$

Proposition*: Let E be a vector space over K ,* $F \in E$ *. Then F is a subvector space of E if and only if the following conditions are satisfied:*

1. $0_E \in F$ $(F \neq \varnothing)$

2. $\forall \alpha, \beta \in \mathrm{K}, \forall x, y \in F, \alpha x + \beta y \in F$

Example :

1) $\{0_E\}$ and E are subvector spaces of E.

2) Si $\varnothing \neq A \subset E$ et $0_E \notin A \Rightarrow A$ is not a subvector space of E.

Proposition: If F and G are two subvector spaces of E, then $F \cap G$ is not a subvector space of E, where as $F \cup G$ is not necessarily a subvector space of E in general.

III. Bases and Dimension

III.1. Free Families, Generating Families, and Bases

1. Definitions :

- **Definition of Free Family, Dependent Family, and Linear Independence**:

A family (a collection) $F = (\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$ A family of vectors in a K-vector space E is said to be dependent if there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}$ not all zero such that:

$$\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + ... + \lambda_n \vec{v}_n = \vec{0}$$

It is also said that the vectors... $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ ...are linearly dependent.

In the opposite case, it is said that the family is free. To say that a family. $F = (\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$ To say that a family is free means that if..$\lambda_{1,2}, \ldots, \in \mathbb{K}$ verify $\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + ... + \lambda_n \vec{v}_n = \vec{0}$, then we necessarily have...$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

- **Definition of generating family**

A family $\mathscr{F}$ of E is said to be a generating family of E if $E = \langle \mathscr{F} \rangle$, i.e., every vector $\vec{u}$ in E is a linear combination of elements of $\mathscr{F}$

- **Definition of a Basis**

A family $\mathscr{F}$ of E is a basis of E if and only if $\mathscr{F}$ is linearly independent and generates E.

2. Bases and coordinates

- Proposition: The family $B = (\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$ is a basis of EE if and only if every vector $\vec{v}$ of is uniquely written as a linear combination of the $\vec{v}_i \in \mathcal{B}$. In other words:

$$\forall \vec{v} \in E, \exists! \lambda_1, \lambda_2, ...\lambda_n \in K \text{ such that } \vec{v} = \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + ... + \lambda_n \vec{v}_n$$

The numbers $\lambda_{1,2}, \ldots, \lambda_n \in \mathbb{K}$ are called the coordinates of $\vec{v}$ in the basis B.

- **The canonical basis of $\mathbb{K}^n$ ($\mathbb{R}^n, \mathbb{C}^n$...)**

Definition: The basis $B = (\vec{e}_1, \vec{e}_2, ..., \vec{e}_n)$ is called the canonical basis of $\mathbb{K}^n$.

The coordinates of a vector $\vec{v} = x_1 + x_2 + ... + x_n \in \mathbb{K}^n$ *in this basis are simply the components* x_i *of* $\vec{v}$ *.* ***Note,*** *this only occurs in this particular basis.*

-Free family of $\mathbb{R}$ ***:***

Any free family $\mathscr{F}$ *of* $\mathbb{R}^n$ *is a basis of* $\mathcal{B}= Vect(\mathscr{F})$*.*

For example, two non-collinear vectors of $\mathbb{R}^n$ *form a basis of the plane spanned by these two vectors.*

III.2.– Dimension of a vector space

1. Definitions

- ***Fundamental Theorem: Dimension and Cardinality of Bases*** **:**

Let EE be a vector space $\neq\{\vec{0}\}$ *and spanned by n vectors. Then, all bases of E have the same number of elements. This integer (or cardinality) is called the dimension of E and is denoted* ***dim*** *of E . Furthermore, we have* $\mathbf{dim}\ E \leq n$ *in this case.*

By convention, we set $\dim\{\vec{0}\}=0$*.*

2. Important Consequences

Theorem: *Let E be a finite-dimensional vector space. Then:*

i) *Every free family* $\mathscr{F}$ *of E satisfies* $\mathbf{card}(\mathscr{F}) \leq \mathbf{dim} E$ *et* $(\mathscr{F})=\mathbf{dim} E$ *implies that* $\mathscr{F}$ *is a basis of E.*

ii) *Every generating family of E has at least dim E elements. If a generating family of E has exactly dim E elements, then it is a basis of E.*

3. ***Rank of a System of Vectors:***

4. ***Definition:*** *Let* $\mathcal{F} = (\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$ *a free family of vectors in E. The rank of* $\mathscr{F}$ *is the dimension of the subspace of E spanned by* $\mathscr{F}$ $(\mathscr{F})$*.*

- ***Caution:*** *Do not confuse the rank with the cardinality of a family!*
  The cardinality is simply the number of elements in the family (which is visible), whereas the rank is a more abstract concept based on the dimension.

- ***Proposition*** **:**
  i) *We always have rankrank* $(\mathscr{F}) \leq card(\mathscr{F})$*.*
  ii) *Case of equality: We have* $rank(\mathscr{F})=card(\mathscr{F})$ *if and only if* $\mathscr{F}$ *is free.*

Applications

Exercise n°1 : *the following vector family is free in* $\mathbb{R}^3$: $\vec{e}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \vec{e}_2 = \begin{pmatrix}2\\1\\1\end{pmatrix}, \vec{e}_3 = \begin{pmatrix}3\\1\\1\end{pmatrix}$.

Solution

Family is free if : $\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2 + \lambda_3\vec{v}_3 = \vec{0} \Rightarrow \lambda_1\begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda_2\begin{pmatrix}2\\1\\1\end{pmatrix} + \lambda_3\begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\Rightarrow \begin{cases}\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0\\ \lambda_1 + \lambda_2 + \lambda_3 = 0\\ \lambda_1 + \lambda_2 + \lambda_3 = 0\end{cases}$$

$$\Rightarrow \begin{cases}\lambda_1 = -3\lambda_3\\ \lambda_2 = 2\lambda_3\end{cases}$$

So this family is dependent

Exercise n°2

Is the following family linearly independent?

$e_1 = (1,1,0)$, $e_2 = (4,1,4)$, $e_3 = (2,-1,4)$

Solution

The family is linearly independent : $\alpha e_1 + \beta e_2 + \gamma e_3 = 0$

$\alpha(1,1,0) + \beta(4,1,4) + \gamma(2,-1,4) = (0,0,0)$

$$\Rightarrow \begin{cases}\alpha + 4\beta + 2\gamma = 0\\ \alpha + \beta - \gamma = 0\\ 4\beta + 4\gamma = 0\end{cases} \Rightarrow \begin{cases}\alpha = 2\gamma\\ \beta = -\gamma\end{cases}$$

There is more than just (0, 0, 0) as a solution, so the family is linearly dependent.

By setting $\gamma = 1$, *we find that* $\alpha = 2, \beta = -1$

Exercise n°3

1. show that the vectors $\vec{v}_1=(0,1,1),\vec{v}_2=(1,0,1),\vec{v}_3=(1,1,0)$ form a basis in $\mathbb{R}^3$.find the components of the vector $w=(1,1,1)$ in this basis (v_1,v_2,v_3).
2. show that the vectors $\vec{e}_1=(1,0,0),\vec{e}_2=(0,0,1),\vec{e}_3=(1,2,-3)$ and $w=(1,1,1)$ form a basis in $\mathbb{R}^3$. find the components of the vector $\vec{e}_1=(1,0,0),\vec{e}_2=(0,0,1),\vec{e}_3=(0,0,1)$ in this basis (v_1,v_2,v_3).

Solution

1. To show that the family f is a base we will show that this family is free and generative

a. F is free $\lambda_1\vec{v}_1+\lambda_2\vec{v}_2+\lambda_3\vec{v}_3=\vec{0}\Rightarrow \lambda_1\begin{pmatrix}0\\1\\1\end{pmatrix}+\lambda_2\begin{pmatrix}1\\0\\1\end{pmatrix}+\lambda_3\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\Rightarrow\begin{cases}\lambda_2+\lambda_3=0\\ \lambda_1+\lambda_3=0\\ \lambda_1+\lambda_2=0\end{cases}$$

$$\Rightarrow\begin{cases}\lambda_2=0\\ \lambda_1=0\\ \lambda_3=0\end{cases}$$

So f is free

b. F is generative : $\lambda_1\vec{v}_1+\lambda_2\vec{v}_2+\lambda_3\vec{v}_3=v\Rightarrow \lambda_1\begin{pmatrix}0\\1\\1\end{pmatrix}+\lambda_2\begin{pmatrix}1\\0\\1\end{pmatrix}+\lambda_3\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$

$$\Rightarrow\begin{cases}\lambda_2+\lambda_3=x\\ \lambda_1+\lambda_3=y\\ \lambda_1+\lambda_2=z\end{cases}\Rightarrow\begin{cases}\lambda_1=\frac{1}{2}(-x+y+z)\\ \lambda_2=\frac{1}{2}(x-y+z)\\ \lambda_3=\frac{1}{2}(x+y-z)\end{cases}$$

So the family $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ is generatrice

Therefore the family is basis

c. For write $w=(1,1,1)$ in basis $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$

To solve the system corresponding to the relation

$$\lambda_1\vec{v}_1+\lambda_2\vec{v}_2+\lambda_3\vec{v}_3=w=(1,1,1)\Rightarrow\begin{cases}\lambda_1=\frac{1}{2}(-x+y+z)\\ \lambda_2=\frac{1}{2}(x-y+z)\\ \lambda_3=\frac{1}{2}(x+y-z)\end{cases}\Rightarrow\begin{cases}\lambda_1=\frac{1}{2}\\ \lambda_2=\frac{1}{2}\\ \lambda_3=\frac{1}{2}\end{cases}$$

In other words, $\frac{1}{2}\vec{v}_1+\frac{1}{2}\vec{v}_2+\frac{1}{2}\vec{v}_3=w$. *the coordinates of* w *in the base* $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ *are* $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$

2. ***To show that the family is free and generative the calculations are similar to those in the previous question.***

- *The calculations give:* $\frac{1}{3}\vec{v}_1-\frac{1}{3}\vec{v}_2+\frac{1}{3}\vec{v}_3=e_1$. *the coordinates of* e_1 *in the base* $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ *are* $(\frac{1}{3},-\frac{1}{3},\frac{1}{3})$
- *The calculations give:* $\frac{1}{3}\vec{v}_1-\frac{1}{3}\vec{v}_2+\frac{1}{3}\vec{v}_3=e_2$. *the coordinates of* e_2 *in the base* $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ *are* $(\frac{1}{3},\frac{2}{3},\frac{1}{3})$
- *The calculations give:* $\frac{1}{3}\vec{v}_1-\frac{1}{3}\vec{v}_2+\frac{1}{3}\vec{v}_3=e_3$. *the coordinates of* e_3 *in the base* $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ *are* $(\frac{1}{3},-\frac{1}{3},-\frac{2}{3})$

Exercise n°3

Let $E=\mathbb{R}^4$ *we consider in E a free family of 4 vectors, show that the following families of vectors of E are related?*

1. $(\vec{e}_1,\vec{e}_3)$
2. $(\vec{e}_1,\vec{e}_2+\vec{e}_3,\vec{e}_4)$
3. $(\vec{e}_1,\vec{e}_1+\vec{e}_3,\vec{e}_3)$
4. $(2\vec{e}_1+\vec{e}_2,\vec{e}_1-3\vec{e}_2,\vec{e}_4,\vec{e}_2-\vec{e}_1)$

Solution

1. *Let* x_1 *and* x_3 *such as:* $x_1\vec{e}_1+x_3\vec{e}_3=\vec{0}$

so we have $x_1\vec{e}_1+x_2\vec{e}_2+x_3\vec{e}_3+x_4\vec{e}_4=\vec{0}$

as the family $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$ *is free, we have:* $x_1 = x_2 = x_3 = x_4 = 0$ *.especially* $x_1 = x_3 = 0$.

Donc $(\vec{e}_1, \vec{e}_3)$ *is free.*

2. *Let* x_1, x_2 *and* x_3 *such as:* $x_1\vec{e}_1 + x_2(\vec{e}_2 + \vec{e}_3) + x_3\vec{e}_4 = \vec{0} \Leftrightarrow x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4 = \vec{0}$

we have: $x_1 = x_2 = x_3 = 0$ *.especially* $x_1 = x_3 = 0$.

Donc $(\vec{e}_1, \vec{e}_3 + \vec{e}_2, \vec{e}_4)$ *is free.*

3. *Let* x_1, x_2 *and* x_3 *such as:* $x_1\vec{e}_1 + x_2(\vec{e}_2 + \vec{e}_3) + x_3\vec{e}_3 = \vec{0} \Leftrightarrow (x_1 + x_2)\vec{e}_1 + (x_2 + x_3)\vec{e}_3 = \vec{0}$

$\Rightarrow \begin{cases} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$ *is impossible that* $x_1 = x_2 = x_3 = 0$ *so this family is dependent.*

4. *Let* x_1, x_2, x_3 *and* x_4 *such as:*

$$x_1(2\vec{e}_1 + \vec{e}_2) + x_2(\vec{e}_1 - 3\vec{e}_2) + x_3\vec{e}_4 + x_4(\vec{e}_2 - \vec{e}_1) = \vec{0} \Leftrightarrow (2x_1 + x_2 - x_4)\vec{e}_1 + (x_1 - 3x_2 + x_4)\vec{e}_2 + 0\vec{e}_3 + x_3\vec{e}_4 = \vec{0}$$

$$\Rightarrow \begin{cases} 2x_1 + x_2 - x_4 = 0 \\ x_1 - 3x_2 + x_4 = 0 \\ x_3 = 0 \end{cases}$$

is impossible that $x_1 = x_2 = x_3 = x_4 = 0$ *so this family is dependent.*

Chapter 3:

Linear Applications

VI. Definition

We call a linear application (or linear transformation) $f: E \to E$, an application with the following properties :

- $f(u+v) = f(u) + f(v), \quad \forall u, v \in E$
- $f(\lambda u) = \lambda f(u), \quad \forall \lambda \in \mathrm{K}$

- A bijective linear map is called **Isomorphism**
- A linear map from E to E is called **Endomorphism**
- An isomorphism from E to E is called **Automorphism**
- A linear map from E to K is called **linear form**

Remark :

F is a linear **application** if and only if:

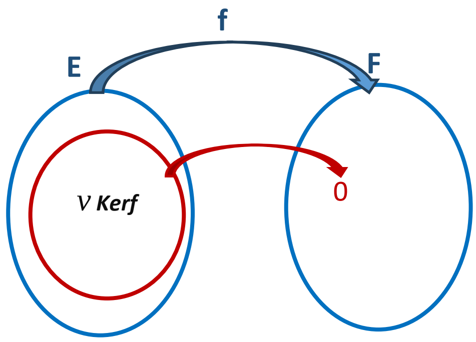
$$\forall u, v \in E, \forall \alpha, \beta \in \mathrm{K}, f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$$

II. Kernel of a linear application

The **kernel** (or **null space**) of a linear application, denoted **Kerf**, is the set of all input vectors that the linear application maps to the zero vector E that f cancels :

If $f: E \to F$

Kerf:={v∈E|f(v) = 0}.

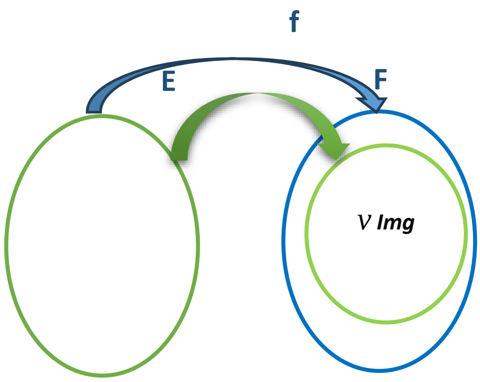


III. Image of a linear application

The image of a linear application, denoted Imf, it represents all vectors in the codomain that can be reached by applying the linear transformation to some vector in the domain

if $f: E \to F$ is a linear application

Imf:={f(v)|v ∈E}.



IV. ***General operations on linear applications***

Notation: *We note* $\mathcal{L}(E,F)$ *the space of linear applications of E in F*

Proposition: $\mathcal{L}(E,F)$ *is a vector space.*

$\rightarrow$ *Si* $f, g \in \mathcal{L}(E,F)$, *so* $\lambda f + g \in \mathcal{L}(E,F)$

V. ***Restriction à un sous espace vectoriel*** :

if $f \in \mathcal{L}(E,F)$ *and G is a vector subspace of E, then* $f : G \rightarrow F$ *is a linear application*

$x \mapsto f(x)$

Composition :

If $(g \circ f)(\lambda x + y) = g(f(\lambda x + y))$

$$= g(\lambda f(x) + f(y))$$
$$= \lambda g(f(x)) + g(f(y))$$
$$= \lambda g(g \circ f)(x) + (g \circ f)(y)$$

Applications

Exersice 1

Consider the following maps defined from $\mathbb{R}^2 \to \mathbb{R}^2$. *Which of them are linear?*

1. $f:(x,y)\mapsto(x+2y,2x-y)$
2. $g:(x,y)\mapsto(x+2y,2x-y+1)$
3. $h:(x,y)\mapsto(x+y,2xy)$
4. $k:(x,y)\mapsto(x+y,x-y^2)$

Solution

1. a. $f(u+v)=f(x+x',y+y')$

$$f(x+x',y+y')=((x+x')+2(y+y'),2(x+x')-(y+y'))$$
$$=(x+2y+x'+2y',2x-y+2x'-y')$$
$$f(x,y)+f(x'+y')=(x+2y,2x-y)+(x'-2y',2x'-y')$$
$$=f(u)+f(v)$$

b. $f(\lambda u)=f(\lambda x,\lambda y)=(\lambda x+2\lambda y,2\lambda x+\lambda y)$

$$=\lambda(x+2y,2x\text{-}y)=\lambda f(x,y)$$

So f is linear

2. *a.* $g(u+v)=g(x+x',y+y')$

$$g(x+x',y+y')=((x+x')+2(y+y'),2(x+x')-(y+y')+1)$$
$$=(x+2y+x'+2y',2x-y+2x'-y'+1)$$
$$g(x,y)+g(x'+y')=(x+2y,2x-y+1)+(x'-2y',2x'-y'+1)$$
$$=(x+2y+x'-2y',2x-y+2x'-y'+2)$$

Since $g(x+x',y+y')\neq g(x,y)+g(x'+y')$ *due to the extra* $+1$, ***additivity fails***

Thus, g ***is not linear****.*

3. *a.* $h(u+v)=h(x+x',y+y')$

$$h(x+x',y+y')=((x+x')+(y+y'),2(x+x')(y+y'))$$
$$=(x+y+x'+y',2x+2x'+2y+2y')$$
$$h(x,y)+h(x'+y')=(x+y,2xy)+(x'+y',2x'y')$$
$$=(x+y+x'+y',2xy+2x'y')$$

Since $h(x+x',y+y')\neq h(x,y)+h(x'+y')$

Thus, h ***is not linear****.*

4. *a.* $k(u+v)=k(x+x',y+y')$

$k(x+x',y+y')=((x+x')+(y+y'),(x+x')-(y+y')^2)$
$h(x,y)+h(x'+y')=(x+y,x-y^2)+(x'+y',x'-y'^2)$

Since $(y+y')^2 \neq y^2+y'^2$

Thus, k ***is not linear***.

Exersice 2

Let $f:\mathbb{R}^3 \to \mathbb{R}^2$ *defined by* $f(x,y,z) \mapsto (2x+y-z,x+y)$

1. *Demonstrate that f is linear*
2. *Determine its kernel, its image, f is it injective?surjective?*

Solution

1. Let $u=(x,y,z)$ *and* $v=(x',y',z')$ *of* $\mathbb{R}^3$ *and* $\lambda \in \mathbb{R}$

a. $f(u+v)=f(x+x',y+y',z+z')$

Suppose that: $X=x+x'=2x, Y=y+y'=y$ *and* $Z=z+z'=-z$

$$=(2(x+x')+y+y'-(z+z'),(x+x')+(y+y'))$$
$$=(2x+2x'+y+y'-z-z',x+x'+y+y')$$
$$=(2x+y-z+2x'+y'-z',x+y+x'+y')$$
$$=(2x+y-z,x+y)+(2x'+y'-z',x'+y')$$
$$=f(u)+f(v)$$

b. $f(\lambda u)=f(\lambda x,\lambda y,\lambda z)=(2\lambda x+\lambda y-\lambda z,\lambda x+\lambda y)$

$$=\lambda(2x+y-z,x+y)=\lambda f(u)$$

So f is linear

2. *Let* $u=(x,y,z) \in \mathbb{R}^3$

$u \in Ker(f) \Leftrightarrow f(u))=o_{\mathbb{R}^2}$

$\Leftrightarrow (2x+y-z,x+y)=(0;0)$

$$\Leftrightarrow \begin{cases} 2x+y-z=0 \\ x+y=0 \end{cases} \Rightarrow \begin{cases} z=-y \\ x=-y \end{cases}$$

$Ker(f)=\{(-y,y,-y)/y\in\mathbb{R}^2=vect((-1,1,-1))$

So dim $Ker(f)=1$

. *Ker(f) is not reduced at (0,0,0) so f is not injective*

. *rank theorem :* $dimE = dimKer(f) + dimIm(f)$

$dim\mathbb{R}^3 = 3 = 1 + dimIm(f)$

$$\left.\begin{array}{l} dimIm(f) = 2 \\ dimIm(f) \in \mathbb{R}^2 \end{array}\right\} \Rightarrow dimIm(f) = \mathbb{R}^2$$

f is surjective

Exersice 3

we consider the application $f : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by :*

$f(x,y,z) \mapsto (x+2y+z, x-y, 2x+y+z)$

1. *determine a kernel basis of f and its dimension*
2. *f is it injective?surjective? give the rank of f.*
3. *determine image of f.*

Solution

1. *Let* $u = (x,y,z) \in \mathbb{R}^3$

$u \in Ker(f) \Leftrightarrow f(u)) = o_{\mathbb{R}^3} \Leftrightarrow (x+2y+z, x-y, 2x+y+z) = (0,0,0)$

$$\Rightarrow \begin{cases} x+2y+z=0 \\ x-y=0 \\ 2x+y+z=0 \end{cases} \Leftrightarrow \begin{cases} 3x=0 \\ x=y \\ 3x+z=0 \end{cases} \Leftrightarrow \begin{cases} y=x \\ z=-3x \end{cases}$$

$\Leftrightarrow (x,y,z) = (x,x,-3x) \Leftrightarrow (x,y,z) = x(1,1,-3)$

$\Leftrightarrow (x,y,z) \in vect(1,1,-3)$

So dim $Ker(f) = 1$

2. *Ker(f) ≠ (0,0,0) so f is not injective*

- *rank theorem :* $dimE = dimKer(f) + dimIm(f)$

  $dim\mathbb{R}^3 = 3 = 1 + dimIm(f)$

$$\left.\begin{array}{l} dimIm(f) = 2 \\ dimIm(f) \in \mathbb{R}^3 \end{array}\right\} \Rightarrow dimIm(f) \neq \mathbb{R}^3$$

F is not surjective

3. $dimIm(f) = 2$

Exersice 4

Let $f:\mathbb{R}^3 \to \mathbb{R}^3$ *defined by* $f(x,y,z) \mapsto (3x-y+4z, 2x-y+z, 5x-y-z)$

1. *Demonstrate that f is linear.*
2. *Determine the kernel, image.*
3. *f is it injective?surjective?*

Solution

1. *f linear* $\Leftrightarrow \forall u,v \in E, \forall \alpha,\beta \in \mathrm{K},\ f(\alpha u+\beta v)=\alpha f(u)+\beta f(v)$

$f(x,y,z)=(3x-y+4z, 2x-y+z, 5x-y-z)$

Let $u=(x,y,z)$ *and* $v=(x',y',z')$ *of* $\mathbb{R}^3$ *and* $\alpha,\beta \in \mathbb{R}$

$f(\alpha u+\beta v)= f(\alpha x+\beta x', \alpha y+\beta y', \alpha z+\beta z')$

Suppose that: $X=x+x', Y=y+y$ *and* $Z=z+z'$

= (3X-Y+4Z,2X-Y+Z,5X-Y-Z)

$=(3(\alpha x+\beta x')-(\alpha y+\beta y')+4(\alpha z+\beta z'), 2(\alpha x+\beta x')-(\alpha y+\beta y')$
$+(\alpha z+\beta z'), 5(\alpha x+\beta x')-(\alpha y+\beta y')-(\alpha z+\beta z'))$
$=(3\alpha x-\alpha y+4\alpha z+3\alpha x'-\alpha y'+4\alpha z', 2\alpha x-\alpha y$
$+\alpha z+2\alpha x'-\alpha y'+\alpha z', 5\alpha x-\alpha y-\alpha z+5\alpha x'-\alpha y'-\alpha z')$
$=\alpha(3x-y+4z, 2x-y+z, 5x-y-z)+\beta(3x'-y'+4z', 2x'-y'+z', 5x'-y'-z')$

$f(u+v)=\alpha f(x,y,z)+\beta f(x',y',z')$
$f(u+v)=\alpha f(u)+\beta f(v)$

So f is linear application

2. *Let* $u=(x,y,z) \in \mathbb{R}^3$

$u \in Ker(f) \Leftrightarrow f(u))=o_{\mathbb{R}^3} \Leftrightarrow (3x-y+4z, 2x-y+z, 5x-y-z)=(0,0,0)$

$$\Rightarrow \begin{cases} 3x-y+4z=0 \\ 2x-y+z=0 \\ 5x-y-z=0 \end{cases} \Leftrightarrow \begin{cases} x+3z=0 \\ -6z-y+z=0 \\ -15z+5z-z=0 \end{cases} \Leftrightarrow \begin{cases} x=-3z \\ y=-5z \\ z=0 \end{cases}$$

$\Leftrightarrow (x,y,z)=(0,0,0)$

So dim $Ker(f)=0$ and *dimIm* $(f)=3$

EXAMEN

MODEL

University of RELIZANE *Time: 1h30min*

Level: 1st year SM

جامعة غليزان
RELIZANE UNIVERSITY

Math1 Exam -January 2025-

Analysis part:

Exercice n°1 : *(05pts)*

we define on $\mathbb{R}^*$ the binary relation by :

$$\forall\, x, y \in \mathbb{R}^*, x\mathcal{R}y \Leftrightarrow y(x^2+1) = x(y^2+1)$$

1. *show that $\mathcal{R}$ is an equivalence equation.*
2. *give the equivalence class of* ***2***

Exercice n°2 : *(05 pts)*

We consider the application $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x}{x^2+1}$

1. *Calculate f(2) and f(1/2) and solve in $\mathbb{R}$ the equation f(x)=2*
2. *the application f is injective ? surjective ? bijective ?*
3. *What restriction must be made on the domain set so that f becomes a bijective? In this case, explain the reciprocal application.*

Algebra part:

Exercice n°1 : *(05pts)*

we provide $A = \mathbb{R} \to \mathbb{R}$ the two internal composition law $*$ defined by:

$$(x,y) + (x',y') = (x+x', y+y') \quad and \quad (x,y) * (x',y') = ((xx', xy' + x'y)$$

1. *Show that (A,+) is a commutative group*
2. *Show that the law * is commutative.*
3. *Show that * is associative.*
4. *Determine the neutral element of for the law * .*
5. *Show that (A,+,*) is a commutative ring.*

Exercice n°2 : *(05 pts)*

The following family is a base?

$$e_1 = (1,-2,0), \qquad e_2 = (0,1,1), \qquad e_3 = (1,1,2)$$

University of Science and Technology of Oran (MB)

Faculty of Physics

Department of Basic Physics Teaching

1st Year LMD (ST)

Math 1 Exam

Monday, 13/01/2025 (Duration: 1h30min)

Exercise 1 ***(04 pts)***

Let f be the application from the set $I \subseteq [-1,1]$ *to* $J \subseteq [-1,1]$ *, defined by:*

$$f(x) = x^2$$

1. *Give sets I and J such that f is* ***injective but not surjective****.*
2. *Give sets I and J such that f is* ***surjective but not injective****.*
3. *Give sets I and J such that f is* ***neither injective nor surjective****.*
4. *Give sets I and J such that f is* ***both injective and surjective****.*

Exercise 2 ***(05 pts)***

Let: $F = \{(x, y, z, t) \in \mathbb{R}^4 / x - y = \text{t} \text{ and } t = x\}$

1. *Show that F is a* ***vector subspace*** *of* $\mathbb{R}^4$.
2. *Give a* ***basis*** *for F.*
3. ***Calculate the dimension*** *of F*

Exercise 3 ***(05 pts)***

Define on $\mathbb{R}$ *, the internal operation* $*$ *as follows:* $x * y = \ln(e^x + e^y - 1)$

Is $(\mathbb{R}, *)$ an abelian group?

Exercise 4 ***(03 pts)***

Let the function f be defined by:

$$f(x)=\begin{cases}1-x\sin\dfrac{1}{x} & x\neq 0\\ 2 & x=0\end{cases}$$

1. *Give the* ***domain of definition*** *of f*
2. *Study the* ***continuity and differentiability*** *of f over its domain.*
3. ***Give the derivative*** *of f.*

Exercise 5 ***(03 pts)***

Give the limit devellopment to order 3 near 0 for the following functions:

1. $f_1(x)=\dfrac{e^x-1-x}{x^2}$
2. $f_2(x)=\ln(4+x)$

Good luck.Mme BESSAI

University of RELIZANE

Level: 1st year SM

Time: 1h30min

جامعة غليزان
RELIZANE UNIVERSITY

Make-Up Exam Math 1 -May 2025-

Analysis part:

Exercise n°1 : (05pts)

Let f, g and h be three functions defined by: $f(x)=\frac{1}{x^2-4}, g(x)=x^3+3x^2+5, h(x)=\sqrt{4-2x}$.

- Determine the sets $D_f, D_g, D_h, C_{D_f}, C_{D_g}, C_{D_h}, D_f \cup D_h, D_g \setminus D_h, D_g \cap D_h$.

Exercise n°2 : (05 pts)

We consider $f:\mathbb{R}\to\mathbb{R}$ the application defined by:

$$f(x)=x^2+3x+\frac{9}{4}$$

1. Calculate $f(0), f(-3)$ and $f^{-1}(-1)$.
2. Is f injective? surjective? bijective?
3. . Give intervals f, J so $f:I\to J$ that is bijective; then give the reciprocal f^{-1} of f.

Algebra part:

Exercice n°1 : (05pts)

We define $\mathbb{Z}$ a binary operation * as follows :

$$\forall(x,y)\in\mathbb{Z}, x*y=xy(x+y)$$

1) Show that the law * is commutative.

2) Calculate (1 * (-1)) * 2 and 1 * ((-1) * 2). Is the law * associative?

3) Solve the equation in $\mathbb{Z}$: $x*x=16$.

4) Show that * does not admit an identity element.

Exercice n°2 : (05 pts)

The following family is a base?

$$e_1=(1,-2,0),\qquad e_2=(0,1,1),\qquad e_3=(1,1,2)$$

Exam Revision for Math 1, Year 1 (ST.SM.MI)

Professor Mouslim (January 2025)

Exercise 1:

Consider the function f: $\mathbb{R} \to \mathbb{R}$ *defined by:*

$$f(x) = \begin{cases} x+2 & \text{if } x \geq 0 \\ 2 & \text{if } x < 0 \end{cases}$$

1. *Compute* $f(\mathbb{R}), f([-1,1]), f^{-1}(\{0\})$ *and* $f^{-1}(]2,3[)$

2. *Is f injective? Is it surjective?*

Exercise 2:

Consider the function f given by:

$$f(x) = \begin{cases} \dfrac{\arctan(\sinh x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

1. *State the domain of definition of f.*

2. *Investigate the continuity and differentiability of f on its domain.*

3. *Compute the **limited expansion** (Taylor expansion) of f up to order 3 near 00.*

Exercise 3:

Let : $F = \{(x, y, z) \in \mathbb{R}^3 / 5x + 4y - z = 0\}$

1. *Show that F is a vector subspace of* $\mathbb{R}^3$.

2. *Give a basis for F and deduce the dimension of F.*

3. *Is F equal to* $\mathbb{R}^3$ *?*

Exercise 4:

Define the internal law $*$ *on* $\mathbb{R} / \{4\}$ *as follows:*

$$x * y = \frac{1}{4}(xy + x + y)$$

Is $(\mathbb{R} / \{4\}, *)$ an abelian group?

Saad Dahlab University, Blida **First Year LMD (Bachelor's)**

TCST (Science and Technology) **2021/2022**

Mathematics 1 Exam (Wave 1)

Duration: 1 hour

Exercise 1: (4.5 points)

Define a relation $\Re$ *on* $\mathbb{R}\times\mathbb{R}_+$ *as follows:* $\forall(a,b),(c,d)\in\mathbb{R}\times\mathbb{R}_+ \Leftrightarrow a(1+d)=c(1+b)$

1. *Show that* $\Re$ *is an equivalence relation.*
2. *Determine the equivalence classes* $cl(0,0)$ *and* $cl(-1,\frac{1}{2})$.

Problem: (15.5 points = 9.75 + 5.75)

I. *Consider the function f defined by:* $f(x)=\dfrac{1}{1-e^{2x-1}}$

1. *Determine* D_f, *the* ***domain of definition*** *of f.*
2. *Compute* $f(\left\{3,0,-\frac{1}{2}\right\})$ *and* $f^{-1}(\left\{\frac{3}{2},\frac{1}{3},1\right\})$
3. *Show that ff is* ***injective***.
4. *Is f* ***surjective***? *Justify your answer.*
5. *prove that f is* ***bijective*** *from* $\left]-\infty,\frac{1}{2}\right[$ *to an interval I (to be determined).*
6. *Using the* ***Intermediate Value Theorem***, *show that f(x)=−0.4 has a* ***unique solution*** *between 1 and* $\frac{3}{2}$

II. 1. *Calculate the following limits:*

a) $\lim\limits_{x\to\frac{1}{2}}(2x)^{\frac{1}{1-e^{2x-1}}}$

b. $\lim\limits_{x\to\frac{1}{2}}\dfrac{\cos(4\pi x)-1}{(2x-1)^2}$

2. *Consider the function g defined by:* $g(x)\begin{cases}\dfrac{1}{1-e^{2x-1}}.\ln(2x) & if \quad x\succ\dfrac{1}{2}\\ \dfrac{1}{2\pi^2}.\dfrac{\cos(4\pi x)-1}{(2x-1)^2} & if \quad x\prec\dfrac{1}{2}\end{cases}$

- *Show that g admits a continuous extension at* $x=\frac{1}{2}$, *denoted* $\bar{g}$, *and define this extension.*

Bibliography

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