People's Democratic Republic of Algeria
Ministry of higher education and scientific research
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MULTILINEAR ALGEBRA (II)

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Course intended primarily for students of Master 2

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Introduction

There are several ways to construct new vector spaces from a family of vector spaces over the same field. Two of the most important of these constructions are the direct sum and the vector space of all linear transformations.

This course introduces a basic concept which has a major importance in many areas of sciences such as applied mathematics, physics and engineering, called tensor product, that combines two vector spaces V and W into a new vector space $V \otimes W$.

Chapter 1 Tensor products (Part 1)

Chapter contents

In this chapter, we will mainly be concerned with finite dimensional vector spaces over a field $\mathbb{F}$ of characteristic zero. We will give the definition of the tensor product of vector spaces (resp. tensor product of linear mappings). Also various properties of the tensor product are explained in this chapter.

1.1 Linear and bilinear maps

Definition 1.1.1 Linear Transformation (Linear mapping)

Let V, W be two vector spaces over the same field $\mathbb{F}$. A function $f\colon V \longrightarrow W$ is called a *linear transformation* from V to W if the following hold for all vectors u, v in V and for all scalars $\alpha \in \mathbb{F}$.

(1) $f(u+v) = f(u) + f(v)$ (additivity),

(2) $f(\alpha u) = \alpha f(u)$ (homogeneity).

Note 1.1.2

The set of all $\mathbb{F}$-linear transformation $f : V \longrightarrow W$ is a vectors space. If f, g are two linear maps and $\alpha \in \mathbb{F}$, the sum and scalar multiplication are defined by the following formulas.

$$(f+g)(v) = f(v) + g(v),$$

and

$$(\alpha f)(v) = \alpha f(v).$$

We denote the set of all such linear transformations, from V to W, by $\mathcal{L}(V, W)$ or $\mathrm{Hom}(V, W)$.

Definition 1.1.3 Linear Functional (or 1-form)

Let V be a vector space. Define

$$V^* = \mathcal{L}(V, \mathbb{F}).$$

V^* is called the **dual space** of V.

The elements of V^* are called **linear functional**. So a linear functional ϕ on V is a linear transformation $\phi : V \longrightarrow \mathbb{F}$.

Lemma 1.1.4 Dual basis

Suppose that $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis for the finite dimensional vector space V. Define $f_i \in V^*$ by

$$f_i(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Then

$$\mathcal{B}^* = \{f_1, f_2, \ldots, f_n\}$$

is a basis for V^*, and it's called the dual basis of $\mathcal{B}$.

Proof. Let $\alpha_1, \ldots, \alpha_n$ be scalars such that

$$\sum_{i=1}^{n} \alpha_i f_i = 0.$$

Then for all $r \in \{1, ..., n\}$, we have

$$\sum_{i=1}^{n} \alpha_i f_i(v_r) = 0.$$

So

$$\sum_{i=1}^{n} \alpha_i \delta_{ij} = 0.$$

So $\alpha_r = 0$. Therefore the set $\{f_1, \ldots, f_n\}$ is linearly independent. Clearly for all $f \in V^*$, we have

$$h = \sum_{i=1}^{n} h(v_i) f_i.$$

□

Corollary 1.1.5

If V is a finite dimensional vector space, then $\dim V^* = \dim V$.

Definition 1.1.6 Bilinear maps

Let U, V and W be $\mathbb{F}$-vectors spaces. A mapping $f : V \times W \longrightarrow U$ is called a bilinear mapping, if it is linear in each variable. That means : for all $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$ and $a \in \mathbb{F}$, we have

$$f(au_1 + u_2, v) = af(u_1, v) + f(u_2, v),$$
$$f(u, av_1 + v_2) = af(u, v_1) + f(u, v_2).$$

Note 1.1.7

The set of all $\mathbb{F}$-bilinear map $f : U \times V \longrightarrow W$ is a vectors space. If f, $g \in \mathcal{L}(V, W; U)$ are bilinear maps and $a \in \mathbb{F}$, the sum and scalar multiplication are defined by the following formulas.

$$(f + g)(v, w) = f(v, w) + g(v, w),$$

and

$$(af)(v, w) = af(v, w).$$

We denote the set of all $\mathbb{F}$-bilinear maps from $U \times V$ into W by $\mathrm{Bil}(U \times V, W)$ or $\mathcal{L}(U, V; W)$.

Lemma 1.1.8 A basis $\mathcal{L}(V, W; \mathbb{F})$

Let V and W be two $\mathbb{F}$ vector spaces. Take bases $\{v_1, \ldots, v_n\}$ for V and $\{w_1, \ldots, w_m\}$ for W. Let $\{f_1, \ldots, f_n\}$ for V and $\{g_1, \ldots, g_m\}$ be their dual bases.

For all $1 \leq i \leq n$ and $1 \leq j \leq m$, we define the mapping $h_{ij} : V \times W \longrightarrow \mathbb{F}$ by

$$h_{ij}(v, w) = f_i(v) g_j(w).$$

Then, the set

$$\{h_{ij} \mid 1 \leq s \leq n \text{ and } 1 \leq r \leq m\}$$

form a basis for $\mathcal{L}(V, W; \mathbb{F})$.

Proof. Clearly h_{ij} is bilinear, and for all $1 \leq s \leq n$ and $1 \leq r \leq m$,

$$h_{ij}(v_r, w_s) = \begin{cases} 1 & \text{if} \quad i = r \quad \text{and} \quad j = s \\ 0 & \text{otherwise} \end{cases}$$

The set $\{h_{ij} \mid 1 \leq i \leq n$ is linearly independent and $1 \leq j \leq m\}$ form a basis for $\mathcal{L}(V, W; \mathbb{F})$, and every element $h \in \mathcal{L}(V, W; \mathbb{F})$, can be written as

$$h = \sum_{i,j} h(v_i, w_j) h_{ij}.$$

□

Corollary 1.1.9

If $\dim V = n$ and $\dim W = m$, then $\dim \mathcal{L}(V, W; \mathbb{F}) = nm$.

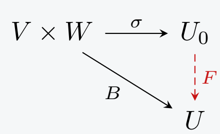
1.2 Linearization of bilinear mappings

Definition 1.2.1 Conditions **(T1)** and **(T2)**

Let V and W be finite dimensional $\mathbb{F}$-vector spaces. We say that V and W satisfies the condition **(T1)** and **(T2)**, if there exist a $\mathbb{F}$-vector space U_0 and a bilinear mapping $\sigma \in \mathcal{L}(V, W; U_0)$ for which such that

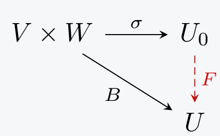
(T1) U_0 is generated by the image $\sigma(V \times W)$ of σ.

(T2) For any $B \in \mathcal{L}(V, W; U)$, there exists a $\mathbb{F}$-linear mapping $F : U_o \longrightarrow U$ such that $B = F \circ \sigma$:



Definition 1.2.2 Universality Property: condition **(T)**

Let V and W be finite dimensional $\mathbb{F}$-vector spaces. We say that V and W satisfies the condition **(T)**, if there exist a $\mathbb{F}$-vector space U_0 and a bilinear mapping $\sigma \in \mathcal{L}(V, W; U_0)$ such that for any $B \in \mathcal{L}(V, W; U)$, there exists one and only $\mathbb{F}$-linear mapping $F : U_o \longrightarrow U$ for witch $B = F \circ \sigma$:



Remark 1.2.3. Let V be a vector space and S be a subset of V. The intersection of all subspaces of V containing S is also a subspace containing S and is the smallest among them. This space is called the subspace generated (or spanned) by S and is denoted by $\text{span}(S)$. It is easy to see that $\text{span}(S)$ is

the set of all finite linear combination of elements of S. When $\operatorname{span}(S) = V$, V is said to be generated by S and S is called a set of generators (or a generating set) of V,

Lemma 1.2.4

$$(\ \mathbf{T1} \wedge \mathbf{T1}\ ) \iff (\mathbf{T})$$

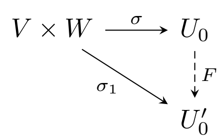
Proof. Suppose that (U_0, σ) satisfies **(T1** and **(T2)**. The existence of F follows from **(T2)**. Suppose that F and F' are linear mappings $U_0 \longrightarrow U$ such that

$$B = F \circ \sigma = F' \circ \sigma.$$

Since F and F' are linear mappings that coincide on the generating set $(V \times W)$ of U_0, we have $F = F'$, which shows that (U_0, σ) satisfies **(T)**.

Conversely, suppose that (U_0, σ) satisfies **(T)**. Clearly we have **(T2)**. Let U_0' be the subspace of U_0 generated by $\sigma(V \times W)$. Since the image of is contained in U_0', σ can be considered as a mapping of $V \times W$ into U_0', which we denote by σ_1.

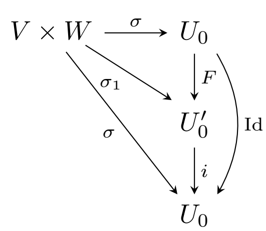
Applying **(T2)** to σ_1, we have a linear mapping F such that $\sigma_1 = F \circ \sigma$



Let i be the inclusion mapping of U_0' into U. Then

$$\sigma = i \circ \sigma_1$$

We have



Therefore

$$\sigma = (i \circ F) \circ \sigma \tag{1.1}$$

Clearly

$$\sigma = \operatorname{Id} \circ \sigma \tag{1.2}$$

By the uniqueness of the linear mapping F, we get from (1.1) and (1.2)

$$i \circ F = \operatorname{Id}$$

Hence i is surjective, and so

$$U_0' = U_0.$$

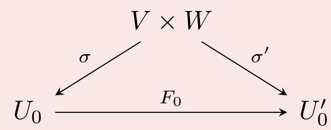
□

Theorem 1.2.5 Linearization of bilinear mappings

Let V and W be finite dimensional $\mathbb{F}$-vector spaces.

(1) There exist a $\mathbb{F}$-vector space U_0 and a bilinear mapping $\sigma : V \times W \longrightarrow U_0$ which satisfy the condition **(T)** .

(2) The pair (U_0, σ) is unique in the following sense: If the pairs (U_0, σ) and (U_0', σ') consisting of a $\mathbb{F}$-vector space and a $\mathbb{F}$-bilinear mapping satisfy condition **(T)**, then there exists a unique linear isomorphism $F_0 : U_0 \longrightarrow U_0'$ such that $\sigma' = F \circ \sigma$:



Proof. **(1)** Assume that $\dim V = n$ and $\dim W = m$. By using Lemma 1.2.4, we will prove that any vector space U_0 of dimension nm satisfies the conditions **(TI)** and **(T2)** for an appropriate σ. Take

$$\mathcal{B}_V = \{v_1, \dots, v_n\} \quad \text{a basis for} \quad V$$

$$\mathcal{B}_W = \{w_1, \dots, w_m\} \quad \text{a basis for} \quad W$$

$$\mathcal{S} = \{u_{ij} \mid 1 \leq i \leq n \quad \text{and} \quad 1 \leq j \leq m\} \quad \text{a basis for} \quad U_0.$$

Define the bilinear mapping $\sigma : V \times W \longrightarrow U_0$ as follows:

$$\sigma(v_i, w_j) = u_{ij}$$

That means for all

$$v = \sum_{i=1}^{n} \alpha_i v_i \quad \text{and} \quad w = \sum_{j=1}^{m} \beta_j w_j,$$

we have

$$\sigma(v, w) = \sum_{i,j} \alpha_i \beta_j u_{ij}$$

By construction of σ, it's clear that $\sigma \in \mathcal{L}(V, W; U_0)$ and $\operatorname{span}(\sigma(V \times W)) = U_0$. So the condition **(T1)** is satisfied. It remain to show that the condition **(T2)** is also satisfied.

Let $B : V \times W \longrightarrow U$ be a bilinear mapping. Define the function $F : U_0 \longrightarrow U$ by

$$F(u) = F\Big(\sum_{i,j} \gamma_{ij} u_{ij}\Big) = \sum_{i,j} \gamma_{ij} B(v_i, w_j).$$

We have for all i, j

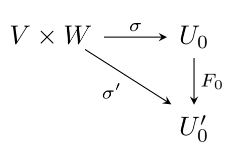
$$F \circ \sigma(v_i, w_j) = F(u_{ij}) = B(v_i, w_j).$$

Hence

$$B = F \circ \sigma.$$

Therefore (U_0, σ) has the condition **(T2)**.

(2) Assume that (U_0, σ) and (U_0', σ') have the property mentioned in **(1)**.

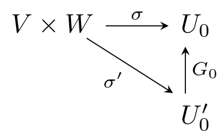


Since σ' is a bilinear mapping: $V \times W \longrightarrow U_0'$, applying **(T)** to (U_0, σ), we have a linear mapping $F_0 : U_0 \longrightarrow U_0'$ such that

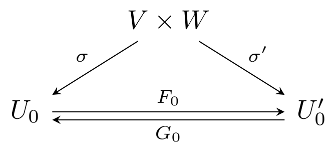
$$F_0 \circ \sigma = \sigma' \tag{1.3}$$

Similarly, there is a linear mapping $G_0 : U_0' \longrightarrow U_0$ such that

$$G_0 \circ \sigma' = \sigma \tag{1.4}$$



Hence, we have the following commutative diagram:



From (1.3) and (1.5), we get

$$\sigma = G_0 \circ F_0 \circ \sigma \tag{1.5}$$

Clearly,

$$\sigma = \mathrm{Id}_{U_0} \circ \sigma. \tag{1.6}$$

By the uniqueness in the condition **(T)** we obtain from (1.5) and (1.6),

$$G_0 \circ F_0 = \mathrm{Id}_{U_0}$$

Similarly, we can show that

$$F_0 \circ G_0 = \mathrm{Id}_{U_0'}$$

Therefore F_0 is an isomorphism.

□

Corollary 1.2.6

Let $\mathcal{B}_V = \{v_1, \ldots, v_n\}$ and $\mathcal{B}_W = \{w_1, \ldots, w_b\}$ be respectively basis for V and W.

Let $\{f_1, \ldots, f_n\}$ and $\{g_1, \ldots, g_m\}$ are respectively the dual basis of $\mathcal{B}_V$ and $\mathcal{B}_W$

Consider $U_0 = \mathcal{L}(V, W; \mathbb{F})$.

For all $1 \leq i \leq n$, and $1 \leq j \leq m$, let $h_{ij} : V \times W \longrightarrow \mathbb{F}$ be the bilinear form given by

$$h_{ij}(v, w) = f_i(v) g_j(w)$$

The set

$$\{h_{ij} \mid 1 \leq i \leq i,\ 1 \leq j \leq m\}$$

form a basis for U_0.

Then $U_0 = \mathcal{L}(V, W; \mathbb{F})$ is a vector space of dimension nm satisfies the conditions **(TI)** and **(T2)**, where the bilinear mapping $\sigma : V \times W \longrightarrow \mathcal{L}(V, W; \mathbb{F})$ is given by

$$\sigma(u_i, v_j) = h_{ij}.$$

1.3 Tensor products of two vector spaces

We are now ready to define the tensor product of $\mathbb{F}$-vector spaces.

Definition 1.3.1 Tensor product

Let V and W be $\mathbb{F}$-vector spaces. The pair (U_0, σ) consisting of a $\mathbb{F}$-vector space U_0 and a bilinear mapping $\sigma : V \times W \longrightarrow U_0$, satisfying the property **(T)**, the existence of which is assured by Theorem 1.2.5 is called a tensor product of V and W.
We write

$$U_0 = V \otimes W \quad \text{and} \quad \sigma(u, w) = v \otimes w.$$

The mapping σ is called the canonical mapping of a tensor product $V \otimes W$.

Example 1.3.2

Let $n, m \in \mathbb{N}$, $V = \mathbb{F}^n$ and $W = \mathbb{F}^m$ Then $V \otimes W = \mathbb{F}^{nm}$ is a tensor product of V and W whose canonical bilinear mapping σ is given by:

$$\begin{aligned} \sigma \ : & \mathbb{F}^n \times \mathbb{F}^m \longrightarrow \mathbb{F}^{nm} \\ & \big((x_i)_{i=1}^n, (y_j)_{j=1}^m\big) \longmapsto \quad (x_i y_j)_{1\leq i\leq n, 1\leq j\leq m}. \end{aligned}$$

Remark 1.3.3.

In the following, we sometimes say that a vector space U_0 is a tensor V and W. Implicitly this means that there exists a bilinear mapping $\sigma : V \times W \longrightarrow U_0$ satisfying the property **(T)**).

The property **(T)** can be restated as follows: a tensor product U_0 of V and W is generated by

$$\{u \otimes w \mid v \in V \quad \text{and} \quad w \in W\}$$

That means, every vector $u \in V \otimes W$ can be written as

$$u = \sum_{i,j} \gamma_{ij} \ (v_i \otimes w_j)$$

for some vectors $v_i \in V$, $w_j \in W$ and scalars $\gamma_{ij} \in \mathbb{F}$.

The uniqueness property (2) of Theorem 1.2.5 can be restated as follows: if U_0 and U_0' are tensor products of V and W, then there exists a unique linear isomorphism $F : U_0 \longrightarrow U_0'$ such that F associates $v \otimes w$ in U_0 to $v \otimes w$ in U_0' for all $v \in V$ and $w \in W$.

Remark 1.3.4.

In the proof of existence in Theorem 1.2.5, we used bases for V and W Therefore, it might be difficult to understand the meaning of the tensor product.

Thus, we give another construction of a tensor product (U_0, σ) free from bases:

Let V^* and W^* be the dual spaces of V and W respectively and let U_0 be defined by

$$U_0 = \mathcal{L}(V^*, W^*; \mathbb{F}).$$

For fixed $v \in V$ and $w \in W$, the mapping $B : V^* \times W^* \longrightarrow \mathbb{F}$ defined by

$$B_{(v,w)}(f, g) = f(v)g(w)$$

is bilinear (cf. Exercise 1.7.6). So it is an element of U_0.

Consider the following map: $\sigma : V \times W \longrightarrow U_0$ defined by

$$\sigma(v, w) = B_{(v,w)}$$

which is also bilinear (cf. Exercise 1.7.6)..

Then we can show that (U_0, σ) satisfies conditions **(T1)** and **(T2)**. Take

$$\mathcal{B}_V = \{v_1, \dots, v_n\} \quad \text{a basis for} \quad V$$

$$\mathcal{B}_W = \{w_1, \dots, w_m\} \quad \text{a basis for} \quad W$$

and let $\mathcal{B}_V^* = \{f_1, \dots, f_n\}$, $\mathcal{B}_W^* = \{g_1, \dots, g_m\}$ be receptively the dual basis of V^* and W^*. We construct a basis

$$\mathcal{S} = \{u_{rs} \mid 1 \leq r \leq n \quad \text{and} \quad 1 \leq s \leq m\}$$

for U_0, where

$$u_{rs}(f, g) = \begin{cases} 1 & \text{if} \quad i = r \quad \text{and} \quad j = s \\ 0 & \text{otherwise} \end{cases}$$

Clearly

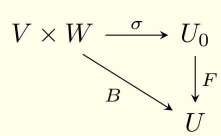
$$u_{rs} = B_{(v_r, w_s)}$$

from which we obtain condition **(T1)**.

For every $B \in \mathcal{L}(V, W; U)$, define a linear mapping $F : U_0 \longrightarrow U$ by

$$F\Big(\sum_{r,s} \gamma_{rs} u_{rs}\Big) = \sum_{r,s} \gamma_{rs} B(v_r, w_s).$$

Hence $B = F \circ \sigma$.



which implies condition **(T2)**.

Using the bilinearity of the canonical mapping σ, we can prove the following properties :

Proposition 1.3.5 Bilinearity of $\otimes$

For a, $a, b \in \mathbb{F}$, $v, v_1, v_2 \in V$ and $w, w_1, w_2 \in W$, we have

(i) $(av_1 + bv_2) \otimes w \;=\; a(v_1 \otimes w) \;+\; b(v_2 \otimes w)$.

(ii) $v \otimes (aw_1 + bw_2) \;=\; a(v \otimes w_1) \;+\; b(v \otimes w_2)$.

Proposition 1.3.6 Basis for $V \otimes W$

Let $\mathcal{B}_V = \{v_1, \ldots, v_n\}$ be a basis for V, and $\mathcal{B}_W = \{w_1, \ldots, w_m\}$ a basis for W. Then the nm vectors

$$\{v_i \otimes w_j \quad | \quad 1 \leq i \leq n \quad \text{and} \quad 1 \leq j \leq m\}$$

form a basis for $V \otimes W$. In particular,

$$\dim(V \otimes W) = \dim V \times \dim W.$$

Corollary 1.3.7

Let $v \in V$ and $w \in W$ be nonzero vectors. Then $v \otimes w \neq 0$.

Proof. If we take in the previous proposition $v_1 = v$ and $w_1 = w$, we get $v \otimes w$ is a vector in the basis of $V \otimes W$. Therefore $v \otimes w \neq 0$. □

Proposition 1.3.8

Every vector $u \in V \otimes W$ can be written as

$$u = \sum_{i,j} (e_i \otimes f_j)$$

for some vectors $e_i \in V$, $f_j \in W$.

Proof. Let $u \in V \otimes W$. Since

$$\{v_i \otimes w_j \quad | \quad 1 \leq i \leq n \quad \text{and} \quad 1 \leq j \leq m\}$$

form a basis of $V \otimes W$, the vector u can be written as

$$u = \sum_{i,j} \gamma_{ij}(u_i \otimes w_j) = \sum_i \Big(u_i \otimes \sum_j \gamma_{ij} w_j\Big) = \sum_j \Big(\sum_i \gamma_{ij} u_i \otimes w_j\Big).$$

□

Proposition 1.3.9

Let $\mathcal{S}_1 = \{v_1, \ldots, v_r\} \subseteq V$ and $\mathcal{S}_2 = \{w_1, \ldots, w_r\} \subseteq W$ and

$$u = \sum_{i=1}^{r} v_i \otimes w_i.$$

Then

(1) If the $\mathcal{S}_1$ is linearly independent, the vectors $w_1, \ldots, w_r$ are uniquely determined. Namely, if

$$\sum_{i=1}^{r} v_i \otimes w_i = \sum_{i=1}^{r} v_i \otimes w_i'$$

then $w_i' = w_i$ for all i.

(2) If the $\mathcal{S}_2$ is linearly independent, the vectors $v_1, \ldots, v_r$ are uniquely determined. Namely, if

$$\sum_{i=1}^{r} v_i \otimes w_i = \sum_{i=1}^{r} v_i' \otimes w_i,$$

then $v_i' = v_i'$ for all i.

Proof. Assume that $\dim V = n$ and $\dim W = m$.

(1) If $\mathcal{S}_1 = \{v_1, \ldots, v_r\} \subseteq V$ is linearly independent, then we can choose a basis for V of the form $\mathcal{B}_1 = \{v_1, \ldots, v_r, v_{r+1}, \ldots, v_n\}$. Let $\mathcal{B}_2 = \{f_1, \ldots, f_m\}$ a basis for W. Suppose that

$$\sum_{i=1}^{r} v_i \otimes w_i = \sum_{i=1}^{r} v_i \otimes w_i'. \tag{1.7}$$

Then

$$\sum_{i=1}^{r} v_i \otimes (w_i - w_i') = 0. \tag{1.8}$$

Since $w_i - w_i'$ is a vector in W, it's can be expressed as linear combination in its basis:

$$w_i - w_i' = \sum_{j=1}^{m} \alpha_{ij} f_j$$

Hence form (1.8), we get

$$\sum_{i=1}^{r} v_i \otimes \Big(\sum_{j=1}^{m} \alpha_{ij} f_j \Big) = 0.$$

So

$$\sum_{i=1}^{r} \sum_{j=1}^{m} \alpha_{ij} \left(v_i \otimes f_j \right) = 0.$$

Since $\{v_i \otimes f_j\}_{ij}$ is a basis for $V \otimes W$, this implies $\alpha_{ij} = 0$ for all i, j. Then by (1.7), we obtain $w_i = w_i'$ for all i.

(2) Use the same ideas as in the first item.

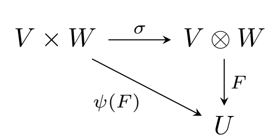
□

Proposition 1.3.10 Bilinear $\longrightarrow$ Linear

As $\mathbb{F}$-vector spaces, we have the following isomorphic

$$\mathcal{L}(V, W; U) \cong \mathcal{L}(V \otimes W, U).$$

Proof. Consider the following mapping: $\psi : \mathcal{L}(V \otimes W, U) \longrightarrow \mathcal{L}(V, W; U)$ defined by $\psi(F) = F \circ \sigma$:



where σ is the canonical mapping of a tensor product $V \otimes W$.

First, we show that F is linear. Clearly for all $F_1, F_2 \in \mathcal{L}(V \otimes W, U)$ and $\alpha \in \mathbb{F}$,

$$\psi(F_1 + \alpha F_2) = \sigma \circ (F_1 + \alpha F_2) = \sigma \circ F_1 + \alpha(\sigma \circ F_2).$$

Then

$$\psi(F_1 + \alpha F_2) = \psi(F_1) + \alpha\psi(F_2)$$

Using the property **(T)**, for all $B \in \mathcal{L}(V, W; U)$, there exists a unique $F \in \mathcal{L}(V \otimes W, U)$ such that

$$B = F \circ \sigma = \psi(F),$$

$$\begin{array}{ccc} V \times W & \xrightarrow{\ \sigma\ } & V \otimes W \\ & \searrow^{B} & \big\downarrow F \\ & & U \end{array}$$

Hence, the property **(T)** confirm that ψ is bijective, hence we have the following isomorphic of $\mathbb{F}$-vector spaces:

$$\mathcal{L}(V, W; U) \cong \mathcal{L}(V \otimes W, U).$$

$\square$

Corollary 1.3.11 Dual space of the tensor product

We have the following isomorphic :

$$(V \otimes W)^* \ \cong \ (V^* \otimes W^*).$$

The element F of $(V \otimes W)^*$ corresponding to $f \otimes g \in V^* \otimes W^*$ is given by

$$F(v \otimes w) = f(v)g(w).$$

Proof. Using the Proposition 1.3.10, we obtain

$$\mathcal{L}(V \otimes W, \mathbb{F}) \ \cong \ \mathcal{L}(V, W; \mathbb{F})$$

Let

$$V^* \otimes W^* \xrightarrow{\ \psi\ } \mathcal{L}(V, W; \mathbb{F}) \xrightarrow{\ t\ } (V \otimes W)^*$$

$$\psi(f_i \otimes g_j) = h_{ij}$$

where $h_{ij}(v, w) = f_i(v)\, g_j(w)$, and

$$t(h_{ij}) = F_{ij}$$

where $F_{ij}(v_i \otimes w_j) = f_i(v)\, g_j(w)$. Since ψ and t are isomorphisms, their composition $F = t \circ \psi$ is also an isomorphism,and hence

$$V^* \otimes W^* \cong (V \otimes W)^*.$$

Clearly, for all $f \otimes g \in V^* \otimes W^*$, we have

$$F(f \otimes g)(v \otimes w) = f(v)\, g(w).$$

$\square$

Proposition 1.3.12 Tensor with $\mathbb{F}$

Let V be a vector space over a field $\mathbb{F}$. By the correspondence $(\alpha \otimes v \longleftrightarrow \alpha v\ )$, where $\alpha \in \mathbb{F}$ and $v \in V$,

$$\mathbb{F} \otimes V \ \cong\ V.$$

Proof. Let $\sigma : \mathbb{F} \times V \longrightarrow V$ the bilinear map defined by

$$\sigma(\alpha \times v) = \alpha v.$$

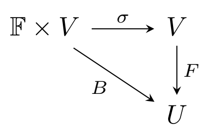
Using Theorem 1.2.5 **(2)**, to prove that $\mathbb{F} \otimes V \ \cong\ V$, we will show that the pair (V, σ) satisfies the property **(T1)** and **(T2)**. Since $\sigma(\mathbb{F} \times V) = V$, the pair (V, σ) satisfies **(T1)** . Let $B\mathcal{L}(\mathbb{F}, V; \mathbb{F})$ be a bilinear mapping. Define the mapping $F : V \longrightarrow U$ by

$$F(v) = B(1, v).$$

Then B is linear and for all $(\alpha, v) \in V$, we have

$$(F \circ \sigma)(\alpha, v) = F(\alpha v) = B(1, (\alpha v) = \alpha B(1, v) = B(\alpha, v).$$

That means $F \circ \sigma = B$



Hence the pair (V, σ) satisfies the property **(T2)**.

Consequently

$$\mathbb{F} \otimes V \ \cong\ V.$$

By Theorem 1.2.5 **(2)**, the isomorphism $F_0 : \mathbb{F} \otimes V \longrightarrow V$ is given by

$$\mathbb{F}_0(\alpha \otimes v) = \alpha v.$$

□

Proposition 1.3.13 Commutativity of the tensor product

By the correspondence $(v \otimes w \longleftrightarrow w \otimes v\ )$,

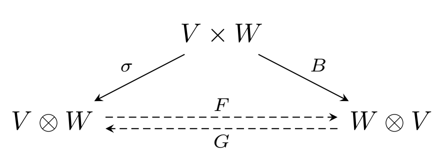
$$V \otimes W \ \cong\ W \otimes V.$$

Proof. By the property **(T)** for the tensor product $V \otimes W$, the bilinear $B : V \times W \longrightarrow W \otimes V$ defined by $B(v, w) = w \otimes v$, induces a linear mapping $F : V \otimes W \longrightarrow W \otimes V$ such that Therefore

$$F(v \otimes w) = w \otimes v.$$

Similarly, we can find a linear mapping $G : W \otimes V \longrightarrow V \otimes W$ such that

$$G(w \otimes v) = v \otimes w.$$



Clearly

$$F \circ G = \mathrm{Id} \quad \text{and} \quad G \circ F = \mathrm{Id}.$$

Then F is an isomorphism, and so

$$V \otimes W \;\cong\; W \otimes V.$$

□

1.4 Tensor products of more than two vector spaces

Definition 1.4.1 Multilinear mapping

Let $V_1, V_2, \ldots, V_n$ and U be $\mathbb{F}$-vector spaces. A mapping $f : V_1 \times V_2 \times \cdots V_n \longrightarrow U$ is called a multilinear mapping (or n-multilinear mapping) if it is linear in each variable.

More precisely:

For for each $k = 1, 2, \ldots, n$, and for all $(v_1, \ldots, v_n) \in V_1 \times V_2 \times \cdots \times \cdots \times V_n, v'_k \in V_k$ and $\alpha, \beta \in \mathbb{F}$,

$$\begin{aligned} f(v_1, \ldots, v_{i-1},\ \alpha v_k + \beta v'_k,\ v_{k+1}, \ldots, v_n\,) =& \alpha f(v_1, \ldots, v_{k-1},\ v_k,\ v_{k+1}, \ldots, v_n\,) \\ &+ \beta f(v_1, \ldots, v_{k-1},\ v'_k,\ v_{k+1}, \ldots, v_n\,). \end{aligned}$$

Definition 1.4.2 Multilinear form

In the previous definition, when $U = \mathbb{F}$, the function f is called multilinear form (or n-multilinear form)

Example 1.4.3

Let for all $i = 1, ..., n$, $f_i \in V^*$, and define $f : V_1 \times V_2 \times \cdots V_n \longrightarrow \mathbb{F}$ by

$$f(v_1, \ldots, v_n) = f_1(v_1) f_2(v_2) \cdots f_n(v_n).$$

Then f is n-multilinear form.

Note 1.4.4 $\mathcal{L}(V_1, \ldots, V_n; U)$

The set of all multilinear mappings of $V_1 \times V_2 \times \cdots V_n$ into U is a $\mathbb{F}$-vector space and it is is denoted by $\mathcal{L}(V_1, \ldots, V_n; U)$.

Definition 1.4.5 Conditions **(T1)** and **(T2)**

Let $V_1, \ldots, V_n$ be finite dimensional $\mathbb{F}$-vector spaces.

We say that $V_1, \ldots, V_n$ satisfies the condition **(T1)** and **(T2)**, if there exist a $\mathbb{F}$-vector space U_0 and an n-multilinear mapping $\sigma \in \mathcal{L}(V_1, \ldots, V_n; U_0)$ for which

(T1) U_0 is generated by the image $\sigma(V_1 \times V_2 \times \cdots \times V_n)$ of σ.

(T2) For any $B \in \mathcal{L}(V_1, \ldots, V_n; U_0)$, there exists a $\mathbb{F}$-linear mapping $F : U_o \longrightarrow U$ such that $B = F \circ \sigma$:

$$\begin{array}{ccc} V_1 \times V_2 \times \cdots \times V_n & \xrightarrow{\sigma} & U_0 \\ & \underset{B}{\searrow} & \downarrow F \\ & & U \end{array}$$

Theorem 1.4.6 Linearization of n-multilinear mappings

Let $V_1, \ldots, V_n$ be finite dimensional $\mathbb{F}$-vector spaces. There exist a $\mathbb{F}$-vector space U_0 and a multilinear mapping $\sigma : V_1 \times V_2 \times \ldots V_n \longrightarrow U_0$ which satisfy the the conditions **(T1)** and **(T2)**.

Definition 1.4.7 Tensor product of more than two vector spaces

Let$V_1, \ldots, V_n$ be $\mathbb{F}$-vector spaces. The pair (U_0, σ) consisting of a $\mathbb{F}$-vector space U_0 and a multilinear mapping $\sigma : V_1 \times V_2 \times \ldots V_n \longrightarrow U_0$, satisfying the conditions **(T1)** and **(T2)**, the existence of which is assured by Theorem 1.4.6 is called a tensor product of $V_1, \ldots, V_n$.

We write

$$U_0 = V_1 \otimes V_2 \otimes \ldots V_n \quad \text{and} \quad \sigma(v_1, \ldots, v_n) = v_1 \otimes v_2 \otimes \ldots v_n$$

The mapping σ is called the canonical mapping of a tensor product $V_1 \otimes V_2 \otimes \ldots \otimes V_n$.

Proposition 1.4.8

The correspondence

$$v_1 \otimes v_2 \otimes v_3 \longleftrightarrow (v_1 \otimes v_2) \otimes v_3$$

gives an isomorphism

$$V_1 \otimes V_2 \otimes V_3. \cong (V_1 \otimes V_2) \otimes V_3.$$

Proof. Consider the multilinear mapping $B : V_1 \times V_2 \times V_3 \longrightarrow (V_1 \otimes V_2) \otimes V_3$ given by

$$(v_1, v_2, v_3) \longmapsto (v_1 \otimes v_2) \otimes v_3.$$

Apply **(T2)** for the tensor product $V_1 \otimes V_2 \otimes V_3$, there is a linear mapping F for which the following diagram is commutative:

$$\begin{array}{ccc} V_1 \times V_2 \times V_3 & \xrightarrow{\sigma} & V_1 \otimes V_2 \otimes V_3 \\ & \underset{B}{\searrow} & \downarrow F \\ & & (V_1 \otimes V_2) \otimes V_3 \end{array}$$

Then

$$F(v_1 \otimes v_2 \otimes v_3) = (v_1 \otimes v_2) \otimes v_3.$$

Fix $v \in V_3$, and consider the bilinear mapping $B_v : V_1 \times V_2 \longrightarrow V_1 \otimes V_2 \otimes V_3$ given by

$$B_v(v_1, v_2) = v_1 \otimes v_2 \otimes v.$$

Apply **(T2)** for the tensor product $V_1 \otimes V_2$, there is a linear mapping F_v for which the following diagram is commutative:

$$\begin{array}{ccc} V_1 \times V_2 & \xrightarrow{\sigma} & V_1 \otimes V_2 \\ & \searrow^{B_v} & \downarrow F_v \\ & & V_1 \otimes V_2 \otimes V_3 \end{array}$$

Then

$$F_v(v_1 \otimes v_2) = v_1 \otimes v_2 \otimes v.$$

For $v.v' \in V_3$ and $\alpha \in \mathbb{F}$, we have

$$F_{v+v'} = F_v + F_{v'} \quad \text{and} \quad F_{\alpha v} = \alpha F_v$$

Using these facts, we define a bilinear mapping $\omega : (V_1 \otimes V_2) \times V_3 \longrightarrow V_1 \otimes V_2 \otimes V_3$ by

$$\omega(x, v) = F_v(x)$$

Apply **(T2)** for the tensor product $(V_1 \otimes V_2) \otimes V_3$, there is a linear mapping G for which the following diagram is commutative:

$$\begin{array}{ccc} (V_1 \otimes V_2) \times V_3 & \xrightarrow{\sigma} & (V_1 \otimes V_2) \otimes V_3 \\ & \searrow^{\omega} & \downarrow G \\ & & V_1 \otimes V_2 \otimes V_3 \end{array}$$

Then

$$G((v_1 \otimes v_2) \otimes v_3) = \omega(v_1 \otimes v_2, v_3) = F_{v_3}(v_1 \otimes v_2) = v_1 \otimes v_2 \otimes v_3.$$

Clearly

$$G \circ F = \text{Id} \quad \text{and} \quad F \circ G = \text{Id}.$$

Hence F is an isomorphism of vector spaces. Consequently

$$V_1 \otimes V_2 \otimes V_3. \cong (V_1 \otimes V_2) \otimes V_3.$$

□

Proposition 1.4.9

The correspondence

$$v_1 \otimes v_2 \otimes v_3 \longleftrightarrow v_1 \otimes (v_2 \otimes v_3)$$

gives an isomorphism

$$V_1 \otimes V_2 \otimes V_3. \cong V_1 \otimes (V_2 \otimes V_3).$$

Corollary 1.4.10 Associativity of the tensor product

The correspondence

$$(v_1 \otimes v_2) \otimes v_3 \longleftrightarrow v_1 \otimes (v_2 \otimes v_3)$$

gives an isomorphism

$$(V_1 \otimes V_2) \otimes V_3. \cong V_1 \otimes (V_2 \otimes V_3).$$

Proposition 1.4.11 Multilinearity of $\otimes$

Let $v_i, v_i' \in V_i$, and $\alpha, \beta \in \mathbb{F}$. For any $i = 1, \dots, n$.

$$v_1 \otimes \cdots \otimes v_{i-1} \otimes (\alpha v_i + \beta v_i') \otimes v_{i+1} \otimes \cdots \otimes v_n = \alpha(v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n) + \beta(v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i' \otimes v_{i+1} \otimes \cdots \otimes v_n).$$

Proposition 1.4.12 Basis for $V_1 \otimes \cdots \otimes V_n$

Let $\mathcal{B}_i = \{e_1^{(i)}, \dots, e_{m_i}^{(i)}\}$ be a basis for V_i, where $m_i = \dim V_i$. Then the $m_1 m_2 \cdots m_n$ vectors

$$e_{j_1}^{(1)} \otimes e_{j_2}^{(2)} \otimes \cdots \otimes e_{j_n}^{(n)}, \quad 1 \leq j_i \leq m_i \quad \text{and} \quad 1 \leq i \leq n$$

Corollary 1.4.13 Dimension of $V_1 \otimes \cdots \otimes V_n$

$$\dim(V_1 \otimes \cdots \otimes V_n) = \dim V_1 \dim V_2 \cdots \dim V_n.$$

1.5 Tensor products of linear mappings

Theorem 1.5.1 Tensor products of linear mappings : $F_1 \otimes F_2$

Let $F_1 : V_1 \longrightarrow W_1$ and $F_2 : V_2 \longrightarrow W_2$ be linear mappings. Then there exists a linear mapping $\tilde{F} : V_1 \otimes V_2 \longrightarrow W_1 \otimes W_2$ such that for all $v_1 \in V_1$ and $v_2 \in V_2$

$$\tilde{F}(v_1 \otimes v_2) = F_1(v_1) \otimes F_2(v_2).$$

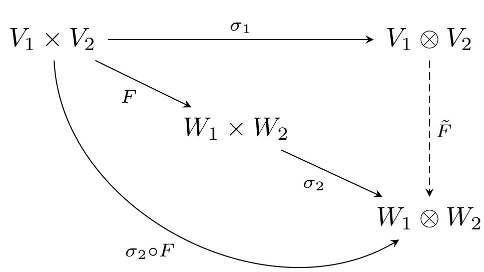
The mapping $\tilde{F}$ is called the tensor product of F_1 and F_2 and is denoted by $F_1 \otimes F_2$.

Proof. Let σ_1 and σ_2 be the canonical mappings of $V_1 \otimes V_2$ and $W_1 \otimes W_2$ respectively.

Consider the bilinear mapping $F = F_1 \times F_2 : V_1 \times V_2 \longrightarrow W_1 \times W_2$ given by

$$(F_1 \times F_2)(v_1, v_2) = (F_1(v_1), F_2(v_2)).$$

Apply the property **(T)** for the tensor product $V_1 \otimes V_2$, there is a linear mapping $\tilde{F}$ for which the following diagram is commutative:



Hence

$$\tilde{F}(v_1 \otimes v_2) = \sigma_2(F(v_1, v_2)) = \sigma_2(F_1(v_1), F_2(v_2)) = F_1(v_1) \otimes F_2(v_2).$$

□

Proposition 1.5.2 Properties of the tensor product of linear mappings

Let $V_1, V_2, W_1, W_2, U_1, U_2$ be $\mathbb{F}$-vector spaces and $\alpha \in \mathbb{F}$. Consider the following six linear mappings of vector spaces:

$$V_1 \underset{G_1}{\overset{F_1}{\rightrightarrows}} W_1 \xrightarrow{H_1} U_1$$

$$V_2 \underset{G_2}{\overset{F_2}{\rightrightarrows}} W_2 \xrightarrow{H_2} U_2$$

Then

(1) $F_1 \otimes (F_2 + G_2) = F_1 \otimes F_2 + F_1 \otimes G_2$,

(2) $(F_1 + G_1) \otimes F_2 = F_1 \otimes F_2 + G_1 \otimes F_2$,

(3) $(\alpha F_1) \otimes F_2 = F_1 \otimes (\alpha F_2) = \alpha (F_1 \otimes F_2)$,

(4) $(H_1 \circ F_1) \otimes (H_2 \circ F_2) = (H_1 \otimes H_2) \circ (F_1 \otimes F_2)$.

Proof. The proofs of $(\mathbf{1}), (\mathbf{2}), (\mathbf{3})$ and $(\mathbf{4})$ are all similar, so we give here just the proof of the first equality. Since both sides of the equality are linear mappings of the vector space $V_1 \otimes V_2$ into $W_1 \otimes W_2$, it is enough to show that they coincide on the generating set $\{v_1 \otimes v_2 \mid v_1 \in V_1, v_2 \in V_2\}$ of $V_1 \otimes V_2$. For all $v_1 \otimes v_2$, we have

$$\begin{aligned}
(F_1 \otimes (F_2 + G_2))(v_1 \otimes v_2) &= F_1(v_1) \otimes (F_2 + G_2)(v_2) \\
&= F_1(v_1) \otimes \big(F_2(v_2) + G_2(v_2) \big) \quad \text{(definition of the sum of mappings)} \\
&= F_1(v_1) \otimes F_2(v_2) + F_1(v_1) \otimes G_2(v_2) \quad \text{(bilinearity of } \otimes\text{)} \\
&= (F_1 \otimes F_2)(v_1 \otimes v_2) + (F_1 \otimes G_2)(v_1 \otimes v_2) \quad \text{(tensor product of mappings)} \\
&= (F_1 \otimes F_2 + F_1 \otimes G_2)(v_1 \otimes v_2) \quad \text{(definition of the sum of mappings).}
\end{aligned}$$

□

Remark 1.5.3. The tensor product $F_1 \otimes \cdots \otimes F_n$ of n linear mappings $F_i : V_i \longrightarrow W_i$, $(i = 1, .., n)$ is defined similarly.

$$(F_1 \otimes \cdots \otimes F_n)(v_1 \otimes \cdots \otimes v_n) = F_1(v_1) \otimes F_2(v_2) \otimes \cdots \otimes F_n(v_n).$$

1.6 Tensor product of matrices: $A \otimes B$

Let V_1, V_2, W_1, and W_2 be $\mathbb{F}$-vector spaces of dimension r, s, m, n respectively. Take

$$\begin{aligned}
\mathcal{B}_{V_1} &= \{e_1, \ldots, e_r\} \\
\mathcal{B}_{V_2} &= \{e'_1, \ldots, e'_s\} \\
\mathcal{B}_{W_1} &= \{f_1, \ldots, f_m\} \\
\mathcal{B}_{W_2} &= \{f'_1, \ldots, f'_n\}
\end{aligned}$$

be bases of V_1, V_2, W_1, and W_2 respectively.

Then by Proposition 1.3.6).

$$\mathcal{B}_{V_1 \otimes V_2} = \{e_1 \otimes e'_1, e_1 \otimes e'_2, \ldots, e_1 \otimes e'_s, e_2 \otimes e'_1, e_2 \otimes e'_2, \ldots, e_2 \otimes e'_s, \ldots, e_r \otimes e'_1\},$$

and

$$\mathcal{B}_{W_1 \otimes W_2} = \{f_1 \otimes f'_1, f_1 \otimes f'_2, \ldots, f_1 \otimes f'_m, f_2 \otimes f'_1, f_2 \otimes f'_2, \ldots, f_2 \otimes f'_m, \ldots, f_m \otimes f'_n\},$$

are bases of $V_1 \otimes V_2$ and $W_1 \otimes W_2$ respectively.

Let $A = (\alpha_{ij})$ and $B = (\beta_{ij})$ be the matrices for F_1 and F_2 with respect to the bases $\mathcal{B}_{V_1}, \mathcal{B}_{V_2}, \mathcal{B}_{W_1}$ and $\mathcal{B}_{W_2}$, Namely,

$$F_1(e_i) = \sum_{l=1}^{m} \alpha_{li} f_l, \quad F_2(e'_j) = \sum_{h=1}^{n} \beta_{hj} f'_h.$$

Thus, for all i and j, we have

$$\begin{aligned}(F_1 \otimes F_2)(e_i \otimes e'_j) &= F_1(e_i) \otimes F_2(e'_j) \\ &= \sum_{l=1}^{m} \sum_{h=1}^{n} \alpha_{li}\beta_{hj}(f_l \otimes f'_h).\end{aligned}$$

The matrix of $F_1 \otimes F_2$ with respect to the bases $\mathcal{B}_{V_1 \otimes V_2}$ and $\mathcal{B}_{W_1 \otimes W_2}$ is given as follows:

$$\begin{bmatrix} \alpha_{11}\beta_{11} & \alpha_{11}\beta_{12} & \cdots \\ \alpha_{11}\beta_{21} & \alpha_{11}\beta_{22} & \cdots \\ \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} \alpha_{11}B & \alpha_{12}B & \cdots & \alpha_{1r}B \\ \alpha_{21}B & \alpha_{22}B & \cdots & \alpha_{2r}B \\ \vdots & & & \vdots \\ \alpha_{m1}B & \alpha_{m2}B & \cdots & \alpha_{mr}B \end{bmatrix}.$$

Definition 1.6.1 Tensor product of matrices (Kronecker product)

Let $A = (\alpha_{ij})$ and $B = (\beta_{ij})$ be matrices. The matrix

$$\begin{bmatrix} \alpha_{11}B & \alpha_{12}B & \cdots & \alpha_{1n}B \\ \alpha_{21}B & \alpha_{22}B & \cdots & \alpha_{2n}B \\ \vdots & & & \vdots \\ \alpha_{m1}B & \alpha_{m2}B & \cdots & \alpha_{mn}B \end{bmatrix}$$

is called the tensor product (or Kronecker product) of A and B. It is denoted by $A \otimes B$. If A is an $m \times n$ matrix and B is an $m' \times n'$ matrix, $A \otimes B$ is an $mm' \times nn'$ matrix.

Example 1.6.2

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} 0 & 5 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 0 & 5 \\ 6 & 7 \end{pmatrix} & 2\begin{pmatrix} 0 & 5 \\ 6 & 7 \end{pmatrix} \\ 3\begin{pmatrix} 0 & 5 \\ 6 & 7 \end{pmatrix} & 4\begin{pmatrix} 0 & 5 \\ 6 & 7 \end{pmatrix} \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ \hline 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{array}\right).$$

According to this definition, the matrix of $F_1 \otimes F_2$ with respect to the bases above is the tensor product of the matrices of F_1 and F_2.

Proposition 1.6.3 Properties of the tensor product of matrices

Let A_i be $m \times n$ matrices, B_i be $m' \times n'$ matrices; let C_1 be an $n \times l$ matrix and D_1 be an $n' \times l'$ matrix. Then we have

$$\begin{aligned}
&A_1 \otimes (B_1 + B_2) = A_1 \otimes B_1 + A_1 \otimes B_2, \\
&(A_1 + A_2) \otimes B_1 = A_1 \otimes B_1 + A_2 \otimes B_1, \\
&(\alpha A_1) \otimes B_1 = A_1 \otimes (\alpha B_1) = \alpha (A_1 \otimes B_1) \quad (\alpha \in k), \\
&(A_1 \otimes B_1)^{\mathrm{t}} = A_1^{\mathrm{t}} \otimes B_1^{\mathrm{t}}, \\
&A_1 C_1 \otimes B_1 D_1 = (A_1 \otimes B_1)(C_1 \otimes D_1).
\end{aligned}$$

Corollary 1.6.4

If A and B are regular matrices, then $A \otimes B$ is regular, and we have

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

This follows from the last formula in Proposition 1.6.3.

Definition 1.6.5 Unitary matrices

An invertible complex square matrix U is unitary if its conjugate transpose U^* is also its inverse, that is:

$$U^* = U^{-1}.$$

Lemma 1.6.6 Schur's Triangularization Theorem

Given A a square $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ counting multiplicities, there exists a unitary matrix U such that

$$A = U \begin{pmatrix} \lambda_1 & \star & \cdots & \star \\ 0 & \lambda_2 & \cdots & \star \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} U^*$$

Proposition 1.6.7

Let A be an $n \times n$ matrix whose eigenvalues are $\alpha_1, \ldots, \alpha_n$ and let B be an $m \times m$ matrix whose eigenvalues are $\beta_1, \ldots, \beta_m$. Then the eigenvalues of $A \otimes B$ are $\alpha_i \beta_j$, $(i = 1, \ldots, n, j = 1, \ldots, m)$.

Proof. Using Schur's Triangularization Theorem. There exist unitary matrices S and T such that

$$S^{-1} A S = \begin{pmatrix} \alpha_1 & \star & \cdots & \star \\ 0 & \alpha_2 & \cdots & \star \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix}$$

and

$$T^{-1}BT = \begin{pmatrix} \beta_1 & \star & \cdots & \star \\ 0 & \beta_2 & \cdots & \star \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_m \end{pmatrix}$$

By Corollary 1.6.4, the matrix $S \otimes T$ is invertible and

$$\begin{aligned}(S\otimes T)^{-1}(A\otimes B)(S\otimes T) &= (S^{-1}\otimes T^{-1})(A\otimes B)(S\otimes T)\\ &= (S^{-1}AS)\otimes(T^{-1}BT)\\ &= \begin{pmatrix} \alpha_1 & \star & \cdots & \star \\ 0 & \alpha_2 & \cdots & \star \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix} \otimes \begin{pmatrix} \beta_1 & \star & \cdots & \star \\ 0 & \beta_2 & \cdots & \star \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_m \end{pmatrix}\\ &= \begin{pmatrix} \alpha_1\beta_1 & \star & \cdots & \star \\ 0 & \alpha_2\beta_2 & \cdots & \star \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n\beta_m \end{pmatrix}\end{aligned}$$

Hence $A \otimes B$ is similar to the following an upper triangular matrix:

$$\begin{pmatrix} \boxed{\begin{matrix} \alpha_1\beta_1 & \star & \cdots & \star \\ 0 & \alpha_1\beta_2 & \cdots & \star \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_1\beta_m \end{matrix}} & & & \huge\star \\ & \boxed{\begin{matrix} \alpha_2\beta_1 & \star & \cdots & \star \\ 0 & \alpha_2\beta_2 & \cdots & \star \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_2\beta_m \end{matrix}} & & \\ & & \ddots & \\ \huge 0 & & & \boxed{\begin{matrix} \alpha_n\beta_1 & \star & \cdots & \star \\ 0 & \alpha_n\beta_2 & \cdots & \star \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n\beta_m \end{matrix}} \end{pmatrix}$$

Therefore the eigenvalues of $A \otimes B$ are $\alpha_i\beta_j$, $(i = 1, \dots, n, j = 1, \dots, m)$. □

1.7 Exercises set

Exercise 1.7.1

Let f be an element of $\mathcal{L}(V, W; U)$. The set

$$\mathrm{Im} f = f(V \times W) = \{f(v, w) \mid u \in U \quad \text{and} \quad w \in W\}$$

is not necessarily a vector subspace. Give an example of such that $\mathrm{Im} f$ is not a vector subspace. (Compare with the case of linear mappings.)

Solution. Let $f : \mathbb{R}[x] \times R[y] \longrightarrow \mathbb{R}[x, y]$ defined by

$$f(p, q) = pq.$$

The image of f contains x and y, but not $x + y$.

Exercise 1.7.2

Let U and V be two $\mathbb{F}$-vector spaces and S a subset V. Show that if $f, g \in \mathcal{L}(V, U)$ such that

$$f(v) = g(v), \quad \text{for all} \quad v \in S$$

then $f(v) = g(v)$ for all $v \in \text{span}(S)$.

Solution. Assume that

$$f(v) = g(v), \quad \text{for all} \quad v \in S.$$

Let $v \in \text{span}(S)$, then the vector v can be expressed as :

$$v = \sum_{i=1}^{n} \alpha_i s_i$$

for some scalars α_i and $s_i \in S$. Since f and g are linear, we have

$$f(v) = f\Big(\sum_{i=1}^{n} \alpha_i s_i\Big) = \sum_{i=1}^{n} f(\alpha_i s_i) = \sum_{i=1}^{n} \alpha_i f(s_i),$$

and

$$g(v) = g\Big(\sum_{i=1}^{n} \alpha_i s_i\Big) = \sum_{i=1}^{n} g(\alpha_i s_i) = \sum_{i=1}^{n} \alpha_i g(s_i).$$

But $f(s_i) = g(s_i)$ for all $i = 1, \dots, n$. Hence

$$f(v) = g(v) \quad \text{for all} \quad v \in \text{span}(S).$$

Exercise 1.7.3

Let V be a $\mathbb{R}$-vector space de dimension 2, and $\mathcal{B} = \{v_1, v_2\}$ a basis for V.

(1) What is the dimension of $V \otimes V$?

(2) Construct a basis $\mathcal{S}$ of $V \otimes V$ from $\mathcal{B}$.

(3) Find the coordinates of $(2v_1 - 3v_2) \otimes (4v_1 - v_2)$ relative to the basis $\mathcal{S}$.

(4) Show that the tensor $X = 11v_1 \otimes v_1 + 8v_1 \otimes v_2 + 3v_2 \otimes v_2$ cannot be written as tensor product of two vectors in V.

Solution.

(1) $\dim V \otimes V = 4$.

(2) $\mathcal{S} = \{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}$.

(3) We have $(2v_1 - 3v_2) \otimes (4v_1 - v_2) = 8v_1 \otimes v_1 - 2v_1 \otimes v_2 - 12v_2 \otimes v_1 + 3v_2 \otimes v_2$. So, the coordinates of $(2v_1 - 3v_2) \otimes (4v_1 - v_2)$ relative to the basis $\mathcal{S}$ are $(8, -2, -12, 3)$.

(4) Suppose that $X = v \otimes v'$ for some vectors v and v' in V. Let $v = \alpha v_1 + \beta v_2$ and $v' = \alpha' v_1 + \beta' v_2$. Then

$$X = \alpha\alpha' v_1 \otimes v_1 + \alpha\beta' v_1 \otimes v_2 + \beta\alpha' v_2 \otimes v_1 + \beta\beta' v_2 \otimes v_2$$

Hence, by comparison

$$\begin{cases} \alpha\alpha' = 11 \\ \alpha\beta' = 8 \\ \beta\alpha' = 0 \\ \beta\beta' = 3 \end{cases}$$

Clearly this system has no solution, and hence X can't be written as $v \otimes v'$.

Exercise 1.7.4

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the standard basis of the real vector spaces $W = \mathbb{R}^3$ and $V = \mathbb{R}^2$ respectively:

$$\mathcal{S}_1 = \{w_1, w_2, w_3\} \quad \text{and} \quad \mathcal{S}_2 = \{v_1, v_2\}$$

where

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

(1) Let x be the element of $W \otimes V$ given by $x = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Express x as a linear combination of the basis elements $(w_i \otimes v_j)$.

(2) Let y be the element of $W \otimes V$ given by

$$y = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Is it possible to express y as the form $w \otimes v$ for some $w \in W$ and $v \in V$?

Solution. **(1)** We have

$$\begin{aligned} x &= \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= (-w_1 + 2w_2 + 3w_3) \otimes (v_1 - 2v_2) \\ &= -w_1 \otimes v_1 + 2w_1 \otimes v_2 + 2w_2 \otimes v_1 - 4w_2 \otimes v_2 + 3w_3 \otimes v_1 - 6w_3 \otimes v_2 \end{aligned}$$

(2) Let y be the element of $V \otimes W$ given by $y = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

Consider $y = v \otimes w$, where

$$w = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} r \\ s \end{pmatrix}$$

We have

$$y = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 9 \\ 9 \\ 9 \\ 9 \\ 6 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ -5 \\ -5 \\ -4 \\ -3 \\ -3 \end{pmatrix} \quad \text{and} \quad w \otimes v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \otimes v = \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} ar \\ as \\ br \\ bs \\ cr \\ cs \end{pmatrix}$$

Hence

$$y = w \otimes v \Longleftrightarrow \begin{cases} ar = -4 & (1.9) \\ as = -5 & (1.10) \\ br = -5 & (1.11) \\ bs = -4 & (1.12) \\ cr = -3 & (1.13) \\ cs = -3 & (1.14) \end{cases}$$

If this system has a solution, then a, b must be not equal zero. From the equations (1.9) and (1.10), we get $r + s = \frac{-9}{a}$. Similarly form the equations (1.11) and (1.12), we get $r + s = \frac{-9}{b}$. Therefore

$$r + s = \frac{-9}{a} = \frac{-9}{b}$$

So $a = b$. Hence $ar = -4$ and $ar = -5$ which is a contradiction. Consequently, y can not be written as $v \otimes w$.

Exercise 1.7.5

Let $\sigma : \mathbb{R}^3 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^6$ the bilinear mapping defined by :

$$\text{for all} \quad w = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \quad \text{and} \quad v = \begin{pmatrix} r \\ s \end{pmatrix} \in \mathbb{R}^2 : \qquad \sigma(w, v) = \begin{pmatrix} ar \\ as \\ br \\ bs \\ cr \\ cs \end{pmatrix}$$

(1) Let $\mathcal{S}_1 = \{w_1, w_2, w_3\}$ and $\mathcal{S}_1 = \{v_1, v_2\}$ be the standard basis of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Compute $e_{ij} = \sigma(w_i, v_j)$ for all $1 \leq i, j \leq 3$.

(2) Find span(Im σ).

(3) Let $B : \mathbb{R}^3 \times \mathbb{R}^2 \longrightarrow U$ be a bilinear mapping. Consider $F : \mathbb{R}^6 \longrightarrow U$ defined by:

$$F\left(\sum_{1 \leq i,j \leq 3} x_{ij} e_{ij} \right) = \sum_{1 \leq i,j \leq 3} x_{ij} B(w_i, v_j)$$

(4.1) Show that F is linear

(4.2) Find the relation between $F \circ \sigma$ and B.

(4) Show that $\mathbb{R}^3 \otimes \mathbb{R}^2 = \mathbb{R}^6$.

Solution. **(1)** $e_{ij} = \sigma(w_i, v_j)$ for all $1 \leq i, j \leq 3$.

$$e_{11} = \sigma(w_1, v_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad e_{12} = \sigma(w_1, v_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$e_{21} = \sigma(w_2, v_1) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad e_{22} = \sigma(w_2, v_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$e_{31} = \sigma(w_3, v_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad e_{32} = \sigma(w_3, v_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(2) $\text{span}(\text{Im}\,\sigma) = \mathbb{R}^6$

(3) Let $B : \mathbb{R}^3 \times \mathbb{R}^2 \longrightarrow U$ be a bilinear mapping. Consider $F : \mathbb{R}^6 \longrightarrow U$ defined by:

$$F\left(\sum_{1 \le i,j \le 3} x_{ij} e_{ij} \right) = \sum_{1 \le i,j \le 3} x_{ij} B(w_i, v_j)$$

We have the following diagram:

$$\begin{array}{ccc} \mathbb{R}^3 \times \mathbb{R}^2 & \xrightarrow{\sigma} & \mathbb{R}^6 \\ & {\scriptstyle B} \searrow & \downarrow {\scriptstyle F} \\ & & U \end{array}$$

(4.1) F is linear : Let $\alpha \in \mathbb{R}$ and $X = \sum_{1 \le i,j \le 3} x_{ij} e_{ij}$ and $Y = \sum_{1 \le i,j \le 3} y_{ij} e_{ij}$ be two vectors in $\mathbb{R}^6$. Then

$$\begin{aligned} F(\alpha X + Y) &= F(\sum_{1 \le i,j \le 3} (\alpha x_{ij} + y_{ij}) e_{ij}) \\ &= \sum_{1 \le i,j \le 3} (\alpha x_{ij} + y_{ij}) B(w_i, v_j) \\ &= \sum_{1 \le i,j \le 3} (\alpha x_{ij}) B(w_i, v_j) + \sum_{1 \le i,j \le 3} y_{ij} B(w_i, v_j) \\ &= \alpha \sum_{1 \le i,j \le 3} x_{ij} B(w_i, v_j) + \sum_{1 \le i,j \le 3} y_{ij} B(w_i, v_j) \\ &= \alpha F(X) + F(Y) \end{aligned}$$

(4.2) $F \circ \sigma = B$. Let

$$w = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \quad \text{and} \quad v = \begin{pmatrix} r \\ s \end{pmatrix} \in \mathbb{R}^2$$

We have

$$
\begin{aligned}
F\circ\sigma(w,v) &= F\begin{pmatrix} ar\\ as\\ br\\ bs\\ cr\\ cs\end{pmatrix}\\
&= F(are_{11}+ase_{12}+bre_{21}+bse_{22}+cre_{31}+cse_{32})\\
&= arB(w_1,v_1)+asB(w_1,v_2)+brB(w_2,v_1)+bsB(w_2,v_2)+crB(w_3,v_1)+csB(w_3,v_2)\\
&= B(aw_1,rv_1)+B(aw_1,sv_2)+B(bw_2,rv_1)+B(bw_2,sv_2)+B(cw_3,rv_1)+B(cw_3,sv_2)\\
&= B(aw_1,rv_1+sv_2)+B(bw_2,rv_1+sv_2)+B(cw_3,rv_1+sv_2)\\
&= B(aw_1+bw_2+cw_3,rv_1+sv_2)\\
&= B(w,v)
\end{aligned}
$$

(4) Using the universal property of the tensor product, we get $\mathbb{R}^3\otimes\mathbb{R}^2 = \mathbb{R}^6$.

Exercise 1.7.6

Let V and W be $\mathbb{F}$-vector space and V^* and W^* the dual spaces of V and W respectively.

(1) For fixed $v\in V$ and $w\in W$, let $B_{(v,w)}: V^*\times W^*\longrightarrow\mathbb{F}$ defined by

$$B_{(v,w)}(f,g)=f(v)g(w)\quad\text{for all}\quad f\in V^*\quad\text{and}\quad g\in W^*$$

Show that the mapping $B_{(v,w)}$ is a bilinear form.

(2) Consider the map $\sigma: V\times W\longrightarrow\mathcal{L}(V^*,W^*;\mathbb{F})$ defined by

$$\sigma(v,w)=B_{(v,w)}.$$

Show that σ is bilinear.

Solution. **(1)** Let $v\in V$ and $w\in W$. For any α, $f_1,f_2\in V^*$ and $g\in W^*$, we have

$$
\begin{aligned}
B_{(v,w)}(\alpha f_1+f_2,g) &= (\alpha f_1+f_2)(v)g(w)\\
&= (\alpha f_1(v)+f_2(v))g(w)\\
&= \alpha f_1(v)g(w)+f_2(v)g(w)\\
&= \alpha B_{(v,w)}(f_1,g)+B_{(v,w)}(f_2,g).
\end{aligned}
$$

Similarly, we can show that For any α, $g_1,g_2\in W^*$ and $f\in V^*$,

$$B_{(v,w)}(f,\alpha g_1+g_2)=\alpha B_{(v,w)}(f,g_1)+B_{(v,w)}(f,g_2).$$

Hence, the mapping $B_{(v,w)}$ is a bilinear form.

(2) Consider the map $\sigma: V\times W\longrightarrow\mathcal{L}(V^*,W^*;\mathbb{F})$ defined by

$$\sigma(v,w)=B_{(v,w)}.$$

Let α, $v_1,v_2\in V$ and $w\in W$. Then by definition,

$$\sigma(\alpha v_1+v_2,w)=B_{(\alpha v_1+v_2,w)}.$$

Therefore, for all $f \in V^*$ and $g \in W^*$, we have

$$\begin{aligned}
\sigma(\alpha v_1 + v_2, w)(f, g) &= B_{(\alpha v_1 + v_2, w)}(f, g) \\
&= f(\alpha v_1 + v_2)g(w)) \\
&= \big(f(\alpha v_1) + f(v_2)\big)g(w)) \\
&= \alpha f(v_1)g(w) + f(v_2)g(w) \\
&= \alpha B_{(v_1, w)}(f, g) + B_{(v_2, w)}(f, g) \\
&= \alpha\sigma(v_1, w)(f, g) + \sigma(v_2, w)(f, g) \\
&= \big(\alpha\sigma(v_1, w) + \sigma(v_2, w)\big)(f, g).
\end{aligned}$$

Hence

$$\sigma(\alpha v_1 + v_2, w) = \alpha\sigma(v_1, w) + \sigma(v_2, w).$$

Similarly, we can show that, for all α, $w_1, w_2 \in W$ and $v \in W$. we have

$$\sigma(v, \alpha w_1 + w_2) = \alpha\sigma(v, w_1) + \sigma(v, w_2).$$

Exercise 1.7.7

Let $\mathcal{B}_V = \{v_1, \ldots, v_n\}$ be a basis for V and $\mathcal{B}_W = \{w_1, \ldots, w_m\}$ a basis for W.

Put $\mathcal{B}_V^* = \{f_1, \ldots, f_n\}$, $\mathcal{B}_W^* = \{g_1, \ldots, g_m\}$ be respectively the dual basis of V^* and W^*.

Consider the linear mapping $\Phi : V^* \otimes W^* \longrightarrow \mathcal{L}(V, W; \mathbb{F})$ given by $f_i \otimes g_j \longmapsto \Phi(f_i \otimes g_j)$, where

$$\Phi(f_i \otimes g_j)(v, w) = f_i(v)g_j(w)$$

for all $u \in U$ and $w \in W$.

Prove that Φ is an isomorphic of $\mathbb{F}$-vector spaces.

Solution. We know that the vector spaces $V^* \otimes W^*$ and $\mathcal{L}(V, W; \mathbb{F})$ have the same dimension, so to prove that Φ is an isomorphic, we need only to show that Φ is onto (subjective). Let $h \in \mathcal{L}(V, W; \mathbb{F})$. Using Lemma 1.1.8, the set

$$\{h_{ij} \mid 1 \leq s \leq n \text{ and } 1 \leq r \leq m\}$$

form a basis for $\mathcal{L}(V, W; \mathbb{F})$, where

$$h_{ij}(v, w) = f_i(v)g_j(w),$$

and

$$h = \sum_{i,j} h(v_i, w_j)h_{ij}.$$

Hence

$$h = \sum_{i,j} h(v_i, w_j)\Phi(f_i \otimes g_j).$$

Therefore

$$h = \Phi\Big(\sum_{i,j} h(v_i, w_j)(f_i \otimes g_j)\Big).$$

So Φ is onto, and hence it is an isomorphism of $\mathbb{F}$-vector spaces.

Exercise 1.7.8

Let V, W and U be $\mathbb{F}$-vector spaces. Show that

$$\mathcal{L}(V, W; U) \;\cong\; \mathcal{L}(V, \mathcal{L}(W, U)).$$

Deduce that

$$\mathcal{L}(V \otimes W, U) \;\cong\; \mathcal{L}(V, \mathcal{L}(W, U)),$$

and

$$(V \otimes W)^* \;\cong\; \mathcal{L}(V, W^*).$$

Solution. For any bilinear mapping $B : V \times W \longrightarrow U$. Let $\phi(f) : V \longrightarrow \mathrm{Hom}(W, U)$ the mapping defined by

$$\phi(f)(v)(w) = f(v, w), \quad \text{for all} \quad v \in V, w \in W.$$

Conversely, given a linear map, $g \in \mathcal{L}(V, \mathcal{L}(W, U))$, we get the bilinear map $\psi(g) : V \times W \longrightarrow U$, given by

$$\psi(g)(v, w) = (g(v))(w), \quad \text{for all} \quad v \in V, w \in W.$$

It is clear that

$$\phi \circ \psi(g)(v, w) = \phi(\ \psi(g)(v, w)\ ) = \phi(\ g(v)(w)\ ) = g(v, w).$$

and

$$(\psi \circ \phi(f))(v, w) = \psi(\ \phi(f))(v, w)\ ) = \phi(\ g(v)(w)\ ) = g(v, w).$$

So

$$\phi \circ \psi = \mathrm{Id} \quad \text{and} \quad \psi \circ \phi = \mathrm{Id}.$$

Consequently, we have the following isomorphism:

$$\mathcal{L}(V, W; U) \quad \cong \quad \mathcal{L}(V, \mathcal{L}(W, U)).$$

Exercise 1.7.9

Let V and W be $\mathbb{F}$-vector spaces. Prove that

$$V^* \otimes W \quad \cong \quad \mathrm{Hom}(V, W).$$

Solution. Recall that $\mathrm{Hom}(V, W)$ is the $\mathbb{F}$-vector space of all linear mappings of V into W . Consider the mapping $B : V^* \times W \longrightarrow \mathrm{Hom}(V, W)$ given by

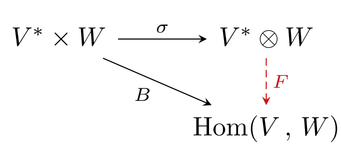
$$B(\varphi, w) = B_{\varphi, w}$$

where $B_{\phi, w}$ is defined by

$$B_{\varphi,\, w}(v) = \varphi(v)w \qquad \text{for all } v \in V$$

It is easy to see that $B_{\varphi\,, w}$ is an element of $\mathrm{Hom}(V\,,\, W)$, and B is a bilinear mapping on $V^* \times W$.

Therefore, by condition **(T2)** applied to $V^* \otimes W$, there exists a linear mapping $F : V^* \otimes W \longrightarrow \mathrm{Hom}(V\,,\, W)$ such that $F \circ \sigma = B$.



We will show that $\widehat{F}$ is an isomorphism.

First, we will prove that $\widehat{F}$ is one-to-one (injective). Let $t \in V^* \otimes W$, such that $F(t) = 0$. If $\{w_1, \ldots, w_m\}$, $\{\varphi_1, \ldots, \varphi_m\}$ is a basis for W and V^* respectively, the element t can be written as

$$t = \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha_{ij} \varphi_i \otimes w_j = \sum_{j=1}^{m} \Big(\sum_{i=1}^{n} \alpha_{ij} \varphi_i \Big) \otimes w_j = \sum_{j=1}^{m} f_j \otimes w_j$$

where $f_j = \sum_{i=1}^{n} \alpha_{ij} \varphi_i$. Therefore

$$\begin{aligned} F(t) &= F\Big(\sum_{j=1}^{m} f_j \otimes w_j \Big) \\ &= \sum_{j=1}^{m} F(f_j \otimes w_j) \\ &= \sum_{j=1}^{m} (F \circ \sigma)(f_j, w_j) \\ &= \sum_{j=1}^{m} B(f_j, w_j) \\ &= \sum_{j=1}^{m} B_{f_j, w_j}. \end{aligned}$$

So

$$\begin{aligned} F(t) = 0 &\Longrightarrow (F(f))\,(v) = 0 \quad \text{for all } v \in V \\ &\Longrightarrow \sum B_{f_j, w_j}(v) \quad \text{for all } v \in V \\ &\Longrightarrow \sum f_j(v) w_j \quad \text{for all } v \in V. \end{aligned}$$

Since w_j are linearly independent, for all $j = 1, ..., m$:

$$f_j(v) = 0 \quad \text{for all } v \in V.$$

So $f_j = 0$ for all $j = 1, ..., m$. Therefore $t = 0$. That means F is injective, and hence it is surjective because the vector spaces $V^* \otimes W$ and $\mathrm{Hom}(V, W)$ have the same dimension:

$$\dim V^* \otimes W \quad = \quad \dim \mathrm{Hom}(V, W) \quad = \quad nm.$$

Exercise 1.7.10

Consider V and W are two finite dimensional vector spaces over a field $\mathbb{F}$. Let $v_1, v_2 \in V \backslash \{0\}$ and $w_1, w_2 \in W \backslash \{0\}$. Show that the following conditions are equivalent:

(1) $v_1 \otimes w_1 = v_2 \otimes w_2$

(2) there exists $\alpha \in \mathbb{F} \backslash \{0\}$ such that $v_2 = \alpha v_1$ and $w_1 = \alpha w_2$

Solution.
(2) $\Longrightarrow$ (1): Suppose that, there exists $\alpha \in \mathbb{F} \backslash \{0\}$ such that $v_2 = \alpha v_1$ and $w_1 = \alpha w_2$. Hence

$$\begin{aligned} v_2 = \alpha v_1 \quad \text{and} \quad w_1 = \alpha w_2 &\Longrightarrow v_1 \otimes w_1 = \alpha^{-1} v_2 \otimes \alpha w_2 \\ &\Longrightarrow v_1 \otimes w_1 = \alpha^{-1} \alpha (v_2 \otimes w_2) \\ &\Longrightarrow v_1 \otimes w_1 = v_2 \otimes w_2 \end{aligned}$$

(1) $\implies$ (2): Conversely, assume that $v_1 \otimes w_1 = v_2 \otimes w_2$. Then v_1, v_2 or w_1, w_2 are linearly dependent. Otherwise, by using the incomplete basis theorem, we can construct a basis $\mathcal{B} = \{v_1, v_2, ...v_n\}$ of V and a basis $\mathcal{S} = \{w_1, w_2, ...w_m\}$ of W, therefore $v_1 \otimes w_1$ and $v_2 \otimes w_2$ are in the basis of $V \otimes W$ obtained from $\mathcal{B}$ and $\mathcal{S}$. Which is a contradiction with the hypothesis $v_1 \otimes w_1 = v_2 \otimes w_2$.

Consider for example v_1, v_2 are linrearly dependent, so there exists $v_2 = \alpha v_1$

$$
\begin{aligned}
v_1 \otimes w_1 = v_2 \otimes w_2 &\implies v_1 \otimes w_1 = \alpha v_1 \otimes w_2 \\
&\implies v_1 \otimes w_1 = \alpha v_1 \otimes w_2 \\
&\implies v_1 \otimes w_1 = v_1 \otimes \alpha w_2 \\
&\implies v_1 \otimes w_1 - v_1 \otimes \alpha w_2 = 0 \\
&\implies v_1 \otimes (w_1 - \alpha w_2) = 0 \\
&\implies w_1 - \alpha w_2 = 0 \\
&\implies w_1 = \alpha w_2
\end{aligned}
$$

Exercise 1.7.11

Let A be a matrix. Find

$$I_n \otimes I_m \quad \text{and} \quad A \otimes 0.$$

Solution. By the definition of Kronecker product of matrices:

$$I_n \otimes I_m = I_{nm}$$

and

$$A \otimes 0 = 0 \otimes A = 0.$$

Exercise 1.7.12

Let A be an $n \times n$ matrix and B an $m \times m$ matrix. Show that

(1) $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \cdot \operatorname{tr}(B)$,

(2) $\det(A \otimes B) = (\det A)^m \cdot (\det B)^n$.

Solution. **(1)** Let $A = (\alpha_{ij})$. We have

$$
\begin{aligned}
\operatorname{tr}(A \otimes B) &= \operatorname{tr} \begin{pmatrix} \alpha_{11}B & \cdots & \alpha_{1n}B \\ \vdots & \ddots & \vdots \\ \alpha_{n1}B & \cdots & \alpha_{nn}B \end{pmatrix} \\
&= \sum_{k=1}^{n} \operatorname{tr}(\alpha_{kk}B) \\
&= \sum_{k=1}^{n} \alpha_{kk} \operatorname{tr}(B) \\
&= \operatorname{tr}(B) \sum_{k=1}^{n} \alpha_{kk} \\
&= \operatorname{tr}(B)\operatorname{tr}(A).
\end{aligned}
$$

(2) Let A be an $n \times n$ matrix whose eigenvalues are $\alpha_1, \ldots, \alpha_n$ and let B be an $m \times m$ matrix whose eigenvalues are $\beta_1, \ldots, \beta_m$. Using Schur's Triangularization Theorem, there exist unitary matrices R and S such that

$$A = RT_1R^{-1} \quad \text{and} \quad B = ST_2S^{-1}$$

where

$$T_1 = \begin{pmatrix} \alpha_1 & \star & \cdots & \star \\ 0 & \alpha_2 & \cdots & \star \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} \beta_1 & \star & \cdots & \star \\ 0 & \beta_2 & \cdots & \star \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_m \end{pmatrix}$$

Then

$$\begin{aligned} A \otimes B &= (RT_1R^{-1}) \otimes (ST_2S^{-1}) \\ &= (R \otimes S)(T_1 \otimes T_2)(R^{-1} \otimes S^{-1}) \\ &= (R \otimes S)(T_1 \otimes T_2)(R \otimes S)^{-1}. \end{aligned}$$

Hence

$$\det(A \otimes B) = \det(T_1 \otimes T_2),$$

and since

$$T_1 \otimes T_2 = \begin{pmatrix} \boxed{\begin{matrix} \alpha_1\beta_1 & \star & \cdots & \star \\ 0 & \alpha_1\beta_2 & \cdots & \star \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_1\beta_m \end{matrix}} & & & \star \\ & \boxed{\begin{matrix} \alpha_2\beta_1 & \star & \cdots & \star \\ 0 & \alpha_2\beta_2 & \cdots & \star \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_2\beta_m \end{matrix}} & & \\ & & \ddots & \\ 0 & & & \boxed{\begin{matrix} \alpha_n\beta_1 & \star & \cdots & \star \\ 0 & \alpha_n\beta_2 & \cdots & \star \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n\beta_m \end{matrix}} \end{pmatrix}$$

$$\det(A \otimes B) = \det(T_1 \otimes T_2) = (\prod_{i=1}^{n} \alpha_i)^m (\prod_{j=1}^{m} \beta_j)^n.$$

Therefore

$$\det(A \otimes B) = (\det A)^m (\det B)^n.$$

Exercise 1.7.13

Let A and B be two matrices. Show that

$$\operatorname{tr}(A \otimes I_n + I_m \otimes B) = n\operatorname{tr}(A) + m\operatorname{tr}(B).$$

Solution.

$$\operatorname{tr}(A \otimes I_n + I_m \otimes B) = \operatorname{tr}(A \otimes I_n) + \operatorname{tr}(I_m \otimes B) = n\operatorname{tr}(A) + m\operatorname{tr}(B).$$

Exercise 1.7.14

Show that $A \otimes B = 0$ if and only if $A = 0$ or $B = 0$.

Solution. Clearly, if $A = 0$ or $B = 0$, then $A \otimes B = 0$. Conversely, assume that $A \otimes B = 0$ and $A = (a_{ij})$, then $a_{ij}B = 0$, for all i and j. We have two cases:

(i) If $a_{ij} = 0$, for all i and j, then $A = 0$.

(ii) If there is r and s such that $a_{rs} \neq 0$, then the equation $a_{rs}B = 0$ implies that $B = 0$.

Exercise 1.7.15

Let A be an $m \times n$ matrix. what size matrix is $A^{\otimes k}$? where

$$A^{\otimes k} = \underbrace{A \otimes A \otimes \cdots \otimes A}_{k \text{ times}}.$$

Solution. We know that, if A is an $m \times n$ matrix and B is an $m' \times n'$ matrix, then $A \otimes B$ is an $mm' \times nn'$ matrix. Hence the size matrix is $A^{\otimes k}$ is $m^k \times n^k$.

Exercise 1.7.16

Let A be an $m \times m$ and B be an $n \times n$ matrix. Recall that, the direct sum is the $(m+n) \times (m+n)$ matrix

$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Find the 2×2 matrices X such that $X \oplus X = X \otimes X$.

Solution. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

such that $A \oplus A = A \otimes A$. Then

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^2 & a_{11}a_{12} & a_{12}a_{11} & a_{12}^2 \\ a_{11}a_{21} & a_{11}a_{22} & a_{12}a_{21} & a_{12}a_{22} \\ a_{21}a_{11} & a_{21}a_{12} & a_{22}a_{11} & a_{22}a_{12} \\ a_{21}^2 & a_{21}a_{22} & a_{22}a_{21} & a_{22}^2 \end{pmatrix}$$

By comparison:

$$\begin{cases} a_{11} = a_{11}^2 \\ a_{22} = a_{22}^2 \\ a_{12}(a_{11} - 1) = 0 \\ a_{21}(a_{11} - 1) = 0 \\ a_{22}(a_{11} - 1) = 0 \\ a_{11}(a_{22} - 1) = 0 \\ a_{12}(a_{22} - 1) = 0 \\ a_{21}(a_{22} - 1) = 0 \\ a_{12}a_{11} = a_{12}^2 = a_{12}a_{21} = a_{12}a_{22} = 0 \\ a_{11}a_{21} = a_{12}a_{21} = a_{21}^2 = a_{21}a_{22} = 0. \end{cases}$$

Thus $a_{12} = a_{21} = 0$, and $a_{11} = a_{22} = 0$ or $a_{11} = a_{22} = 1$. Therefore

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Exercise 1.7.17

Show that if A and B are two Hermitian matrices of the same size, then $A \otimes B$ is Hermitian.

Solution. We know that

$$(A \otimes B)^{\mathsf{h}} = A^{\mathsf{h}} \otimes B^{\mathsf{h}}.$$

So, if A and B are Hermitian, we get $A^{\mathsf{h}} = A$ and $B^{\mathsf{h}} = B$. Hence

$$(A \otimes B)^{\mathsf{h}} = A \otimes B.$$

This implies that $A \otimes B$ is Hermitian.

Exercise 1.7.18

Let A be an $n \times n$ matrix and B an $m \times m$ matrix. Prove that, if $A \otimes B = \lambda I_{nm}$ such that $\lambda \neq 0$, then there exist a scalars α and β such that $A = \alpha I_n$, $B = \beta I_m$ and $\alpha\beta = \lambda$.

Solution.

$$a_{rr} b_{ss} = \lambda \quad \text{for all } r, s$$

and

$$a_{rk} b_{ij} = 0 \quad \text{for all } r \neq k, i, j$$

We have $A \otimes B \neq 0 \implies B \neq 0$, and hence $a_{rk} = 0 \quad$ for all $r \neq k$. That means A is diagonal matrix. Put

$$A = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix}$$

So

$$A \otimes B = \begin{pmatrix} \alpha_1 B & 0 & \cdots & 0 \\ 0 & \alpha_2 B & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n B \end{pmatrix} = \lambda I_{nm}.$$

Consequently, for all i,

$$\alpha_i B = \lambda I_m. \tag{1.15}$$

So $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, and hence

$$A = \alpha I_n,$$

and form the equality (1.15), we get

$$B = \alpha^{-1} \lambda I_m = \beta I_m,$$

Remark that $\alpha \neq 0$, because if $\alpha = 0$, then $A = 0$, and hence $A \otimes B = 0$, which is a contradiction.

Exercise 1.7.19

Let $V_1, V_2, W_1, W_2, U_1, U_2$ be $\mathbb{F}$-vector spaces. Consider the following linear mappings of vector spaces:

$$V_1 \xrightarrow{F_1} W_1 \xrightarrow{G_1} U_1$$

$$V_2 \xrightarrow{F_2} W_2 \xrightarrow{G_2} U_2$$

Show that $(G_1 \circ F_1) \otimes (G_2 \circ F_2) = (G_1 \otimes G_2) \circ (F_1 \otimes F_2)$.

Solution. Let $v_1 \in V_1$ and $v_2 \in V_2$. We have:

$$\begin{aligned}
\Big((G_1 \otimes G_2) \circ (F_1 \otimes F_2)\Big)(v_1 \otimes v_2) &= (G_1 \otimes G_2)\left((F_1 \otimes F_2)(v_1 \otimes v_2)\right) \\
&= (G_1 \otimes G_2)\left(F_1(v_1) \otimes F_2(v_2)\right) \\
&= G_1\left(F_1(v_1)\right) \otimes G_2\left(F_2(v_2)\right) \\
&= (G_1 \circ F_1)(v_1) \otimes (G_2 \circ F_2)(v_2) \\
&= \Big((G_1 \circ F_1) \otimes (G_2 \circ F_2)\Big)(v_1 \otimes v_2).
\end{aligned}$$

Exercise 1.7.20

Let V_1, V_2, W_1, W_2 be $\mathbb{F}$-vector spaces and $\alpha \in \mathbb{F}$. Consider the following linear mappings of vector spaces:

$$V_1 \underset{G_1}{\overset{F_1}{\rightrightarrows}} W_1$$

$$V_2 \underset{G_2}{\overset{F_2}{\rightrightarrows}} W_2$$

Prove the following properties:

(1) $(F_1 + G_1) \otimes F_2 = (F_1 \otimes F_2) + (G_1 \otimes F_2)$

(2) $(\alpha F_1) \otimes F_2 = \alpha (F_1 \otimes F_2)$.

Solution. **(1)** Let $v_1 \in V_1$ and $v_2 \in V_2$. We have:

$$\begin{aligned}
\Big((F_1 + G_1) \otimes F_2\Big)(v_1 \otimes v_2) &= (F_1 + G_1)(v_1) \otimes F_2(v_2) \\
&= \Big(F_1(v_1) + G_1(v_2)\Big) \otimes F_2(v_2) \\
&= F_1(v_1) \otimes F_2(v_2) + G_1(v_1) \otimes F_2(v_2) \\
&= (F_1 \otimes F_2)(v_1 \otimes v_2) + (G_1 \otimes F_2)(v_1 \otimes v_2) \\
&= \Big(\ (F_1 \otimes F_2) + (G_1 \otimes F_2)\ \Big)(v_1 \otimes v_2)
\end{aligned}$$

Hence

$$(F_1 + G_1) \otimes F_2 = (F_1 \otimes F_2) + (G_1 \otimes F_2).$$

(2) Let $v_1 \in V_1$ and $v_2 \in V_2$. We have:

$$\begin{aligned}\Big((\alpha F_1)\otimes F_2\Big)(v_1\otimes v_2) &= (\alpha F_1)(v_1)\otimes F_2(v_2)\\ &= \alpha F_1(v_1)\otimes F_2(v_2)\\ &= \alpha\Big(F_1(v_1)\otimes F_2(v_2)\Big)\\ &= \Big(\alpha(F_1\otimes F_2)\Big)(v_1\otimes v_2)\end{aligned}$$

Hence

$$(\alpha F_1)\otimes F_2 = F_1\otimes(\alpha F_2) = \alpha\,(F_1\otimes F_2)\,.$$

Exercise 1.7.21

Let A, B be square matrices of order n, and C, D be square matrices of order m.

(1) Show that for all integers $k \geq 0$ and $m \geq 1$, we have

$$(I_m\otimes A)^k = I_m\otimes A^k.$$

(2) Recall that the exponential of a square matrix A of order n is defined by :

$$\mathrm{e}^A = I_n + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots + \frac{1}{k!}A^k + \cdots$$

Show that:

$$\mathrm{e}^{A\otimes I_m} = \mathrm{e}^A\otimes I_m \quad \text{and} \quad \mathrm{e}^{I_m\otimes A} = I_m\otimes \mathrm{e}^A.$$

Solution.

(1)

$$\begin{aligned}(I_m\otimes A)^k &= \underbrace{(I_m\otimes A)(I_m\otimes A)\cdots(I_m\otimes A)}_{k\text{ times}}\\ &= \underbrace{(I_mI_m\cdots I_m)}_{k\text{ times}}\otimes\underbrace{(AA\cdots A)}_{k\text{ times}}\\ &= I_m\otimes A^k.\end{aligned}$$

(2)

$$\begin{aligned}\mathrm{e}^{A\otimes I_m} &= (I_n\otimes I_m) + (A\otimes I_m) + \frac{1}{2!}\,(I_m\otimes A)^2 + \ldots\\ &= (I_n\otimes I_m) + (A\otimes I_m) + \frac{1}{2!}\left(A^2\otimes I_m\right) + \ldots\\ &= \left(I_n + A + \frac{1}{2!}A^2 + \ldots\right)\otimes I_m\\ &= \mathrm{e}^A\otimes I_m\end{aligned}$$

Exercise 1.7.22

Let A, B be square matrices of order n and m respectively. We define the Kronecker sum of matrices by

$$A \oplus B = (A \otimes I_m) + (I_n \otimes B).$$

Show that

(1) $(A \otimes I_m)$ and $(I_n \otimes B)$ commute.

(2) $\mathrm{e}^{A \oplus B} = \mathrm{e}^A \otimes \mathrm{e}^B$

Solution.

$$\begin{aligned} \mathrm{e}^{(A \oplus B)} \quad &= \mathrm{e}^{(A \otimes I_m + I_n \otimes B)} \\ &= (\mathrm{e}^A \otimes I_m)(I_n \otimes \mathrm{e}^B) \\ &= (\mathrm{e}^A I_n) \otimes (I_m \mathrm{e}^B) \\ &= \mathrm{e}^A \otimes \mathrm{e}^B \end{aligned}$$

Chapter

2 Tensor products (Part 2)

Chapter contents

Modules over a ring are a generalization of the concept of vector spaces. In this chapter, we will give the definition of the tensor product of R-modules (resp. tensor product of morphisms). Also various properties of these tensor products are explained in this chapter.

2.1 Modules over a ring

Definition 2.1.1 Left R-module

Let M be an abelian group and R a ring with unity 1_R. We say that M is a left R-module, if there is a scalar product

$$\begin{aligned} \cdot\ :R \times M &\longrightarrow M \\ (r,m) &\longmapsto rm, \end{aligned}$$

satisfying the following axioms.

- $\alpha(\beta m) = (\alpha\beta)m$
- $(\alpha+\beta)m = \alpha m + \beta \cdot m$
- $\alpha(m+n) = \alpha m + \alpha n$
- $1_R m = m$

where $\alpha, \beta \in R$ and $m, n \in M$.

Similarly, the right R-modules are defined as follow :

Definition 2.1.2 right R-module

Let M be an abelian group and R a ring with unity 1_R. We say that M is a right R-module, if there is a scalar product

$$\begin{aligned} \cdot\ :M \times R &\longrightarrow M \\ (m,r) &\longmapsto mr, \end{aligned}$$

satisfying the following axioms.

- $(m\alpha)\beta = m(\alpha\beta)$
- $m(\alpha+\beta) = m\alpha + m\beta$
- $(m+n)\alpha = m\alpha + n\alpha$
- $m1_R = m$

where $\alpha, \beta \in R$ and $m, n \in M$.

Remark 2.1.3. If R is a commutative ring, every left R module is right module, and conversely. In fact, let M be a left R-module. Define a mapping $M \times R \longrightarrow R$ by $mr = rm$ and we can show directly that the axioms of the right module are satisfied. Therefore, if a ring R is commutative, it is not necessary to distinguish between left and right.

Example 2.1.4

If R is a field, a R-module is a R-vector space.

Example 2.1.5

If G is an abelian group, the G can be viewed as $\mathbb{Z}$-module with scalar multiplication defined as, for $g \in G$ and $n \in \mathbb{Z}$,

$$ng = \begin{cases} \underbrace{g + g + \ldots + g}_{n \text{ times}} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ \underbrace{(-g) + (-g) + \ldots + (-g)}_{n \text{ times}} & \text{if } n < 0 \end{cases}$$

where $-g$ is the inverse of g.

Definition 2.1.6 Homomorphism of R-modules

Let R be a ring and let M and N be R-modules. A function $f : M \longrightarrow N$ is an R-module homomorphism if and only if the following conditions hold:

- $f(m_1 + m_2) = f(m_1) + f(m_2) \quad$ for all $m_1, m_2 \in M$
- $f(\alpha m) = \alpha f(m) \quad$ for all $\alpha \in R, m \in M$.

Definition 2.1.7 Isomorphism of R-modules

Let R be a ring M and N be R-modules and let $f : M \longrightarrow N$ be an R-module homomorphism. The function f is an R-module isomorphism if and only if f is one-to-one and onto.

As a generalization of bilinear mapping, we define the concept of a balanced mapping.

Definition 2.1.8 Balanced mapping

For a ring R, a right R-module M, a left R-module N, and an abelian group G, a map $\varphi : M \times N \longrightarrow G$ is said to be R-balanced mapping, if for all $m, m' \in M$, $n, n' \in N$, and $r \in R$ the following hold:

$$\varphi(m, n + n') = \varphi(m, n) + \varphi(m, n')$$
$$\varphi(m + m', n) = \varphi(m, n) + \varphi(m', n)$$
$$\varphi(mr, n) = \varphi(m, rn)$$

Note 2.1.9

The set of all such balanced mapping over R from $M \times N$ to G is denoted by $\text{Hom}(M, N; G)$, and it is an abelian group (see, Exercise 2.3.1).

Definition 2.1.10 Generating set for R-modules

Let M be a left R-module. A subset S of M is called a set of generators (or a generating set) of M, if every element of Mcan be expressed as a linear combination of a finite number of $\{s_i\}$ with coefficients in R, That mean

$$m = \sum_{i=1}^{k} r_i s_i,$$

for some $r_i \in R$ and $s_i \in S$.

Definition 2.1.11 Free R-basis / R-free subset

Let M be a left R-module. A subset S of M is called R-free, if for all $\{s_1, \dots, s_k\} \subset S$, we have

$$\sum_{i=1}^{k} r_i s_i = 0 \quad \Longrightarrow \quad r_i = 0 \quad \text{for all} \quad i = 1, ..., k.$$

Definition 2.1.12 Free R-module

A R-free set of generators of M is called an R-basis of M and an R- module M which has an R-basis is called a free R-module.

Definition 2.1.13 Then free module R^S

Let S be a set and $L(S)$ the set of mappings from $S \longrightarrow R$ with finite support, where

$$\text{support}(f) = \{s \in S \mid f(s) \neq 0\}.$$

If $\boldsymbol{I}$ is the finite set of $s \in S$ with a non-zero image, we can denote $f(s) = r_s \in R$ and identifying a map with the set of its values, write the map as

$$f = \sum_{s \in I} r_s e_s,$$

where $e_s : S \longrightarrow R$ is the mapping given by

$$e_s(t) = \begin{cases} 1 & \text{if} \quad t = s, \\ 0 & \text{otherwise} \end{cases}$$

Clearly $(L(S), +)$ is an abelian group, and it's can be considered as R-module with the scalar multiplication defined as follow: for all $r \in R$ and $f \in L(S)$, the mapping rf is given by

$$(rf)(s) = r(f(s)), \quad \text{for all} \quad s \in S.$$

In addition, the set $\{e_s \mid s \in S\}$ form a basis for the R-module $L(S)$.

Note 2.1.14

The free module $L(S)$ will be denoted by R^S, and it's called by the module of the finite formal linear combinations of elements of S.

Definition 2.1.15 Submodule / quotient module

- A submodule of an R-module M is a subgroups N of M which is closed under by the scalar multiplication of M, that means, for all $r \in R$ and $n, n' \in N$, we have $rn \in N$, and $n - n' \in N$.
- The quotient group M/N becomes an R-module by defining $a(x + N) = ax + N$. The R-module M/N is the quotient of M by N.

2.2 Tensor product of modules

In these section, we define tensor products of modules over a commutative ring with unity and various properties this tensor products are given.

Definition 2.2.1 R-linear, Homomorphism of modules

Let M and N be two R-modules. A mapping $f : M \longrightarrow N$ is called an R-module homomorphism or an R-linear mapping if

(1) $f(m + m') = f(m) + f(m')$,

(2) $f(rm) = rf(m)$.

The set of all module homomorphisms from M to N is denoted by $\mathrm{Hom}_R(M, N)$.

Definition 2.2.2 R-bilinear mapping

Let M, B and G be R-modules. A mapping $f : M \times N \longrightarrow G$ is called a R-bilinear, if it is linear in each variable. That means : for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in V$ and $r \in \mathbb{R}$, we have

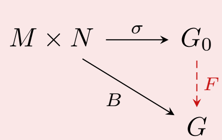
(1) $f(rm_1 + m_2, n) = rf(m_1, v) + f(m_2, n)$

(2) $f(m, rn_1 + n_2) = rf(m, n_1) + f(m, n_2)$,

The set of all R-bilinear mappings from $M \times N$ to G is denoted by $\mathrm{Hom}_R(M, N; G)$.

Theorem 2.2.3

Let R be a ring, M and N two R-modules. Then there exist a pair (G_0, σ) R-module G_0 and an R-bilinear mapping $\sigma : M \times N \longrightarrow G_0$ such that, for every R-bilinear mapping $B : M \times N \longrightarrow G$, there exists a unique homomorphism of $\mathbb{R}$-modules $F : G_0 \longrightarrow G$ such that $B = F \circ \sigma$.



Definition 2.2.4 Tensor product of R-modules

Let M and N be two R-modules. We say that (M,N) satisfy the property **(T)**, if here exist a pair (G_0,σ) consisting of an R-module G_0 and an R-bilinear mapping $\sigma : M\times N \longrightarrow G_0$ such that, for every R-bilinear mapping $B : M\times N \longrightarrow G$, there exists a unique homomorphism of $\mathbb{R}$-modules $F : G_0 \longrightarrow G$ such that $B = F\circ\sigma$.

$$\begin{array}{ccc} M\times N & \xrightarrow{\ \sigma\ } & G_0 \\ & {\scriptstyle B}\searrow & \downarrow{\scriptstyle F} \\ & & G \end{array}$$

The existence of which is assured by Theorem 2.2.5 is called a tensor product of M and N.
We write

$$G_0 = M\otimes_R M \quad \text{and} \quad \sigma(m,n) = m\otimes n.$$

The mapping σ is called the canonical R-bilinear mapping of a tensor product $V\otimes W$.

Theorem 2.2.5 Tensor Product of Modules

If M is a right R-module and N is a left R-module. Then their tensor product $M\otimes_R N$ is the quotient of the free R-module $R^{M\times N}$ by the R-submodule T generated by the elements

(a) $e_{(m,n+n')} - e_{(m,n)} - e_{(m,n')}$

(b) $e_{(m+m',n)} - e_{(m,n)} - e_{(m',n)}$

(c) $e_{(mr,n)} - e_{(m,rn)}$

Remark 2.2.6. We have the following natural mappings:

$$\begin{array}{ccccc} M\times N & \xrightarrow{\quad e \quad} & R^{M\times N} & \xrightarrow{\quad \pi \quad} & R^{M\times N}/T = M\otimes_R N \\ (m\times n) & \longmapsto & e_{(m,n)} & \longmapsto & m\otimes n = \overline{e_{(m,n)}} = e_{(m,n)} + T \end{array}$$

So, for all $m,m'\in M, n,n'\in N, r\in R$. We have a natural mapping $M\times N \longrightarrow M\otimes_R N$, where write $m\otimes n$ for the image of (m,n) in $M\otimes_R N$. Hence we have

(1) $m\otimes(n+n') = m\otimes n + m\otimes n'$

(2) $(m+m')\otimes n = m\otimes n + m'\otimes n$

(3) $mr\otimes n = m\otimes rn$.

Example 2.2.7

$\mathbb{Z}/2\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/3\mathbb{Z} = 0$ because

$$\begin{aligned} m\otimes n &= m1\otimes n = m3\otimes n \\ &= m\otimes 3n = m\otimes 0 \\ &= m\otimes 0\cdot 0 = m\cdot 0\otimes 0 \\ &= 0. \end{aligned}$$

Theorem 2.2.8 Tensor products of R-linear mappings : $F_1 \otimes F_2$

Let $F_1 : M_1 \longrightarrow N_1$ and $F_2 : M_2 \longrightarrow N_2$ be R-module homomorphisms. Then there exists a linear mapping $\tilde{F} : M_1 \otimes M_2 \longrightarrow N_1 \otimes N_2$ such that for all $m_1 \in M_1$ and $m_2 \in M_2$

$$\tilde{F}(m_1 \otimes m_2) = F_1(m_1) \otimes F_2(m_2).$$

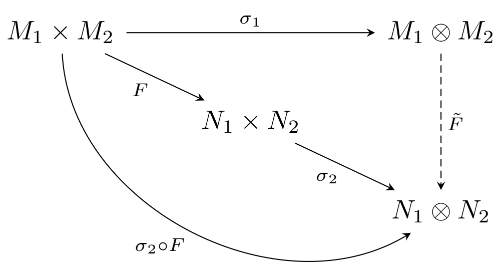
The mapping $\tilde{F}$ is called the tensor product of F_1 and F_2 and is denoted by $F_1 \otimes F_2$.

Proof. Let σ_1 and σ_2 be the canonical mappings of $M_1 \otimes M_2$ and $N_1 \otimes N_2$ respectively.

Consider the R-bilinear mapping $F = F_1 \times F_2 : M_1 \times M_2 \longrightarrow N_1 \times N_2$ given by

$$(F_1 \times F_2)(m_1, m_2) = (F_1(m_1), F_2(m_2)).$$

Apply the property **(T)** for the tensor product $M_1 \otimes M_2$, there is an R-linear mapping $\tilde{F}$ for which the following diagram is commutative:



Hence

$$\tilde{F}(v_1 \otimes v_2) = \sigma_2(F(v_1, v_2)) = \sigma_2(F_1(v_1), F_2(v_2)) = F_1(v_1) \otimes F_2(v_2).$$

□

Proposition 2.2.9 Commutativity of the tensor product

Let N and M be R-modules. By the correspondence $(m \otimes n \longleftrightarrow w \otimes m\ )$, we have

$$M \otimes N \;\cong\; N \otimes M.$$

Proposition 2.2.10 Associativity of the tensor product

The correspondence

$$(m_1 \otimes m_2) \otimes m_3 \longleftrightarrow m_1 \otimes (m_2 \otimes m_3)$$

gives an isomorphism

$$(M_1 \otimes M_2) \otimes M_3. \cong M_1 \otimes (M_2 \otimes M_3).$$

2.3 Exercises set

Exercise 2.3.1

Let R be a commuatative ring and f and g an R-bilinear mappings from $N \times M \longrightarrow G$.

(1) Show that $f+g$ and $-f$ are R-bilinear mappings.

(2) Deduce that $\mathrm{Hom}((N, M; G)$ is an R-module.

Solution. **(1)** For all $n, n_1, n_2 \in N$, $m, m_1, m_2 \in M$ and $r \in R$, we have

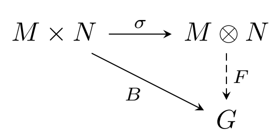
$$\begin{aligned}(f+g)(n_1+n_2, m) &= f(n_1+n_2, m) + g(n_1+n_2, m)\\ &= f(n_1, m) + f(n_2, m) + g(n_1, m) + g(n_2, m)\\ &= f(n_1+n_2, m) + g(n_1+n_2, m)\end{aligned}$$

and

Exercise 2.3.2

Let R be a ring, M, N be two R-modules. Show that $m \otimes n = 0$, if and only if, for every R-balanced mapping $B : M \times N \longrightarrow G$, we have $B(m, n) = 0$.

Solution. Assume that $m \otimes n = 0$. Using the property **(T)** of the tensor product $M \otimes N$, for every R-balanced mapping $B : M \times N \longrightarrow G$, there exists a homomorphism of $\mathbb{Z}$-modules $F : G_0 \longrightarrow G$ such that $B = F \circ \sigma$.



So

$$B(m, n) = F(m \otimes n) = F(0) = 0.$$

Reciprocally, assume that for every R-balanced mapping $B : M \times N \longrightarrow G$, we have $B(m, n) = 0$. hence if we take $B = \sigma$, we get the canonical mapping $\sigma(m, n) = 0$, so $m \otimes n = 0$.

Exercise 2.3.3

Show that, in the tensor product of modules $M \otimes N$, we have

$$m \otimes 0 = 0 \otimes n = 0 \quad \text{for all} \quad m \in M \quad \text{and} \quad n \in N.$$

Solution.

$$m \otimes 0 = m \otimes (0+0) = m \otimes 0 + m \otimes 0$$

Subtracting $m \otimes 0$ from both sides, $m \otimes 0 = 0$.

Exercise 2.3.4

Let M and N be R-modules with respective generating sets $\{m_i\}_{i\in I}$ and $\{n_j\}_{j\in J}$. Show that the tensor product $M \otimes N$ is generated linearly by the elementary tensors $m_i \otimes n_j$.

Solution. Let $m \otimes n \in M \otimes N$. Then

$$m \otimes n = \sum_{i\in I} r_i m_i \quad \otimes \quad \sum_{j\in J} s_j n_j \quad = \quad \sum_{i\in I}\sum_{j\in J} r_i s_j (m_i \otimes n_j).$$

Exercise 2.3.5

For positive integers a and b relatively prime. Show that

$$\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z} \quad = \quad 0.$$

Solution. Since a and b are relatively prime , there exist two integers r and s such that

$$1 = ar + bs.$$

Let B be an arbitrary R-balanced mapping $B : M \times N \longrightarrow G$ to an abelian group G. Then for any $(m, n) \in \mathbb{Z}/a\mathbb{Z} \times_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z}$, we have

$$\begin{aligned} B(n, m) &= (ar + bs)B(n, m) \\ &= arB(n, m) + bsB(n, m) \\ &= rB(an, m) + sB(n, bm) \\ &= rB(0, m) + sB(n, 0) \\ &= 0 \end{aligned}$$

Using Exercise **??**, we get $m \otimes n = 0$ for all $(m, n) \in \mathbb{Z}/a\mathbb{Z} \times_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z}$. Therefore

$$\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z} \quad = \quad 0.$$

Exercise 2.3.6

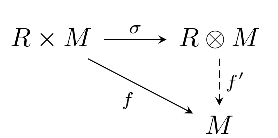
Let M be an R-module. Then on regarding R as a module over itself, show that

$$R \otimes M \cong M.$$

Solution. Define a map $f : R \times M \longrightarrow M$ by

$$f(r, m) = rm, \quad r \in R, \; m \in M.$$

By properties of an R-module, it can be easily to show that f is R-bilinear. Then by the universal property of the tensor product $R \otimes M$ l, there exist a unique R-module homomorphisnr $f' : R \otimes M \longrightarrow M$ such that $f = f' \circ \sigma$ i.e. the following diagram commutes



For any $(r, m) \in R \times M$, we have that

$$f(r, m) = f'(\sigma(r, m)) = f'(r \otimes m).$$

Hence by the definition of f,

$$f'(r \otimes n) = rm.$$

We now claim that f' is an isomorphism:

Surjectivity of f : For any $m \in M$. Since R is a ring with unity 1, we have that $1 \otimes m \in R \otimes M$ and then

$$f'(1 \otimes m) = 1m = m.$$

Therefore f' is surjective.

Injectivity of f : An arbitrary element of $R \otimes M$ is a finite sum of the form

$$\sum_i r_i \otimes m_i = \sum_i r_i (1 \otimes m_i) = \sum_i 1 \otimes (r_i m_i) = 1 \otimes \sum_i r_i m_i = 1 \otimes m,$$

for some $r_i \in R$ and $m_i \in M$. Therefore, every element in $R \otimes M$ can be written as $1 \otimes m$ for some $m \in M$. Now if $1 \otimes m \in \ker f$, then

$$f'(1 \otimes m) = 0 \Longrightarrow 1m = 0 \Longrightarrow m = 0,$$

$1 \otimes m = 1 \otimes 0 = 0$. Hence $\ker f = \{0\}$, so f is injective.

Consequently f' is an isomorphism.

Chapter 3

Tensors and Tensor Algebras

Chapter contents

In this chapter, we give the definition of the tensor algebra $T^p(V)$ generated by a vector space V over a field $\mathbb{F}$. It is also denoted as

$$\bigotimes_{i=0}^{p} V \quad \text{or} \quad V^{\otimes p}$$

and is called the p-th tensor power of V (with $V^{\otimes 1} = V$, and $V^{\otimes 0} = \mathbb{F}$). We can pack all the tensor powers of V into the "big" vector space,

$$T(V) = \bigoplus_{p\geq 0} V^{\otimes p}.$$

This is one of the most important associative algebra defined from V. The elements of this new vector space are called "*Tensors*". Also we present in this chapter the definition of symmetric and alternating tensors with their properties.

3.1 Tensor spaces

Let us now see how tensor products behave under duality. For this, we define a pairing between $V_1^* \otimes \cdots \otimes V_n^*$ and $V_1 \otimes \cdots \otimes V_n$. For any fixed

$$(f_1, \ldots, f_n) \in V_1^* \times \cdots \times V_n^*,$$

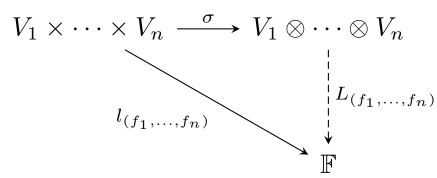
we have a multilinear form $l_{(f_1,\ldots,f_n)} : V_1 \times \cdots \times V_n \longrightarrow \mathbb{F}$ defined by

$$l_{(f_1,\ldots,f_n)}(v_1, \ldots, v_n) = f_1(v_1) \cdot f_2(v_2) \cdots f_n(v_n).$$

Using the property **(T)** of the tensor product $V_1 \otimes \cdots \otimes V_n$, there exist a linear mapping

$$L_{(f_1,\ldots,f_n)} : V_1 \otimes \cdots \otimes V_n \longrightarrow \mathbb{F}$$

such that the following diagram commutes:



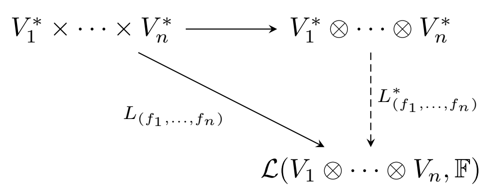
Therefore, we have a multilinear mapping

$$\begin{aligned} L : \qquad V_1^* \times \cdots \times V_n^* &\longrightarrow \mathcal{L}(V_1 \otimes \cdots \otimes V_n, \mathbb{F}) \\ (f_1, \ldots, f_n) &\longmapsto L_{(f_1,\ldots,f_n)} \end{aligned}$$

Using also the property **(T)** of the tensor product $V_1^* \otimes \cdots \otimes V_n^*$, there exist a linear mapping

$$L^*_{(f_1,\ldots,f_n)} : V_1 \otimes \cdots \otimes V_n \longrightarrow \mathbb{F}$$

such that the following diagram commutes:



Finlay, we have constructed a linear mapping:

$$L^* : V_1^* \otimes \cdots \otimes V_n^* \longrightarrow \mathcal{L}(V_1 \otimes \cdots \otimes V_n, \mathbb{F}).$$

Therefore

$$L^* \in \mathcal{L}(V_1^* \otimes \cdots \otimes V_n^*, \mathcal{L}(V_1 \otimes \cdots \otimes V_n, \mathbb{F})).$$

By the fact that (see Exercise 1.7.8), for any $\mathbb{F}$-vector spaces. V, W and U,

$$\mathcal{L}(V \otimes W, U) \;\cong\; \mathcal{L}(V, \mathcal{L}(W, U)),$$

Hence

$$\mathcal{L}\Big(V_1^* \otimes \cdots \otimes V_n^*, \mathcal{L}(V_1 \otimes \cdots \otimes V_n, \mathbb{F})\Big) \;\cong\; \mathcal{L}\Big(\big(V_1^* \otimes \cdots \otimes V_n^*\big) \otimes \big(V_1 \otimes \cdots \otimes V_n\big), \mathbb{F}\Big)$$

So L^* can be viewed a linear form on

$$\big(V_1^* \otimes \cdots \otimes V_n^*\big) \otimes \big(V_1 \otimes \cdots \otimes V_n\big).$$

Definition 3.1.1 Tensor space

If T is a tensor product of p copies of V and q copies of V^*, we call T the tensor space of type (p,q) and denote it by $T^p_q(V)$. More precisely:

$$T^p_q(V) = \underbrace{V \otimes V \otimes V \otimes \cdots \otimes V}_{p \text{ factors}} \otimes \underbrace{V^* \otimes V^* \otimes V^* \otimes \cdots \otimes V^*}_{q \text{ factors}}$$

Definition 3.1.2 Covariant/contravariant

• Elements of the tensor space $T^p_q(V)$ are called tensors of type (p,q) or tensors which are contravariant of degree p and covariant of degree q.

• In particular, tensors of type $(p,0)$ are called contravariant tensors of degree p and those of type $(0,q)$ covariant tensors of degree q.

• Moreover the elements of $T^1_0(V) = V$ are called contravariant vectors, those of $T^0_1(V) = V^*$ covariant vectors, and those of $T^0_0(V) = \mathbb{F}$ scalars.

Remark 3.1.3. Sometimes we write

$$T^p(V) = T^p_0(V) = \underbrace{V \otimes V \otimes V \otimes \cdots \otimes V}_{p \text{ factors}}$$

and

$$T_q(V) = T^0_q(V) = \underbrace{V^* \otimes V^* \otimes V^* \otimes \cdots \otimes V^*}_{q \text{ factors}}$$

Example 3.1.4 $T^1_1(V)$

From Exercise 1.7.9, we know that $V^* \otimes V \cong \mathcal{L}(V,V)$. Hence

$$T^1_1(V) \cong \mathcal{L}(V,V).$$

That is, the linear transformations of V can be regarded as tensors of type $(1,1)$.

Example 3.1.5 $T_2(V)$

Since $V \otimes W \cong \mathcal{L}(V^*, W^*; \mathbb{F})$ and $(V^*)^* \cong V$, we have

$$T_2(V) = V^* \otimes V^* \cong \mathcal{L}(V, V; \mathbb{F}).$$

More general, we can show that

$$T_q(V) = \underbrace{V^* \otimes V^* \otimes V^* \otimes \cdots \otimes V^*}_{q \text{ factors}} \cong \mathcal{L}(\underbrace{V, V, \cdots, V}_{q \text{ factors}}; \mathbb{F}).$$

Example 3.1.6 $T_2^1(V)$

Setting $W = U = V$ in the following formula

$$\mathcal{L}(V\,,\,W\,;\,U) \cong \mathcal{L}(V\otimes W\,,\,U) \cong (V\otimes W)^* \otimes U \cong V^*\otimes W^*\otimes U\,,$$

we have

$$\mathcal{L}(V,\,V:\,V) \cong V^*\otimes V^*\otimes V = T_2^1(V).$$

Example 3.1.7 Dual space of $T_q^p(V)$

$$\left(T_q^p(V)\right)^* \;\cong\; T_p^q(V).$$

3.2 Properties of tensor spaces

Proposition 3.2.1

If V is a $\mathbb{F}$-vector space of dimension n, then

$$\dim T_q^p(V) \;=\; n^{p+q}.$$

Proof. Since $\dim V\otimes W = \dim V\times \dim W$ and $\dim V^* = \dim V$, we have

$$\begin{aligned}
\dim T_q^p(V) &= \dim\Big(\underbrace{V\otimes V\otimes V\otimes\cdots\otimes V}_{p\text{ factors}}\otimes\underbrace{V^*\otimes V^*\otimes V^*\otimes\cdots\otimes V^*}_{q\text{ factors}}\Big)\\
&= \underbrace{\dim V\times\dim V\times\cdots\times\dim V}_{p\text{ factors}}\times\underbrace{\dim V^*\times\dim V^*\times\cdots\times\dim V^*}_{q\text{ factors}}\\
&= \underbrace{\dim V\times\dim V\times\cdots\dim V}_{p\text{ factors}}\times\underbrace{\dim V\times\dim V\times\cdots\times\dim V}_{q\text{ factors}}\\
&= \underbrace{\dim V\times\dim V\times\cdots\times\dim V}_{p+q\text{ factors}}\\
&= n^{p+q}.
\end{aligned}$$

□

Remark 3.2.2. If $\{v_1,\ldots,v_n\}$ is a basis for V and $\{f_1,\ldots,f_n\}$ its dual basis, then, the set

$$\left\{v_{i_1}\otimes\cdots\otimes v_{i_p}\otimes f_{j_1}\otimes\cdots\otimes f_{j_q}\quad\middle|\quad 1\le i_k\le n\quad\text{and}\quad 1\le j_l\le n\right\}$$

form a basis for $T_q^p(V)$.

Example 3.2.3

Bilinear forms on V can be considered as covariant tensors of degree 2 because

$$T_2^0(V) = V^* \otimes V^* \cong \mathcal{L}(V, V; \mathbb{F}).$$

Let $\{v_1, \ldots, v_n\}$ be a basis for V and $\{f_1, \ldots, f_n\}$ its dual basis. Then, the set

$$\{f_i \otimes f_j \mid 1 \leq i, j \leq n\}$$

Therefore, every bilinear form B on V can be written as

$$B = \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_{ij} f_i \otimes f_j$$

Then

$$\begin{aligned} B(v_k, v_l) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_{ij} f_i \otimes f_j(v_k, v_l) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_{ij} f_i(v_k) f_j(v_l) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_{ij} \delta_{ik} \delta_{jl} \\ &= \xi_{kl}. \end{aligned}$$

Proposition 3.2.4

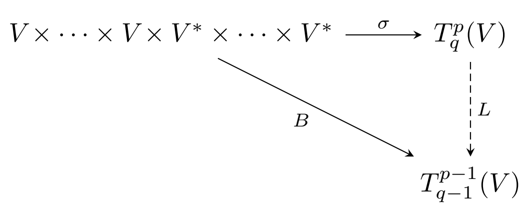
Consider the tensor space $T_q^p(V)$ and assume that $p > 0$ and $q > 0$. Fix integers r and s such that $1 \leq r \leq p$ and $1 \leq s \leq q$. Then there is a unique linear mapping $C_s^r : T_q^p(V) \longrightarrow T_{q-1}^{p-1}(V)$, such that for all $v_i \in V$ and $f_j \in V_j^*$, we have

$$C_s^r(v_1 \otimes \cdots \otimes v_p \otimes f_1 \otimes \cdots \otimes f_q) = f_s(v_r) v_1 \otimes \cdots \otimes v_{r-1} \otimes v_{r+1} \otimes \cdots \otimes v_p \otimes f_1 \otimes \cdots \otimes f_{s-1} \otimes f_{s+1} \otimes \cdots \otimes f_q.$$

Proof. Let $B : V \times \cdots \times V \times V^* \times \cdots \times V^* \longrightarrow T_{q-1}^{p-1}(V)$ the mapping defined by

$$B(v_1 \times \cdots \times v_p \times f_1 \times \cdots \times f_q) = f_s(v_r) v_1 \otimes \cdots \otimes v_{r-1} \otimes v_{r+1} \otimes \cdots \otimes v_p \otimes f_1 \otimes \cdots \otimes f_{s-1} \otimes f_{s+1} \otimes \cdots \otimes f_q.$$

It is easy to see that B is $(p+q)$-multilinear mapping. Using the property **(T)** of the tensor product $T_q^p(V)$, there exists a linear mapping $L : T_q^p(V) \longrightarrow T_{q-1}^{p-1}(V)$, for which the following diagram commutes:



Therefore, we can take $C_s^r = L$.

□

Definition 3.2.5 Contraction

The linear mapping C_s^r is called the contraction with respect to rth contravariant index and sth covariant index.

3.3 Symmetric tensors and alternating tensors

There are families of tensors which are called symmetric or alternating. In this section, we give their definitions and study their properties.

Let S_p be the set of permutations of the set $\{1, ..., p\}$ with p elements. Denote by $\mathrm{sgn}(\sigma)$ the signature of σ, (i.e., $\mathrm{sgn}(\sigma) = 1$ if σ is an even permutation and $\mathrm{sgn}(\sigma) = -1$ if σ is an odd permutation.)

Proposition 3.3.1

Let $\sigma \in S_p$.

(1) There exists a linear mapping $P_\sigma : T^p(V) \longrightarrow T^p(V)$, such that for all $v_1, \dots, v_p \in V$,

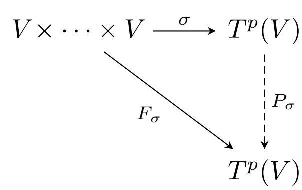
$$P_\sigma(v_1 \otimes \cdots \otimes v_p) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(p)}$$

(2) If $\sigma, \tau \in S_n$, then $P_\sigma \circ P_\tau = P_{\tau\sigma}$

Proof. **(1)** Let $F_\sigma : V_1 \times \cdots \times V_p \longrightarrow T^p(V)$ the mapping defined by

$$F_\sigma(v_1, \dots, v_p) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(p)}$$

This mapping is p-multilinear (see Exercise 3.5.1). Using the property **(T)** of the tensor product $T^p(V)$, there exists a linear mapping P_σ such that the following diagram commutes:



(2)

$$\begin{aligned} P_\sigma \circ P_\tau(v_1 \otimes \cdots \otimes v_p) &= P_\sigma(v_{\tau^{-1}(1)} \otimes v_{\tau^{-1}(2)} \otimes \cdots \otimes v_{\tau^{-1}(p)}) \\ &= v_{\sigma^{-1}\tau^{-1}(1)} \otimes v_{\sigma^{-1}\tau^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}\tau^{-1}(p)} \\ &= v_{(\tau\sigma)^{-1}(1)} \otimes v_{(\tau\sigma)^{-1}(2)} \otimes \cdots \otimes v_{(\tau\sigma)^{-1}(p)} \\ &= P_{\tau\sigma}(v_1 \otimes \cdots \otimes v_p). \end{aligned}$$

□

Remark 3.3.2. Denote by 1 the identity permutation. Then P_1 id the identity transformation of $T^p(V)$.

Definition 3.3.3

(1) An element $t \in T^p(V)$ is called a symmetric tensor, if $P_\sigma(t) = t$ for all $\sigma \in S_p$.

(2) An element $t \in T^p(V)$ is called an alternating tensor, if $P_\sigma(t) = \text{sgn}(\sigma)t$ for all $\sigma \in S_p$.

Note 3.3.4

The set of symmetric tensors and that of alternating tensors are vector subspaces of $T^P(V)$ and are denoted by $S^p(V)$ and $A^p(V)$ respectively.

Example 3.3.5

If $p = 1$, we have
$$S^1(V) = A^1(V) = T^1(V).$$

Example 3.3.6

Let V be a vector space. If $p = 2$, we have
$$S_2 = \{1, (1\ 2)\} \quad \text{and} \quad \text{sgn}(1\ 2) = -1.$$
Then
$$S^2(V) = \{t \in T^2(V) \mid P_{(1\ 2)}(t) = t\}$$
and
$$A^2(V) = \{t \in T^2(V) \mid P_{(1\ 2)}(t) = -t\}$$
We know that if $B = \{v_1, v_2, ..., v_n\}$ is a for V, then the set
$$\{t_{ij} = v_i \otimes v_j \mid 1 \leq i, j \leq n\}$$
form a basis for $T^2(V)$. Clearly for all $1 \leq i, j \leq n$
$$P_{(1\ 2)}(t_{ij}) = t_{ji}.$$
In addition, for all $1 \leq i, j \leq n$, we have
$$P_{(1\ 2)}(t_{ij} + t_{ji}) = t_{ji} + t_{ij} \quad \text{and} \quad P_{(1\ 2)}(t_{ij} - t_{ji}) = t_{ji} - t_{ij} = -(t_{ij} - t_{ji})$$
That means, $t_{ij} + t_{ji}$ are symmetric tensors and $t_{ij} - t_{ji}$ are alternating tensors for all $1 \leq i, j \leq n$.

The set $\{t_{ij} + t_{ji} \mid i \leq j\}$ is a basis for $S^2(V)$ and $\{t_{ij} - t_{ji} \mid i < j\}$ is a basis for $A^2(V)$. Hence
$$\dim S^2(V) = \frac{n(n+1)}{2}$$
and
$$\dim A^2(V) = \frac{n(n-1)}{2}$$
Therefore
$$T^2(V) = S^2(V) \oplus A^2(V).$$

Definition 3.3.7 Symmetrizer and alternator on $T^p(V)$

Consider the following linear transformations on $T^p(V)$:

$$\mathcal{S}_p = \frac{1}{p!}\sum_{\sigma\in S_p} P_\sigma \quad \text{and} \qquad \mathcal{A}_p = \frac{1}{p!}\sum_{\sigma\in S_p} \operatorname{sgn}(\sigma)P_\sigma.$$

The mappings $\mathcal{S}_p$ and $\mathcal{A}_p$ are called respectively the symmetrizer and the alternator on $T^p(V)$

Proposition 3.3.8

(1) For any $\tau \in S_p$, we have

$$P_\tau\mathcal{S}_p = \mathcal{S}_p P_\tau = \mathcal{S}_p \quad \text{and} \quad P_\tau\mathcal{A}_p = \mathcal{A}_p P_\tau = \operatorname{sgn}(\tau)\mathcal{A}_p$$

(2) ${\mathcal{S}_p}^2 = \mathcal{S}_p$ and $\mathcal{A}_p^2 = \mathcal{A}_p$.

(3) Let $t \in T^p(V)$. We have

- **(a)** $t \in S^p(V) \iff \mathcal{S}_p(t) = t$.
- **(b)** $t \in A^p(V) \iff \mathcal{A}_p(t) = t$.

(4) If $p > 1$, then $\mathcal{A}_p\mathcal{S}_p = \mathcal{S}_p\mathcal{A}_p = 0$.

(5) for all $p > 1$, we have

$$S^p(V) \cap A^p(V) = \{0\}.$$

Proof. **(1)** For fixed $\tau \in S_n$, we have

$$\begin{aligned} P_\tau\mathcal{S}_p &= P_\tau\Big(\frac{1}{p!}\sum_{\sigma\in S_p} P_\sigma\Big) \\ &= \frac{1}{p!}\sum_{\sigma\in S_p} P_\tau P_\sigma \\ &= \frac{1}{p!}\sum_{\sigma\in S_p} P_{\sigma\tau} \\ &= \frac{1}{p!}\sum_{\sigma\in S_p} P_\sigma \quad (\text{because } \{\sigma\tau \mid \sigma\in S_n\} = S_n) \\ &= \mathcal{S}_p. \end{aligned}$$

Similarly, we can show that $\mathcal{S}_p P_\tau = \mathcal{S}_p$.

$$\begin{aligned}
P_\tau \mathcal{A}_p &= P_\tau\Big(\frac{1}{p!}\sum_{\sigma\in S_p}\operatorname{sgn}(\sigma)P_\sigma\Big)\\
&= \frac{1}{p!}\sum_{\sigma\in S_p}\operatorname{sgn}(\sigma)P_\tau P_\sigma\\
&= \frac{1}{p!}\sum_{\sigma\in S_p}\operatorname{sgn}(\sigma)P_{\sigma\tau}\\
&= \frac{1}{\operatorname{sgn}(\tau)}\frac{1}{p!}\sum_{\sigma\in S_p}\operatorname{sgn}(\sigma\tau)P_{\sigma\tau}\\
&= \operatorname{sgn}(\tau)\frac{1}{p!}\sum_{\sigma\in S_p}\operatorname{sgn}(\sigma\tau)P_{\sigma\tau}\\
&= \operatorname{sgn}(\tau)\frac{1}{p!}\sum_{\sigma\in S_p}\operatorname{sgn}(\sigma)P_{\sigma}\\
&= \operatorname{sgn}(\tau)\mathcal{A}_p.
\end{aligned}$$

Similarly, we can show that $\mathcal{A}_p P_\tau = \operatorname{sgn}(\tau)\mathcal{A}_p$.

(2) Using **(1)**, we obtain

$$\begin{aligned}
{\mathcal{S}_p}^2 &= \frac{1}{p!}\sum_{\sigma\in S_p} P_\sigma \mathcal{S}_p\\
&= \frac{1}{p!}\sum_{\sigma\in S_p} \mathcal{S}_p\\
&= \mathcal{S}_p,
\end{aligned}$$

and similarly, we get ${\mathcal{A}_p}^2 = \mathcal{A}_p$.

(3) Let $t \in S^p(V)$. We have

(a) Assume that $t \in S^p$. Then by definition $P_\sigma(t) = t$ for all $\sigma \in S_n$. Therefore

$$\mathcal{S}_p(t) = \frac{1}{p!}\sum_{\sigma\in S_p} P_\sigma(t) = \frac{1}{p!}\sum_{\sigma\in S_p} t = t.$$

Conversely, if $\mathcal{S}_p(t) = t$, then for all $\tau \in S_n$, we have

$$P_\tau(t) = P_\tau \mathcal{S}_p(t) = \mathcal{S}_p(t) = t.$$

Hence $t \in S^p$.

(b) Use the same ideas as in **(b)**.

(4)

$$\begin{aligned}
\mathcal{A}_p\mathcal{S}_p &= \frac{1}{p!}\sum_{\sigma\in S_p}\operatorname{sgn}(\sigma)P_\sigma\mathcal{S}_p\\
&= \frac{1}{p!}\sum_{\sigma\in S_p}\operatorname{sgn}(\sigma)\mathcal{S}_p\\
&= \frac{1}{p!}\Big(\sum_{\sigma\in S_p}\operatorname{sgn}(\sigma)\Big)\mathcal{S}_p.
\end{aligned}$$

Since $p > 1$, the number of odd permutation equal the number of even permutation, we get

$$\sum_{\sigma \in S_p} \operatorname{sgn}(\sigma) = 0.$$

Therefore, $\mathcal{A}_p \mathcal{S}_p = 0$

(5) Let $t \in S^p(V) \cap A^p(V)$, then $\mathcal{S}_p(t) = t$. Apply $\mathcal{A}_p$ on both sides, $\mathcal{A}_p\mathcal{S}_p(t) = \mathcal{A}_p(t)$, so $0 = \mathcal{A}_p(t)$, therefore $t = 0$ because $\mathcal{A}_p(t) = t$. Hence

$$S^p(V) \cap A^p(V) = \{0\}.$$

□

Corollary 3.3.9

We have

$$\operatorname{Im}(\mathcal{S}_p) = S^p(V) \quad \text{and} \quad \operatorname{Im}(\mathcal{A}_p) = A^p(V).$$

In particular, for any $t \in T^p(V)$, $\mathcal{S}_p(t)$ is a symmetric tensor and a $\mathcal{A}_p(t)$ is an alternating tensor.

Proof. Clearly form the equivalence $t \in S^p(V) \iff \mathcal{S}_p(t) = t$, we have

$$S^p(V) \subseteq \operatorname{Im}(\mathcal{S}_p).$$

Conversely, let $t \in \operatorname{Im}(\mathcal{S}_p)$, then $\mathcal{S}_p(t') = t$ for some $t' \in T^p(V)$, apply P_σ both sides and since $P_\sigma \mathcal{S}_p = \mathcal{S}_p$, we get $S_p(t') = P_\sigma(t)$, so $t = P_\sigma(t)$, and hence $t \in S^p(V)$. Similarly we show that $\operatorname{Im}(\mathcal{A}_p) = A^p(V)$. □

Lemma 3.3.10

Let V be a $\mathbb{F}$-vector space , where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If $v_1, \ldots, v_k$ are vectors in V such that $v_i = v_j$ for some i and j in $\{1, ..., k\}$, then

$$\mathcal{A}_p(v_1 \otimes \cdots \otimes v_k) = 0$$

Proof. Let τ be the transposition $(i\, j)$. By using Proposition 3.3.8 **(1)**, we have

$$\mathcal{A}_p P_\tau(v_1 \otimes \cdots \otimes v_k) = \operatorname{sgn}(\tau)\mathcal{A}_p(v_1 \otimes \cdots \otimes v_k).$$

That means

$$\mathcal{A}_p(v_1 \otimes \cdots \otimes v_k) = -\mathcal{A}_p(v_1 \otimes \cdots \otimes v_k)$$

Hence $\mathcal{A}_p(v_1 \otimes \cdots \otimes v_k) = 0$. □

Proposition 3.3.11

Let V be a $\mathbb{F}$-vector space , where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then for any $p > n = \dim V$, $A^p(V) = \{0\}$.

Proof. Let $v_1, \dots, v_n$ be a basis for V. We know that, the set

$$\{v_{i_1} \otimes \cdots \otimes v_{i_p} \quad | \quad 1 \leq i_k \leq n\}$$

form a basis for $T_q^p(V)$. Using that fact that $A^p(V) = \mathrm{Im}\mathcal{A}_p$, we obtain

$$A^p(V) = \mathrm{span}\{\mathcal{A}_p(v_{i_1} \otimes \cdots \otimes v_{i_p}) \quad | \quad 1 \leq i_k \leq n\}.$$

Apply Lemma 3.3.10, we get (when $p > n$) :

$$\mathcal{A}_p(v_{i_1} \otimes \cdots \otimes v_{i_p}) = 0 \quad \text{for all} \quad 1 \leq i_k \leq n.$$

Then $A^p(V) = 0$. □

Proposition 3.3.12

Let $\{v_1, \dots, v_n\}$ be a basis for a vector space V over a field $\mathbb{F}$. Then the set

$$\{\mathcal{S}_p(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_p}) \quad | \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_p \leq n\}$$

form a basis for $S^p(V)$. Furthermore

$$\dim S^p(V) = C_p^{n+p-1} = \frac{(n+p-1)!}{(n-1)!p!}.$$

Proposition 3.3.13

Let $\{v_1, \dots, v_n\}$ be a basis for a vector space V over a field $\mathbb{F}$. For all $p \leq n$, the set

$$\{\mathcal{A}_p(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_p}) \quad | \quad 1 \leq i_1 < i_2 < \cdots < i_p \leq n\}$$

form a basis for $A^p(V)$. Furthermore

$$\dim A^p(V) = C_p^n = \frac{n!}{(n-p)!p!}.$$

3.4 Tensor algebras and their properties

Definition 3.4.1 Direct product / Direct sum of vector spaces

Let $\left(V_i\right)_{i=1}^{\infty}$ be infinitely collection of $\mathbb{F}$-vector spaces.

- A direct product $\prod_{i=1}^{\infty} V_i$ is the set of all sequences $(v_1, v_2, ...)$ where each $v_i \in V_i$ with usual pointwise addition

$$(v_1, v_2, \dots) + (w_1, w_2, \dots) = (v_1 + w_1, v_2 + w_2, \dots),$$

and scalar multiplication

$$\lambda(v_1, v_2, \dots) = (\lambda v_1, \lambda v_2, \dots)$$

- The direct sum $\bigoplus_{i=1}^{\infty} V_i$ is the set of all sequences $(v_1, v_2, ...)$ where each $v_i \in V_i$ such that

$$\{i \mid v_i \neq 0\} \quad \text{is finite}$$

with usual pointwise addition and scalar multiplication.

If we identify

$$v_i \in V_i \quad \longleftrightarrow \quad (0, \ldots, 0, v_i, 0, \ldots) \in \bigoplus_{i=1}^{\infty} V_i$$

$$\uparrow$$

$$i\text{th term}$$

then V_i can be considered as a subset of $\bigoplus_{i=1}^{\infty} V_i$.

If $v = (v_1, v_2, ...) \in \bigoplus_{i=1}^{\infty} V_i$, there exists an integer i_0 such that $v_i = 0$ for all $i > i_0$. Thus we can write the element v as

$$v = \sum_{i=1}^{i_0} v_i.$$

Definition 3.4.2 $\mathbb{F}$-algebras

Given a field, $\mathbb{F}$, a $\mathbb{F}$-algebra is a $\mathbb{F}$-vector space A, together with a bilinear operation $\cdot : A \times A \longrightarrow A$, called multiplication, which makes A into a ring with 1. This means that $\cdot$ is associative and that there is a multiplicative identity element, 1, so that $1 \cdot a = a \cdot 1 = a$, for all $a \in A$.

Example 3.4.3

(1) The polynomial ring $\mathbb{F}[X, Y]$ is a $\mathbb{F}$-algebra.

(2) $\mathcal{M}_{n\times n}(\mathbb{F})$ is a $\mathbb{F}$-algebra, This is called a matrix algebra over $\mathbb{F}$.

(3) The set $\mathcal{L}(V, V)$ of linear maps of a $\mathbb{F}$-vector space V to itself is a $\mathbb{F}$-algebra under addition and composition of linear maps.

Recall that if p, q, r, s be positive integers, we have the following isomorphism of $\mathbb{F}$-vector spaces:

$$T_q^p(V) \otimes T_s^r(V) \;\cong\; T_{q+s}^{p+q}(V).$$

Let σ be the canonical mapping of the tensor product $T_q^p(V) \otimes T_s^r(V)$. We have :

$$\begin{array}{ccccc} T_q^p(V) \otimes T_s^r(V) & \xrightarrow{\sigma} & T_q^p(V) \otimes T_s^r(V) & \xrightarrow{\cong} & T_{q+s}^{p+q}(V) \\ (x, y) & \longmapsto & x \otimes y & \longmapsto & xy \end{array}$$

The image of (x, y) in $T_{q+s}^{p+q}(V)$ is denoted by xy.

Definition 3.4.4 Tensor algebra

We define the tensor algebra $T(V)$ of a $\mathbb{F}$-vector space V by

$$T(V) = \bigoplus_{p=0}^{\infty} T^p(V).$$

Next, let us define the product of two elements of $T(V)$. We have the following bilinear mapping:

$$\begin{aligned} T^p(V) \otimes T^q(V) &\longrightarrow T^{p+q}(V) \\ (x, y) &\longmapsto xy \end{aligned}$$

Thus for $t = \sum_{i=1}^{\infty} t_i$ and $t' = \sum_{i=1}^{\infty} t'_i$ be two elements in $T(V)$, where $t_i, t'_i \in T^i(V)$, we define the product tt' by

$$tt' = \sum_{i=1}^{\infty} \sum_{r+s=i} t_r t_s$$

From the associativity of tensor product, the multiplication thus defined satisfies the associativity law, i.e., for $t, t', t" \in T(V)$, we have

$$t(t't'') = (tt')t''$$

If we consider $1 \in \mathbb{F} = T^0(V)$ as an element of $T(V)$, we have, for all $t \in T(V)$,

$$1t = t1 = t.$$

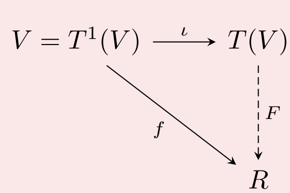
Definition 3.4.5 Homomorphism of associative algebras

Let R and S be two associative algebras over a field $\mathbb{F}$. A linear mapping f from R to S of $\mathbb{F}$-vector spaces is called homomorphism of associative algebras if $f(1_R) = 1_S$. and

$$f(r \cdot r') = f(r) \cdot f(r') \quad \text{for all} \quad r, r' \in R.$$

Theorem 3.4.6 Universal property of the Tensor Algebra $T(V)$

Let V be a $\mathbb{F}$-vector space, R an associative algebra with the unit element 1_R, and f a linear mapping of V into R. There exists a unique associative algebra homomorphism $F : T(V) \longrightarrow R$ such that $F(1_{\mathbb{F}}) = 1_R$ and $F \circ \iota = f$, where ι denotes the natural inclusion mapping of V into $T(V)$.



3.5 Exercise set

Exercise 3.5.1

Let $\sigma \in S_p$ and $F_\sigma : V_1 \times \cdots \times V_p \longrightarrow T^p(V)$ the mapping defined by

$$F_\sigma(v_1, \dots, v_p) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(p)}$$

Show that F_σ is p-multilinear.

Solution. Linearity for the first variable: consider $\sigma(1) = r$. Then

$$\begin{aligned} F_\sigma(\alpha v_1, v_2, \dots, v_p) &= v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(r-1)} \otimes \alpha v_1 \otimes v_{\sigma^{-1}(r+1)} \otimes \cdots \otimes v_{\sigma^{-1}(p)} \\ &= \alpha\Big(v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(r-1)} \otimes v_{\sigma^{-1}(r)} \otimes v_{\sigma^{-1}(r+1)} \otimes \cdots \otimes v_{\sigma^{-1}(p)}\Big) \\ &= \alpha F_\sigma(v_1, v_2, \dots, v_p) \end{aligned}$$

In addition

$$\begin{aligned} F_\sigma(v_1 + v_1', v_2, \dots, v_p) &= v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(r-1)} \otimes (v_1 + v_1') \otimes v_{\sigma^{-1}(r+1)} \otimes \cdots \otimes v_{\sigma^{-1}(p)} \\ &= v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(r-1)} \otimes v_1 \otimes v_{\sigma^{-1}(r+1)} \otimes \cdots \otimes v_{\sigma^{-1}(p)} \\ &\quad + v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(r-1)} \otimes v_1' \otimes v_{\sigma^{-1}(r+1)} \otimes \cdots \otimes v_{\sigma^{-1}(p)} \\ &= F_\sigma(v_1, v_2, \dots, v_p) + F_\sigma(v_1 + v_1', v_2, \dots, v_p). \end{aligned}$$

Similarly, we can show that F_σ is linear for all of each variable.

Exercise 3.5.2

Show that, the set of symmetric tensors $S^p(V)$ and that of alternating tensors $A^p(V)$ are vector subspaces of $T^P(V)$.

Solution. Recall that:

(1) An element $t \in T^p(V)$ is called a symmetric tensor, if $P_\sigma(t) = t$ for all $\sigma \in S_p$.

(2) An element $t \in T^p(V)$ is called an alternating tensor, if $P_\sigma(t) = \text{sgn}(\sigma)t$ for all $\sigma \in S_p$.

Clearly $0 \in S^p(V)$, because $P_\sigma(0) = 0$ for all $\sigma \in S_p$. Let $t_1, t_2 \in S^p(V)$. Then

$$P_\sigma(t_1) = t_1 \quad \text{and} \quad P_\sigma(t_2) = t_2 \quad \text{for all} \quad \sigma \in S_p.$$

Since P_σ is linear, for all $\sigma \in S_p$ and $\alpha \in \mathbb{F}$,

$$\begin{aligned} P_\sigma(t_1 + \alpha t_2) &= P_\sigma(t_1) + \alpha P_\sigma(t_2) \\ &= t_1 + \alpha t_2. \end{aligned}$$

Hence $t_1 + \alpha t_2 \in S^p(V)$. Therefore $S^p(V)$ is a vector subspace of $T^p(V)$.

Similarly, we have $0 \in A^p(V)$, because $P_\sigma(0) = 0 = \text{sgn}(\sigma)0$ for all $\sigma \in S_p$. Let $t_1, t_2 \in A^p(V)$. Then

$$P_\sigma(t_1) = \text{sgn}(\sigma)t_1 \quad \text{and} \quad P_\sigma(t_2) = \text{sgn}(\sigma)t_2 \quad \text{for all} \quad \sigma \in S_p.$$

Since P_σ is linear, for all $\sigma \in S_p$ and $\alpha \in \mathbb{F}$,

$$\begin{aligned} P_\sigma(t_1 + \alpha t_2) &= P_\sigma(t_1) + \alpha P_\sigma(t_2) \\ &= \text{sgn}(\sigma)t_1 + \alpha \text{sgn}(\sigma)t_2 \\ &= \text{sgn}(\sigma)(t_1 + \alpha t_2). \end{aligned}$$

Hence $t_1 + \alpha t_2 \in A^p(V)$. So $A^p(V)$ is a vector subspace of $T^p(V)$.

Exercise 3.5.3

Let V be a vector space. Show that

$$S^1(V) = A^1(V) = T^1(V) = V.$$

Solution. By definition $T^1(V) = V$. The set of permutations S_1 contains one element $\sigma = \mathrm{Id}$ with signature equals 1. We have, for all $v_1 \in V$

$$\begin{aligned} P_\sigma(v_1) &= v_{\sigma^{-1}(1)} \\ &= v_1 \\ &= \operatorname{sgn}(\sigma) v_1. \end{aligned}$$

Hence

$$S^1(V) = A^1(V) = V.$$

Exercise 3.5.4

Let V be a $\mathbb{F}$-vector space of dimension n, $\mathcal{B} = \{v_1, \ldots, v_n\}$ a basis for V, and

$$t_{ij} = v_i \otimes v_j \quad \text{for all} \quad 1 \le i, j \le n.$$

(1) Show that,

(a) $t_{ij} + t_{ji} \in S^2(V)$ for all $1 \le i, j \le n$.
(b) $t_{ij} - t_{ji} \in A^2(V)$ for all $1 \le i, j \le n$.
(c) $\mathcal{B}_1 = \{t_{ij} + t_{ji} \mid i \le j\}$ form a basis of $S^2(V)$.
(d) $\mathcal{B}_2 = \{t_{ij} - t_{ji} \mid i < j\}$ form a basis of $A^2(V)$.

(2) Deduce that

$$\dim S^2(V) = \frac{n(n+1)}{2}$$

and

$$\dim A^2(V) = \frac{n(n-1)}{2}$$

Solution. Let $t = \sum_{i,j} \alpha_{ij} t_{ij} \in S^2(V)$. Then $P_{(1\ 2)}(t) = t$. So

$$\sum_{i,j} \alpha_{ij} t_{ij} = \sum_{i,j} \alpha_{ij} t_{ji}$$

$$\alpha_{ij} = \alpha_{ij}$$

$$t = \sum_{i} \alpha_{ii} t_{ii} + \sum_{i<j} \alpha_{ij}(t_{ij} + t_{ji})$$

$$\dim S^2(V) = n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}$$

and

$$\dim A^2(V) = (n-1) + (n-1) + \cdots + 1 = \frac{n(n-1)}{2}.$$

Exercise 3.5.5

Let v_1, v_2, v_3 be three vectors in V. Show that

$$\mathcal{S}_3(v_1 \otimes v_2 \otimes v_3) = \mathcal{S}_3(v_1 \otimes v_3 \otimes v_2).$$

Solution. By definition, we have

$$\mathcal{S}_3(v_1 \otimes v_2 \otimes v_3) = \frac{1}{3!} \sum_{\sigma \in S_3} P_\sigma(v_1 \otimes v_2 \otimes v_3).$$

Since S_3 has six permutations $\{(1), (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)\}$,

$$\mathcal{S}_3(v_1 \otimes v_2 \otimes v_3) = \frac{1}{6}\big(v_1 \otimes v_2 \otimes v_3 - v_2 \otimes v_1 \otimes v_3 - v_1 \otimes v_3 \otimes v_2 - v_3 \otimes v_2 \otimes v_1 + v_2 \otimes v_3 \otimes v_1 + v_3 \otimes v_1 \otimes v_2\big).$$

Interchanging v_2 and v_3, we get form the previous equality

$$\mathcal{S}_3(v_1 \otimes v_3 \otimes v_2) = \frac{1}{6}\big(v_1 \otimes v_3 \otimes v_2 - v_3 \otimes v_1 \otimes v_2 - v_1 \otimes v_2 \otimes v_3 - v_2 \otimes v_3 \otimes v_1 + v_3 \otimes v_2 \otimes v_1 + v_2 \otimes v_1 \otimes v_3\big).$$

Hence $\mathcal{S}_3(v_1 \otimes v_2 \otimes v_3) = \mathcal{S}_3(v_1 \otimes v_3 \otimes v_2)$.

Exercise 3.5.6

Let $\{v_1, \ldots, v_n\}$ be a basis for a vector space V over a field $\mathbb{F}$. Show that $\dim A^n(V) = 1$ and give a generator of $A^n(V)$.

Solution. Using the formula

$$\dim A^p(V) = C_p^n = \frac{n!}{(n-p)!p!},$$

we get $\dim A^n(V) = 1$. and

$$A^p(V) = \operatorname{span}\{v_1 \otimes \cdots \otimes v_n\}.$$

Exercise 3.5.7

Let $\{v_1, \ldots, v_n\}$ be a basis for a vector space V over a field $\mathbb{F}$ and $p < n$. Show that

$$\dim A^p(V) = \dim A^{n-p}(V).$$

Solution. Using the formula

$$\dim A^p(V) = C_p^n = \frac{n!}{(n-p)!p!},$$

we get

$$\dim A^{n-p}(V) = C_{n-p}^n = \frac{n!}{p!(n-p)!}.$$

Chapter 4

Symmetric and exterior algebras

Chapter contents

4.1 Symmetric algebra

Recall that if V is a $\mathbb{F}$-vector space, then for any permutation $\sigma \in S_p$, we have a linear mapping

$$P_\sigma : T^p(V) \longrightarrow T^p(V),$$

such that for all $v_1, \ldots, v_p \in V$,

$$P_\sigma(v_1 \otimes \cdots \otimes v_p) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(p)}$$

The vector space of symmetric tensor is

$$S^p(V) = \{t \in T(V) \mid P_\sigma(t) = t\}$$

It known that

$$S^0(V) = \mathbb{F} \quad \text{and} \quad S^1(V) = V.$$

Consider the $\mathbb{F}$-vector space

$$S(V) = \bigoplus_{i=0}^{\infty} S^p(V) = \mathbb{F} \oplus V \oplus S^2(V) \oplus \cdots$$

Clearly $S(V)$ is a vector subspace of $T(V)$.

We will define a multiplication for which $S(V)$ becomes an associative algebra. Let $t \in S^p(V)$ and $t' \in S^p(V)$, then $t \otimes t' \in T^{p+q}(V)$ is not necessarily a symmetric tensor, but $\mathcal{S}_{p+q}(t \otimes t')$ is symmetric regarding the proposition 3.3.8, where $\mathcal{S}_{p+q}$ is the symmetrizer transformation on $T(V)$

$$\mathcal{S}_p = \frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q}} P_\sigma.$$

Hence we can define the multiplication $\odot$ on $S(V)$ by

$$t \odot t' = \mathcal{S}_{p+q}(t \otimes t') = \frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q}} P_\sigma(t \otimes t'),$$

for all $t \in S^p(V)$ and $t' \in T^q(V)$.

In general, for $t = \sum_{p=0}^{\infty} t_p$ and $t' = \sum_{q=0}^{\infty} t_q{}'$ $\quad (t_p \in S^p(V)$ and $t'_q \in S^q(V))$, define

$$t \odot t' = \sum_{p,q} t_p \odot t'_q = \sum_{k=0}^{\infty} \left(\sum_{p+q=k} \mathcal{S}_k(t_p \otimes t'_q) \right) = \sum_{k=0}^{\infty} \mathcal{S}_k \left(\sum_{p+q=k} t_p \otimes t'_q \right). \tag{4.1}$$

Example 4.1.1

Let v_1, v_2 be two vectors in V and $\alpha \in \mathbb{F}$. Then v_1 and v_2 are symmetric tensors in $S^1(V)$ and

$$v_1 \odot v_2 = \frac{1}{2!} \sum_{\sigma \in S_2} P_\sigma(v_1 \otimes v_2).$$

Since S_2 has two permutations $\{\sigma_1 = (1), \sigma_2 = (1\,2)\}$, $v_1 \odot v_2 = \frac{1}{2}\big(P_{\sigma_1}(v_1 \otimes v_2) + P_{\sigma_2}(v_1 \otimes v_2)\big)$. Hence

$$v_1 \odot v_2 = \frac{1}{2}\big(v_1 \otimes v_2 + v_2 \otimes v_1\big).$$

We conclude that , If we take $v_1 = v_2 = v \in V$, then

$$v \odot v = v \otimes v.$$

We have, also, if $\alpha \in S^0(V)$ and $v \in S^1(V)$ are symmetric tensor, and

$$\alpha \odot v = \alpha \otimes v.$$

Example 4.1.2

Let $v_1, v_2.v_3$ be two vectors in V. Put $r = \frac{1}{2}\big(v_1 \otimes v_2 + v_2 \otimes v_1\big)$ and $s = v_3$. Then

$$s \odot r = \frac{1}{3!} \sum_{\sigma \in S_3} \frac{1}{2}\big(P_\sigma(v_1 \otimes v_2 \otimes v_3) + P_\sigma(v_2 \otimes v_1 \otimes v_3)\big) = \frac{1}{3!} \sum_{\sigma \in S_3} P_\sigma(v_1 \otimes v_2 \otimes v_3).$$

Since S_3 has six permutations $\{(1), (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}$,
Hence

$$r \odot s = \frac{1}{6}\big(v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3 + v_1 \otimes v_3 \otimes v_2 + v_3 \otimes v_2 \otimes v_1 + v_2 \otimes v_3 \otimes v_1 + v_3 \otimes v_1 \otimes v_2\big).$$

Proposition 4.1.3

Let $t_p \in T^p(V)$ and $t_q \in T^q(V)$. Then

$$\mathcal{S}_{p+q}(\ \mathcal{S}_p(t_p) \otimes t_q\ ) \ = \ \mathcal{S}_{p+q}(\ t_p \otimes \mathcal{S}_q(t_q)\ ) \ = \ \mathcal{S}_{p+q}(t_p \otimes t_q).$$

Proof. For all $\sigma \in S_p$, we denote $\tilde{\sigma} \in S_{p+q}$ the permutation defined by

$$\tilde{\sigma}(i) = \begin{cases} \sigma(i) & \text{if} \quad 1 \leq i \leq p \\ i & \text{if} \quad p+1 \leq i \leq p+q \end{cases}$$

$$\mathcal{S}_{p+q}(\ \mathcal{S}_p(t_p) \otimes t_q\ ) \ = \ \frac{1}{p!} \sum_{\sigma \in S_p} \mathcal{S}_{p+q}(\ P_\sigma(t_p) \otimes t_q\ ).$$

But $\mathcal{S}_{p+q}(\ P_\sigma(t_p) \otimes t_q\ ) \ = \ \mathcal{S}_{p+q}(\ P_{\tilde{\sigma}}(t_p \otimes t_q)\ ) \ = \ \mathcal{S}_{p+q}(t_p \otimes t_q)$ (see Proposition 3.3.8 **(1)**) . Therefore

$$\begin{aligned} \mathcal{S}_{p+q}(\ \mathcal{S}_p(t_p) \otimes t_q\ ) \ &= \ \frac{1}{p!} \sum_{\sigma \in S_p} \mathcal{S}_{p+q}(\ t_p \otimes t_q\ ) \\ &= \ \mathcal{S}_{p+q}(\ t_p \otimes t_q\ ) \end{aligned}$$

□

The product defined in (4.1) is commutative, bilinear and associative.

Commutativity: this product is commutative, because for all $t_p \in S^p(V)$ and $t_q \in S^q(V)$, we have

$$\sum_{\sigma \in S_{p+q}} P_\sigma(t_p \otimes t_q) = \sum_{\sigma \in S_{p+q}} P_\sigma(t_q \otimes t_p).$$

Bilinearity: by the definition of the multiplication in (4.1), we have clearly

$$(t_p + t_q) \odot t_l = t_p \odot t_l + t_q \odot t_l \quad \text{for all} \quad (t_p, t_q, t_l) \in S^p(V) \times S^q(V) \times S^l(V).$$

and for all $\alpha \in \mathbb{F}$, we have

$$\begin{aligned} (\alpha t_p) \odot t_q &= \sum_{\sigma \in S_{p+q}} P_\sigma(\alpha t_p \otimes t_q) \\ &= \sum_{\sigma \in S_{p+q}} \alpha P_\sigma(t_p \otimes t_q) \\ &= \alpha \sum_{\sigma \in S_{p+q}} P_\sigma(t_p \otimes t_q) \\ &= \alpha(t_p \odot t_q). \end{aligned}$$

Associativity: for all $(t_p, t_q, t_l) \in S^p(V) \times S^q(V) \times S^l(V)$, we have

$$\begin{aligned} (t_p \odot t_q) \odot t_l &= \mathcal{S}_{p+q}(t_p \otimes t_q) \cdot t_l \\ &= \mathcal{S}_{p+q+l}(\mathcal{S}_{p+q}(t_p \otimes t_q) \otimes t_l) \\ &= \mathcal{S}_{p+q+l}\Big((t_p \otimes t_q) \otimes t_l \Big) \quad \text{By Proposition 4.1.3 } \textbf{(1)} \end{aligned} \tag{4.2}$$

Similarly, we can show that

$$(t_p \odot \odot(t_q \odot t_l) = \mathcal{S}_{p+q+l}\Big((t_p \otimes (t_q \otimes t_l)\Big) \tag{4.3}$$

Since $t_p \otimes (t_q \otimes t_l) = (t_p \otimes t_q) \otimes t_l$, we obtain from (4.2) and (4.3),

$$(t_p \odot t_q) \odot t_l = t_p \odot (t_q \odot t_l).$$

Definition 4.1.4 Symmetric algebra $S(V)$

The associative algebra $S(V)$ is called the symmetric algebra of V.

4.2 Exterior algebras

Recall that if V is a $\mathbb{F}$-vector space of dimension n, then the vector space of alternating tensor is

$$A^p(V) = \{t \in T(V) \mid P_\sigma(t) = t\}$$

It known that

$$A^0(V) = \mathbb{F} \quad \text{and} \quad A^1(V) = V.$$

Consider the $\mathbb{F}$-vector space

$$A(V) = \bigoplus_{i=0}^{\infty} A^p(V) = \bigoplus_{i=0}^{n} A^p(V) = \mathbb{F} \oplus V \oplus A^2(V) \oplus \cdots A^n(V).$$

Clearly $A(V)$ is a vector subspace of $T(V)$.

We define now a multiplication on $A(V)$. Let $t \in A^p(V)$ and $t' \in A^q(V)$. Then, $t \otimes t' \in T^{p+q}(V)$. Therefore $\mathcal{A}_{p+q}(t \otimes t')$ is an alternating tensor in $A^{p+q}(V)$, where

$$\mathcal{A}_k = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) P_\sigma.$$

We define exterior product $\wedge$ of t and t' by

$$t \wedge t' = \mathcal{A}_{p+q}(t \otimes t').$$

In general, for $t = \sum_{p=0}^{\infty} t_p$ and $t' = \sum_{q=0}^{\infty} t_q{}'$ $\quad (t_p \in A^p(V)$ and $t'_q \in A^q(V))$, define

$$t \wedge t' = \sum_{p,q} t_p \wedge t'_q = \sum_{k=0}^{\infty}\left(\sum_{p+q=k} \mathcal{A}_k(t_p \otimes t'_q)\right) = \sum_{k=0}^{\infty} \mathcal{A}_k \left(\sum_{p+q=k} t_p \otimes t'_q\right). \tag{4.4}$$

Example 4.2.1

Let v_1, v_2 be two vectors in V and $\alpha \in \mathbb{F}$. Then v_1 and v_2 are alternating tensors in $A^1(V)$ and

$$v_1 \wedge v_2 = \frac{1}{2!} \sum_{\sigma \in S_2} \operatorname{sgn}(\sigma) P_\sigma(v_1 \otimes v_2).$$

Therefore,

$$v_1 \wedge v_2 = \frac{1}{2}\big(v_1 \otimes v_2 - v_2 \otimes v_1\big).$$

We conclude that:

(1) for all $v_1, v_2 \in V$, $v_1 \wedge v_2 = -(v_2 \wedge v_1)$

(2) for all $v \in V$, $v \wedge v = 0$.

Example 4.2.2

Let $v_1, v_2.v_3$ be two vectors in V. Put $r = v_1 \wedge v_2 = \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1)$ and $s = v_3$. Then

$$s \wedge r = \frac{1}{3!}\sum_{\sigma \in S_3} \frac{1}{2}\mathrm{sgn}(\sigma)\big(P_\sigma(v_1 \otimes v_2 \otimes v_3) - P_\sigma(v_2 \otimes v_1 \otimes v_3)\big) = \frac{1}{3!}\sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) P_\sigma(v_1 \otimes v_2 \otimes v_3).$$

Since S_3 has six permutations $\{(1), (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}$,
Hence

$$r \wedge s = \frac{1}{6}\big(v_1 \otimes v_2 \otimes v_3 - v_2 \otimes v_1 \otimes v_3 - v_1 \otimes v_3 \otimes v_2 - v_3 \otimes v_2 \otimes v_1 + v_2 \otimes v_3 \otimes v_1 + v_3 \otimes v_1 \otimes v_2\big).$$

Proposition 4.2.3

Let $t_p \in T^p(V)$ and $t_q \in T^q(V)$. Then

$$\mathcal{A}_{p+q}(\ \mathcal{A}_p(t_p) \otimes t_q\ ) \ = \ \mathcal{A}_{p+q}(\ t_p \otimes \mathcal{A}_q(t_q)\ ) \ = \ \mathcal{A}_{p+q}(t_p \otimes t_q).$$

Proof. For all $\sigma \in S_p$, we denote $\tilde{\sigma} \in S_{p+q}$ the permutation defined by

$$\tilde{\sigma}(i) = \begin{cases} \sigma(i) & \text{if} \quad 1 \leq i \leq p \\ i & \text{if} \quad p+1 \leq i \leq p+q \end{cases}$$

$$\mathcal{A}_{p+q}(\ \mathcal{A}_p(t_p) \otimes t_q\ ) \ = \ \frac{1}{p!}\sum_{\sigma \in S_p} \mathrm{sgn}(\sigma)\mathcal{A}_{p+q}(\ P_\sigma(t_p \otimes t_q)\ ) \ = \ \frac{1}{p!}\sum_{\sigma \in S_p} \mathrm{sgn}(\sigma)\mathcal{A}_{p+q}(\ P_{\tilde{\sigma}}(t_p \otimes t_q)\ )$$

Apply Proposition 3.3.8 **(1),** for any $\tau \in S_{p+q}$, we have

$$P_\tau \mathcal{A}_{p+q} = \mathcal{A}_{p+q} P_\tau = \mathrm{sgn}(\tau)\mathcal{A}_{p+q}.$$

Hence

$$\begin{aligned} \mathcal{A}_{p+q}(\ \mathcal{A}_p(t_p) \otimes t_q\ ) &= \frac{1}{p!}\sum_{\sigma \in S_p} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tilde{\sigma})(\ \mathcal{A}_{p+q}(t_p \otimes t_q)\ ) \\ &= \frac{1}{p!}\sum_{\sigma \in S_p} \mathcal{A}_{p+q}(t_p \otimes t_q) \\ &= \mathcal{A}_{p+q}(t_p \otimes t_q). \end{aligned}$$

□

The product defined in (4.4) is bilinear and associative.

Bilinearity: by the definition of the multiplication in (4.4), we have clearly

$$(t_p + t_q) \wedge t_l = t_p \wedge t_l + t_q \wedge t_l \quad \text{for all} \quad (t_p, t_q, t_l) \in A^p(V) \times A^q(V) \times A^l(V).$$

and for all $\alpha \in \mathbb{F}$, we have

$$\begin{aligned}(\alpha t_p) \wedge t_q &= \sum_{\sigma \in S_{p+q}} \operatorname{sgn}(\sigma) P_\sigma(\alpha t_p \otimes t_q) \\ &= \alpha \sum_{\sigma \in S_{p+q}} \operatorname{sgn}(\sigma) P_\sigma(t_p \otimes t_q) \\ &= \alpha(t_p \wedge t_q).\end{aligned}$$

Associativity: for all $(t_p, t_q, t_l) \in A^p(V) \times A^q(V) \times A^l(V)$, we have

$$\begin{aligned}(t_p \wedge t_q) \wedge t_l &= \mathcal{A}_{p+q}(t_p \otimes t_q) \wedge t_l \\ &= \mathcal{A}_{p+q+l}(\mathcal{A}_{p+q}(t_p \otimes t_q) \otimes t_l) \\ &= \mathcal{A}_{p+q+l}\Big((t_p \otimes t_q) \otimes t_l \Big) \quad \text{By Proposition 4.2.3}\end{aligned} \tag{4.5}$$

Similarly, we can show that

$$t_p \wedge (t_q \wedge t_l) = \mathcal{A}_{p+q+l}\Big((t_p \otimes (t_q \otimes t_l) \Big) \tag{4.6}$$

Since $t_p \otimes (t_q \otimes t_l) = (t_p \otimes t_q) \otimes t_l$, we obtain from (4.5) and (4.6),

$$(t_p \wedge t_q) \wedge t_l = t_p \wedge (t_q \wedge t_l).$$

Definition 4.2.4 Exterior algebra $A(V)$

The associative algebra $A(V)$ is called the exterior algebra of V.

Proposition 4.2.5

For all $t \in A^p(V)$ and $' \in A^q(V)$, we have

$$t \wedge t' = (-1)^{pq} t' \wedge t.$$

Proof. Since $\wedge$ is bilinear, it suffices to prove the result for

$$\begin{cases} t = \mathcal{A}_p(v_1 \otimes \cdots \otimes v_p) = v_1 \wedge v_2 \wedge \cdots \wedge v_p \\ t' = \mathcal{A}_q(v_{p+1} \otimes \cdots \otimes v_{p+q}) = v_{p+1} \wedge v_{p+2} \wedge \cdots \wedge v_{p+q}. \end{cases}$$

We have

$$\begin{aligned}
t \wedge t' &= \big(v_1 \wedge v_2 \wedge \cdots \wedge v_p\big) \wedge \big(v_{p+1} \wedge v_{p+2} \wedge \cdots \wedge v_{p+q}\big) \\
&= (-1)^q \big(v_1 \wedge v_2 \wedge \cdots \wedge v_{p-1}\big) \wedge \big(v_{p+1} \wedge v_{p+2} \wedge \cdots \wedge v_{p+q}\big) \wedge v_p \\
&= (-1)^q (-1)^q \big(v_1 \wedge v_2 \wedge \cdots \wedge v_{p-2}\big) \wedge \big(v_{p+1} \wedge v_{p+2} \wedge \cdots \wedge v_{p+q}\big) \wedge v_{p-1} \wedge v_p \\
&\ \vdots \\
&= \underbrace{(-1)^q (-1)^q \cdots (-1)^q}_{p\,\text{factors}} \big(v_{p+1} \wedge v_{p+2} \wedge \cdots \wedge v_{p+q}\big) \wedge \big(v_1 \wedge v_2 \wedge \cdots \wedge v_p\big) \\
&= (-1)^{pq} \big(v_{p+1} \wedge v_{p+2} \wedge \cdots \wedge v_{p+q}\big) \wedge \big(v_1 \wedge v_2 \wedge \cdots \wedge v_p\big) \\
&= (-1)^{pq} (t' \wedge t).
\end{aligned}$$

□

4.3 Exercise set

Exercise 4.3.1

Let $\mathcal{B} = \{e_1, e_2, e_3\}$ be the standard basis of the real vector space $V = \mathbb{R}^3$.

(1) Give the dimension of the following vector spaces: $T^3(V)$, $S^3(V)$ and $A^3(V)$.

(2) Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ be three vectors in $\mathbb{R}^3$. Show that:

(a) $u \wedge v = (u_1v_2 - u_2v_1)e_1 \wedge e_2 + (u_1v_3 - u_3v_1)e_1 \wedge e_3 + (u_2v_3 - u_3v_2)e_2 \wedge e_3$

(b) $u \wedge v \wedge w = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} e_1 \wedge e_2 \wedge e_3.$

Solution.

$$\begin{aligned} u \wedge v &= (u_1e_1 + u_2e_2 + u_3e_3) \wedge (v_1e_1 + v_2e_2 + v_3e_3) \\ &= u_1v_1\, e_1 \wedge e_1 + u_1v_2\, e_1 \wedge e_2 + u_1v_3\, e_1 \wedge e_3 \\ &\quad + u_2v_1\, e_2 \wedge e_1 u_2v_2\, e_2 \wedge e_2 + u_2v_3\, e_2 \wedge e_3 \\ &\quad + u_3v_1\, e_3 \wedge e_1 + u_3v_2\, e_3 \wedge e_2 + u_3v_3\, e_3 \wedge e_3 \\ &= u_1v_2\, e_1 \wedge e_2 + u_1v_3\, e_1 \wedge e_3 + u_2v_1\, e_2 \wedge e_1 + u_2v_3\, e_2 \wedge e_3 - u_3v_1\, e_1 \wedge e_2 - u_3v_2\, e_2 \wedge e_3 \\ &= u_1v_2\, e_1 \wedge e_2 + u_1v_3\, e_1 \wedge e_3 - u_2v_1\, e_1 \wedge e_2 + u_2v_3\, e_2 \wedge e_3 - u_3v_1\, e_1 \wedge e_3 - u_3v_2\, e_2 \wedge e_3 \\ &= (u_1v_2 - u_2v_1)e_1 \wedge e_2 + (u_1v_3 - u_3v_1)e_1 \wedge e_3 + (u_2v_3 - u_3v_2)e_2 \wedge e_3 \end{aligned}$$

Hence

$$u \wedge v = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} e_1 \wedge e_2 + \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} e_1 \wedge e_3 + \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} e_2 \wedge e_3.$$

$$\begin{aligned} u \wedge v \wedge w &= (u_1e_1 + u_2e_2 + u_3e_3) \wedge (v_1e_1 + v_2e_2 + v_3e_3) \wedge (w_1e_1 + w_2e_2 + w_3e_3) \\ &= \Big((u_1v_2 - u_2v_1)e_1 \wedge e_2 + (u_1v_3 - u_3v_1)e_1 \wedge e_3 + (u_2v_3 - u_3v_2)e_2 \wedge e_3\Big)(w_2e_2 + w_3e_3) \\ &= (u_1v_2w_3 - u_2v_1w_3)e_1 \wedge e_2 \wedge e_3 + (u_1v_3w_2 - u_3v_1w_2)e_1 \wedge e_3 \wedge e_2 + (u_2v_3w_1 - u_3v_2w_1)e_2 \wedge e_3 \wedge e_1 \\ &= (u_1v_2w_3 - u_2v_1w_3)e_1 \wedge e_2 \wedge e_3 - (u_1v_3w_2 - u_3v_1w_2)e_1 \wedge e_2 \wedge e_3 + (u_2v_3w_1 - u_3v_2w_1)e_1 \wedge e_2 \wedge e_3 \\ &= (u_1v_2w_3 - u_2v_1w_3 - u_1v_3w_2 + u_3v_1w_2 + u_2v_3w_1 - u_3v_2w_1)e_1 \wedge e_2 \wedge e_3 \\ &= \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} e_1 \wedge e_2 \wedge e_3. \end{aligned}$$

Exercise 4.3.2

Let V be a vector space of dimension n. Show that

$$\dim A(V) = 2^n.$$

Hint. Use Newton's Binomial Theorem.

Solution. Since $A(V) = \sum_{p=1}^{n} A^p(V)$,

$$\dim A(V) = \sum_{p=1}^{n} \dim A^p(V) = \sum_{p=1}^{n} C_p^n.$$

By Newton's Binomial Theorem, we now that

$$(x+y)^n = \sum_{p=1}^{n} C_p^n x^p y^{n-p}.$$

Therefore

$$2^n = \sum_{p=1}^{n} C_p^n.$$

Consequently,

$$\dim A(V) = 2^n.$$

Exercise 4.3.3

Let

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & p \\ i_1 & i_2 & \cdots & i_p \end{pmatrix}$$

be a permutation in S_p, and $v_1, \ldots, v_p$ be elements in a vector space V. Show that

$$v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_p} \;=\; \operatorname{sgn}(\sigma)\big(v_1 \wedge v_2 \wedge \cdots \wedge v_p\big).$$

Solution.

$$\begin{aligned} v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_p} &= \mathcal{A}_p(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_p}) \\ &= \mathcal{A}_p\big(P_{\sigma^{-1}}(v_1 \otimes v_2 \otimes \cdots \otimes v_p)\big) \\ &= \operatorname{sgn}(\sigma^{-1})\mathcal{A}_p\big(v_1 \otimes v_2 \otimes \cdots \otimes v_p\big) \\ &= \operatorname{sgn}(\sigma)\big(v_1 \wedge v_2 \wedge \cdots \wedge v_p\big) \end{aligned}$$

Exercise 4.3.4

Let $t \in A^p(V)$, where p is odd number. Show that

$$t \wedge t = 0.$$

Solution. We now that, for all $t \in A^p(V)$ and $' \in A^q(V)$,

$$t \wedge t' = (-1)^{pq}\, t' \wedge t.$$

Hence

$$t \wedge t = (-1)^{p^2}\, t \wedge t.$$

If $p = 2k+1$ is odd, then $p^2 = 2(2k^2+2k)+1$ is odd, so

$$t \wedge t = -t \wedge t.$$

Therefore

$$t \wedge t = 0.$$

Exercise 4.3.5

Let $V = \mathbb{R}^2, e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Show that, if $v_1 = ae_1 + be_2$ and $v_2 = ce_1 + de_2$, where a, b, c and d are real numbers, then $v_1 \wedge v_2 = (ad - bc)(e_1 \wedge e_2)$.

Solution.

$$\begin{aligned}(a,b) \wedge (c,d) &= (ax+by) \wedge (cx+dy) \\ &= ac(x \wedge x) + ad(x \wedge y) + bc(y \wedge x) + bd(y \wedge y) \\ &= 0 + ad(x \wedge y) + bc(y \wedge x) + 0 \\ &= ad(x \wedge y) - bc(y \wedge x) \\ &= (ad - bc)(x \wedge y)\end{aligned}$$

Exercise 4.3.6

Let $t \in A^p(V)$ and $t' \in A^q(V)$, where p and q are odd numbers. Show that

$$t \wedge t' = -t' \wedge t.$$

Solution. We now that, for all $t \in A^p(V)$ and $' \in A^q(V)$,

$$t \wedge t' = (-1)^{pq} t' \wedge t.$$

Since p and q are odd numbers, pq is odd. Hence

$$t \wedge t = -t \wedge t.$$

Exercise 4.3.7

Let v and v' be vectors in V. Show that

$$v \wedge v' = 0 \iff v \text{ and } v' \text{ are linearly dependent}$$

Solution. If v and v' are linearly dependent, then $v' = \alpha v$ for some scalar $\alpha \in \mathbb{F}$, so

$$v \wedge v' = v \wedge \alpha v = \alpha(v \wedge v) = 0.$$

Conversely, if v and v' are linearly independent and can be extended to a basis, but then $v \wedge v'$ is a basis vector and so is non-zero.

Exercise 4.3.8

Let $v_1, \ldots, v_k$ be vectors in a finite dimensional $\mathbb{F}$-vector space V. Show that,

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k = 0 \iff \text{the vectors } v_1, \ldots, v_k \text{ are linearly dependent.}$$

Solution. Assume that the vectors $v_1, \ldots, v_k$ are linearly dependent. Without loss of generality, suppose that :

$$v_k = \sum_{i=1}^{k-1} c_i v_i,$$

where $c_1, ..., c_{k-1} \in \mathbb{F}$. Then

$$\begin{aligned} v_1 \wedge v_2 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k &= (v_1 \wedge v_2 \wedge \cdots v_{k-1}) \wedge v_k \\ &= (v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1}) \wedge \sum_{1}^{k-1} c_i v_i \\ &= \sum_{i=1}^{k-1} c_i \underbrace{(v_1 \wedge v_2 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_{k-1}) \wedge v_i}_{=0} \\ &= 0. \end{aligned}$$

Conversely, suppose that the vectors $v_1, \ldots, v_k$ are linearly independent. Then we can extend it to a basis $v_1, \ldots, v_n$ of V. This means the elements

$$v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} \quad \text{where} \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n$$

form a basis for $A^k(V)$, and since $v_1 \wedge v_2 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k$ is an element of this basis,

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k \neq 0.$$

Exercise 4.3.9

Let v_1, v_2 and v_3 be vectors in V. Show that

$$(v_3 \wedge v_1 \wedge v_2) \ + \ (v_2 \wedge v_3 \wedge v_1) = 2(v_1 \wedge v_2 \wedge v_3).$$

Solution. Clearly

$$v_3 \wedge v_1 \wedge v_2 = -v_1 \wedge v_3 \wedge v_2 = -(-v_1 \wedge v_2 \wedge v_3) = v_1 \wedge v_2 \wedge v_3,$$

and

$$v_2 \wedge v_3 \wedge v_1 = -(v_2 \wedge v_1 \wedge v_3) = -(-v_1 \wedge v_2 \wedge v_3) = v_1 \wedge v_2 \wedge v_3.$$

Then

$$(v_3 \wedge v_1 \wedge v_2) \ + \ (v_2 \wedge v_3 \wedge v_1) = 2(v_1 \wedge v_2 \wedge v_3).$$

Exercise 4.3.10

Let v be a nonzero vector in V and $t \in A^k(V)$. Show that $v \wedge t = 0$ if and only if $t = v \wedge t'$ for some $t' \in A^{k-1}(V)$.

Solution. Clearly, if $t = v \wedge t'$ for some $t' \in A^{k-1}(V)$, then

$$v \wedge t = v \wedge (v \wedge t') = (v \wedge v) \wedge t' = 0 \wedge t' = 0.$$

Conversely, assume that $v \wedge t = 0$. Extend v to a basis $v_1, \ldots, v_n$ for V, with $v_1 = v$. Write

$$t = \sum c_J v_J,$$

where the sum runs over all strictly ascending multi-indices $1 \leq j_1 < \cdots < j_k \leq n$, and $v_J = v_{j_1} \wedge v_{j_2} \wedge \cdots \wedge v_{j_k}$. In the sum

$$v \wedge t = \sum_J c_J \, v \wedge v_J$$

all the terms $\alpha \wedge v^J$ with $j_1 = 1$ vanish, since $v = v_1$. Hence,

$$0 = v \wedge \gamma = \sum_{j_1 \neq 1} c_J \, v \wedge v_J.$$

Since $(v \wedge v^J)_{j_1 \neq 1}$ is a subset of a basis for $A_{k+1}(V)$ it is linearly independent, and so all c_J are 0 if $j_1 \neq 1$. Thus,

$$t = \sum_{j_1=1} c_J v_J = v \wedge \left(\sum_{j_1=1} c_J v_{j_2} \wedge \cdots \wedge v_{j_k} \right) = v \wedge t',$$

where

$$t' = \sum_{j_1=1} c_J v_{j_2} \wedge \cdots \wedge v_{j_k}.$$

Appendices

Appendix A

Permutations

Appendix contents

A.1 What is a permutation?

Definition A.1.1

A permutation of a set S is a function from S to S that is both one-to- one and onto. A permutation group of a set S is a set of permutations of S that forms a group under function composition.

Although groups of permutations of any nonempty set S of objects exist, we will focus on the case where S is finite. Furthermore, it is customary, as well as convenient, to take S to be a set of the form $\{1, 2, 3, \cdots, n\}$ for some positive integer n.

For example, we define a permutation σ of the set $\{1, 2, 3, 4\}$ by specifying

$$\sigma(1) = 2, \quad \sigma(2) = 3, \quad \sigma(3) = 1, \quad \sigma(4) = 4.$$

A more convenient way to express this correspondence is to write it in array form as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}.$$

Composition of permutations in the set of permutation is a binary operation. As an example, let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}.$$

Then

$$\begin{aligned}
\tau\sigma(1) &= \tau(\sigma(1)) = \tau(2) = 4 \\
\tau\sigma(2) &= \tau(\sigma(2)) = \tau(4) = 2 \\
\tau\sigma(3) &= \tau(\sigma(3)) = \tau(3) = 1 \\
\tau\sigma(4) &= \tau(\sigma(4)) = \tau(5) = 3 \\
\tau\sigma(5) &= \tau(\sigma(5)) = \tau(1) = 5,
\end{aligned}$$

that is, $\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 3 & 5 \end{pmatrix}$.

A.2 Symmetric Group

The symmetric group is one of the most important examples of a finite group, and we will spend quit a bit of time investigating its properties. It will arise as a special case of the set S_X of bijections from a set X back to itself.

Before we can proceed, we need some preliminaries on functions. We will need some of these facts later on when we discuss homomorphisms, so we will work a little more generally than is absolutely necessary right now.

Now let's formally define S_X, the set of bijections from X to itself. We will then produced prove that S_X is a group under composition.

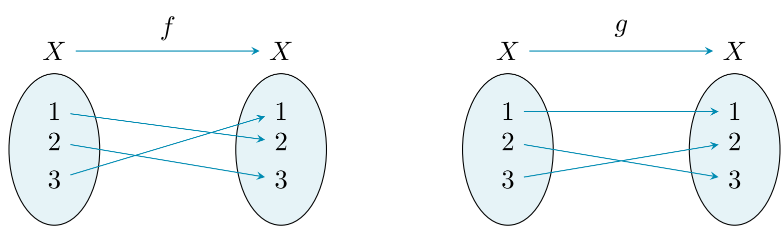
Definition A.2.1

Let X be a set. We define

$$S_X = \{f : X \longrightarrow X : \quad f \quad \text{is a bijection}\}$$

Note that S_X is closed under composition of functions. In other words, the composition operation on S_X is associative (Exercise for student).

The composition is not commutative operation. To see this, let for example $X = \{1, 2, 3\}$, and define f and g by the following diagrams:



Then

$$g \circ f(1) = g(f(1) = g(2) = 3,$$

but

$$f \circ g(1) = f(g(1)) = f(1) = 2,$$

so

$$g \circ f \neq f \circ g$$

Thus S_X will provide a new example of a nonabelian group.

To finish checking that S_X is a group, we need to verify the existence of an identity and inverses. For the first one, recall that any set X has a special bijection from X to X, namely the identity function Id_X:

$$\mathrm{Id}_x(x) = x$$

for all $x \in X$. note for any $f \in S_X$, we have

$$f \circ \mathrm{Id}_X(x) = f(\mathrm{Id}_X(x)) = f(x)$$

and

$$\mathrm{Id}_X \circ f(x) = \mathrm{Id}_X(f(x)) = f(x)$$

for all $x \in X$. Thus $\mathrm{Id}_X \circ f = f \circ \mathrm{Id}_X = f$ for all $f \in S_X$, so Id_X serves as an identity for S_X under composition.

Finally, if $f \in S_X$ and $y \in X$, there is an $x \in X$ such that $f(x) = y$, since f is onto. But f is also one-to-one, so this x is unique. Therefore, we can define $f^{-1}(y) = x$. You can check that

$$\begin{aligned} f \circ f^{-1}(y) &= f(f^{-1}(y)) \\ &= f(x) \\ &= y \\ &= \mathrm{Id}_X(y). \end{aligned}$$

and

$$\begin{aligned} f^{-1} \circ f(x) &= f^{-1}(f(x)) \\ &= f^{-1}(y) \\ &= x \\ &= \mathrm{Id}_X(x). \end{aligned}$$

so f^{-1} really is an inverse for f under composition. Therefore, by making all of these observations, we establishes the following result:

Proposition A.2.2

S_X forms a group under composition of functions.

If X is an infinite set, then S_X is fairly hard to understand. One would have to either very brave or very crazy to try to work with it. Things are much more tractable (and interesting) when X is finite.

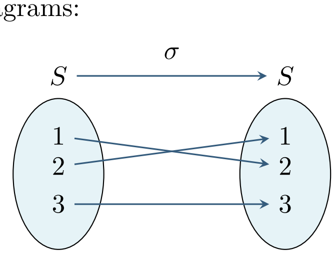
Definition A.2.3 Symmetric group

S_n is the set of all permutations of the set $\{1, 2, \cdots, n\}$ and it is called the symmetric group of n letters.

Pictorially, we represent the following bijection σ of $\{1, 2, 3\}$ defined by

$$\sigma(1) = 2, \quad \sigma(2) = 1 \quad \text{and} \quad \sigma(3) = 3,$$

with the following diagrams:



$$\text{or} \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix}.$$

Proposition A.2.4

The order of the symmetric group S_n is $n!$.

How many permutations of $\{1, 2, \cdots, n\}$ are there? In order to define a permutation f of $\{1, 2, \cdots, n\}$, we need to determine where to send each integer. There are n choices for $\sigma(1)$, and there are $n-1$ choices for $\sigma(2)$. There are $n-2$ choices for $\sigma(3)$, and so on, until we reach $\sigma(n)$, for which we only have one choice. In other words, we have observed that the total number of permutations of $\{1, 2, \cdots, n\}$ is $n(n-1)(n-2)\cdots 2.1$ Phrased in the language of group theory, we have shown that $|S_n| = n!$.

Example A.2.5

Suppose that $\sigma \in S_3$ is given by the picture that we considered earlier, i.e. $\sigma(1) = 2$, $\sigma(2) = 1$, and $\sigma(3) = 3$. Then we have

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

Of course if we are going to represent permutations in this way, it would help to know how multiplication works in this notation. As an example, let

$$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

Then remember that multiplication is really just composition of functions:

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \sigma(\tau(1)) & \sigma(\tau(2)) & \sigma(\tau(3)) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ \sigma(2) & \sigma(3) & \sigma(1) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

On the other hand, what $\tau\sigma$?

$$\tau\sigma = \begin{pmatrix} 1 & \boxed{2} & 3 \\ & \downarrow & \\ 2 & \boxed{3} & 1 \end{pmatrix}\begin{pmatrix} \boxed{1} & 2 & 3 \\ \downarrow & & \\ \boxed{2} & 1 & 3 \end{pmatrix} = \begin{pmatrix} \boxed{1} & 2 & 3 \\ \downarrow & & \\ \boxed{3} & 2 & 1 \end{pmatrix}$$

In other words, one moves right to left when computing the product of two permutations. First one needs to find the number below 1 in the rightmost permutation, then find this number in the top row of the left permutation, and write down the number directly below it. Repeat this process for the rest of the integers 2 and 3.

In example above, note that $\sigma\tau \neq \tau\sigma$, we have actually verified that S_3 in nonabelian.

Proposition A.2.6

For $n \geq 3$, S_n is nonabelian group.

Proof. Let $\sigma, \tau \in S_3$ be defined as in the example, and suppose that $n > 3$. Define $\bar{\sigma}, \bar{\tau} \in S_n$ by

$$\bar{\sigma}(i) = \begin{cases} \sigma(i) & \text{if} \quad 1 \leq i \leq 3, \\ i & \text{if} \quad i > 3 \end{cases}$$

Similarly for $\bar{\tau}$, by

$$\bar{\tau}(i) = \begin{cases} \tau(i) & \text{if} \quad 1 \le i \le 3, \\ i & \text{if} \quad i > 3. \end{cases}$$

We have $\bar{\sigma}, \bar{\tau} \in S_n$. Then the computation that we performed in S_3 shows that $\bar{\sigma}\bar{\tau} \neq \bar{\tau}\bar{\sigma}$, so S_n is nonabelian.

Definition A.2.7

For any permutation σ the unique permutation τ such that $\sigma\tau = \tau\sigma = (1)$ is called the inverse of σ and is denoted by σ^{-1}.

Example A.2.8 Symmetric group S_3

If $S = \{1, 2, 3\}$, the possible permutations can be written as

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1) \qquad \delta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\xi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Thus, $S_3 = \{e, \rho, \sigma, \gamma, \delta, \xi\}$. The Cayley Table of S_3 is given as:

o	e	ρ	ξ	σ	γ	δ
e	e	ρ	ξ	σ	γ	δ
ρ	ρ	ξ	e	γ	δ	σ
ξ	ξ	e	ρ	δ	σ	γ
σ	σ	δ	γ	e	ξ	ρ
γ	γ	σ	δ	ρ	e	ξ
δ	δ	γ	σ	ξ	ρ	e

The following table give the inverse of the permutations in S_3:

Permutation	Inverse	Remarks
$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	$e^{-1} = e$	$e^1 = (1)$
$\delta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	$\delta^{-1} = \delta$	$\delta^2 = (1)$
$\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	$\gamma^{-1} = \gamma$	$\gamma^2 = (1)$
$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	$\sigma^{-1} = \sigma$	$\sigma^2 = (1)$
$\rho = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	$\rho^{-1} = \xi$	$\rho^3 = (1)$
$\xi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	$\xi^{-1} = \rho$	$\xi^3 = (1)$

Example A.2.9

Find the inverse of each of the following permutations:

1. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$.

2. $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$.

Solution

1. The inverse of σ can be obtained by reading the array form the bottom row to the top row. For example, 1 in the bottom row must map to the number above it, which is 2. Similarly for the other numbers, so

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

2. Similar to 1., we read the array form of τ from bottom-to-top to get the array form of τ^{-1} :

$$\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

Notice this is just τ itself. So τ is its own inverse.

Theorem A.2.10

For any $\sigma \in S_n$ there exists an integer $m \geq 1$ for which $\sigma^m = (1)$.

Proof. Consider the list of powers:

$$\sigma, \sigma^2, \sigma^3, \ldots.$$

Since there are only finitely many permutations of any finite set, there must be repetitions within the list. Assume that $\sigma^s = \sigma^r$ for some $0 < r < s$. Then

$$\sigma^{r-s} = (1).$$

Definition A.2.11 Orbits

Let σ be a permutation on a set X. The equivalence classes in X determined by the equivalence relation

$$a \sim b \quad \text{if and only if} \quad b = \sigma^n(a), \quad \text{for all} \quad n \in \mathbb{Z}$$

are the orbits of σ.

Example A.2.12

Find the orbits of the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 4 & 1 & 5 & 2 \end{pmatrix}.$$

Solution. To find the orbit containing 1, we apply σ repeatedly

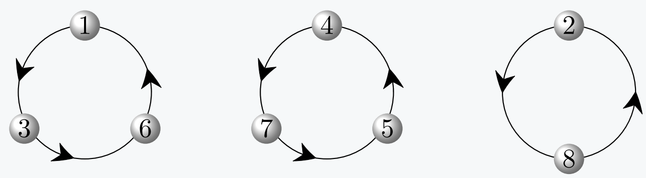
$$1 \xrightarrow{\sigma} 3 \xrightarrow{\sigma} 6 \xrightarrow{\sigma} 1 \xrightarrow{\sigma} 3 \xrightarrow{\sigma} 6 \cdots$$

Since σ^{-1} would simply reverse the directions of the arrow in the chain, we see that the orbit containing 1 is $\{1, 3, 6\}$. We now choose an integer from 1 to 8 not in $\{1, 3, 6\}$, say 2, and similarly find the orbit containing 2 is

$$2 \xrightarrow{\sigma} 8 \xrightarrow{\sigma} 2 \xrightarrow{\sigma} 8 \cdots$$

that is $\{2, 8\}$. Finally, we find the orbit containing 4 is $\{4, 7, 5\}$. Since these three orbits include all integers from 1 to 8. Hence the complete list of orbits of σ is

$$\{1, 3, 6\}, \ \{2, 8\}, \ \{4, 7, 5\}.$$



There is another notation commonly used to specify permutations. It is called **cycle notation**. Cycle notation has theoretical advantages in that certain important properties of the permutation can be readily determined when cycle notation is used.

Definition A.2.13 Cycle

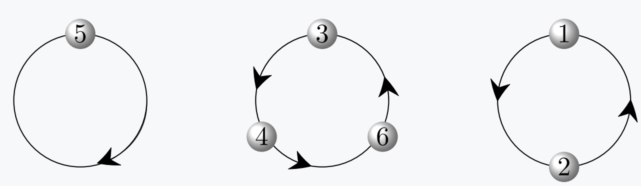
A permutation $\sigma \in S_n$ is a cycle if it has at most one orbit containing more that one element. The length of the cycle is the number of elements in its largest orbit.

Example A.2.14

Let us consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 6 & 5 & 3 \end{pmatrix}.$$

This assignment of values could be presented schematically as follows:



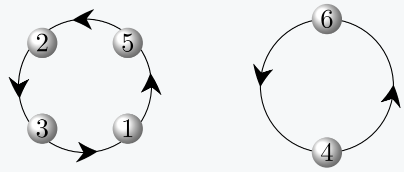
Instead, we leave out the arrows and simply write a

$$\sigma = (1\ 2)(3\ 4\ 6)(5) = (1\ 2)(3\ 4\ 6) = (3\ 4\ 6)(1\ 2).$$

Example A.2.15

As a second example, consider

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 2 & 4 \end{pmatrix}.$$



In cycle notation, τ can be written as

$$\tau = (2\ 3\ 1\ 5)(6\ 4) = (4\ 6)(3\ 1\ 5\ 2).$$

Definition A.2.16

An expression of the form $(a_1, a_2, \cdots, a_m)$ is called a cycle of length m or an m-cycle.

Example A.2.17

To determine the cycle form of the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 1 & 6 & 8 & 4 & 10 & 7 & 2 & 9 & 3 \end{pmatrix}$$

start with the smallest number in the set, in this case it is 1. Since $\sigma(1) = 5$ we begin the cycle by writing

$$(1, 5, \cdots) \cdots$$

Next, 5 maps to 4, so we continue building the cycle

$$(1, 5, 4, \cdots) \cdots$$

Continuing in this way we construct $(1, 5, 4, 8, 2, \cdots) \cdots$, and since 2 maps back to 1 then we close off the cycle:

$$(1, 5, 4, 8, 2) \cdots$$

Next, we pick the smallest number that doesn't appear in any previously constructed cycle. This is the number 3 in this case. We now repeat what we just did and construct the cycle involving 3:

$$(1, 5, 4, 8, 2)(3, 6, 10) \cdots$$

We now pick the smallest number that doesn't appear in any previously constructed cycle, which is 7, and construct the cycle to which it belongs. In this case 7 just maps to itself:

$$(1, 5, 4, 8, 2)(3, 6, 10)(7) \cdots$$

Finally, the only number remaining is 9 and it maps back to itself so the cycle for of σ is

$$(1, 5, 4, 8, 2)(3, 6, 10)(7)(9)$$

which simplifies to

$$\sigma = (1, 5, 4, 8, 2)(3, 6, 10)$$

since our convention is omit 1-cycles. Therefore, σ is the product of a 3-cycle and a 5-cycle.

Definition A.2.18 Support

The support of a k-cycle $\sigma = (a_1, a_2, \ldots, a_k)$, is the set of entries

$$\operatorname{supp}(\sigma) = \{a_1, a_2, \ldots, a_k\}.$$

In particular the support of a 1-cycle (a_1) is the one-point set $\{a_1\}$.

Definition A.2.19 Disjoint cycles

Two cycles σ and τ in S_n are called disjoint if $\operatorname{supp}(\sigma) \cap \operatorname{supp}(\tau) = \emptyset$.

Proposition A.2.20 Inverse of a cycle

If $\sigma = (s_1\ s_2\ \ldots\ s_{k-1}\ s_k)$ be a cycle of length k, then

$$\sigma^{-1} = (s_k\ s_{k-1}\ \ldots\ s_2\ s_1)$$

Proof. Exercise for students. □

Example A.2.21

- Let $\sigma = (2\ 5\ 4\ 6)$, then $\operatorname{supp}(\sigma) = \{2, 5, 4, 6\}$ and $\sigma^{-1} = (6\ 4\ 5\ 2)$.
- The cycles $(1\ 2\ 5\ 6)$ and $(4\ 3)$ are disjoint.
- The cycles $(1, 2, 6)$ and $(4, 3, 1)$ are not disjoint.

Theorem A.2.22 Products of disjoint cycles

Every permutation of a finite set can be written as a cycle or as a product of disjoint cycles.

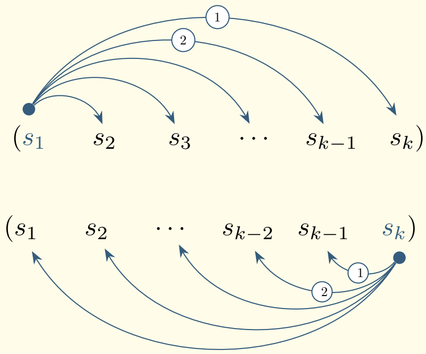
Definition A.2.23 Transposition

A cycle of length 2 is called a transposition.

Remark A.2.24.

1. If σ is a transposition, then $\sigma^{-1} = \sigma$. For example $(2\ \ 5)^{-1} = (5\ 2) = (2\ 5)$.

2. Every k-cycle $(s_1, s_2, \dots, s_{k-1}, s_k)$ can be written as a product transposition:



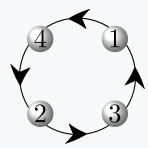
$$(s_1\ s_2\ s_3\ \cdots\ s_{k-1}\ s_k) = (s_1\ s_k)(s_1\ s_{k-1})\ \cdots\ (s_1\ s_3)(s_1\ s_2).$$

or

$$(s_1\ s_2\ \cdots\ s_{k-2}\ s_{k-1}\ s_k) = (s_k\ s_{k-1})(s_k\ s_{k-2})\ \cdots\ (s_k\ s_2)(s_k\ s_1).$$

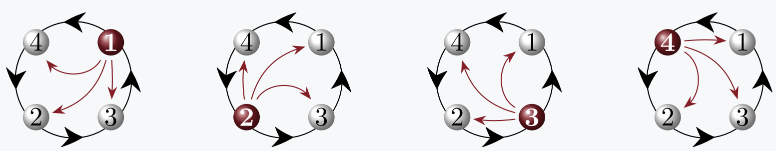
Example A.2.25

Consider the cycle $c = (1\ 4\ 2\ 3) \in S_4$:



Then σ can be written as :

$$\begin{aligned}\sigma &= (1\ 4\ 2\ 3) = (1\ 3)(1\ 2)(1\ 4) = (3\ 2)(3\ 4)(3\ 1)\\ &= (2\ 3\ 1\ 4) = (2\ 4)(2\ 1)(2\ 3) = (4\ 1)(4\ 3)(4\ 2).\end{aligned}$$



Theorem A.2.26

No permutation in S_n can be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions.

Definition A.2.27 Even and odd permutations

A permutation that can be expressed as a product of an even number of transpositions is called an even permutation. A permutation that can be expressed as a product of an odd number of transpositions is called an odd permutation.

Definition A.2.28 Signature of permutations

The signature of a permutation σ is denoted $\operatorname{sgn}(\sigma)$ and defined as 1 if σ is even, and -1 if σ is odd. That means:

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

Remark A.2.29.

1. Every transposition is an odd permutation.

2. The identity permutation $e = (1)$ is even, because for example $e = (1\ 2)(1\ 2)$.

Example A.2.30

Determine whether the following permutation is odd or even and find their signature?

1. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 6 & 5 & 3 \end{pmatrix}$.

2. $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 2 & 4 \end{pmatrix}$.

Solution:

Since

$$\sigma = (1\ 2)(3\ 4\ 6) = \underbrace{(1\ 2)}_{1}\underbrace{(3\ 6)}_{2}\underbrace{(3\ 4)}_{3},$$

is a product of 3 transpositions, σ is an odd permutation, and hence $\operatorname{sgn}(\sigma) = -1$. For the second permutation, we have:

$$\tau = (1\ 5\ 2\ 3)(4\ 6) = \underbrace{(1\ 3)}_{1}\underbrace{(1\ 2)}_{2}\underbrace{(1\ 5)}_{3}\underbrace{(4\ 6)}_{4}.$$

So τ is a product of 4 transpositions. Therefore τ is even, and hence $\operatorname{sgn}(\tau) = 1$.

Proposition A.2.31

Let σ be a cycle of length k. Then

$$\operatorname{sgn}(\sigma) = (-1)^{k+1}.$$

Proposition A.2.32

Let σ and τ be two permutations in S_n. Then

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma) \times \operatorname{sgn}(\tau).$$

Moreover

$$\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\sigma).$$

and

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\tau\sigma).$$

Appendix

B

Graded algebras

Appendix contents

B.1 Preliminaries

Definition B.1.1 $\mathbb{F}$-algebras (associative algebras)

Given a field, $\mathbb{F}$, a $\mathbb{F}$-algebra (or associative algebra over $\mathbb{F}$) is a $\mathbb{F}$-vector space R, together with a bilinear operation $\cdot : R \times R \longrightarrow R$, called multiplication, which makes R into a ring with $1 = 1_R$. This means that $\cdot$ is associative and that there is a multiplicative identity element, 1, so that $1 \cdot r = r \cdot 1 = r$, for all $r \in R$.

Example B.1.2 Algebra of linear transformations

The vector space $\mathcal{L}(V)$ of all linear transformations $T : V \longrightarrow V$ is an algebra, where in this algebra the product fg of two linear transformations $f, g \in \mathcal{L}(V)$ is defined to be their composition; that is, fg is the linear transformation on V defined by

$$(fg)(v) = f(g(v)).$$

The identity map on V, which sends every $v \in V$ to itself, is the identity element $1 \in \mathcal{L}(V)$, and if V has dimension greater than 1, then $\mathcal{L}(V)$ is a noncommutative algebra.

Example B.1.3 Algebra of square matrices

$\mathcal{M}_{n\times n}(\mathbb{F})$ is a $\mathbb{F}$-algebra, This is called a matrix algebra over $\mathbb{F}$, where in this algebra the product is the matrix multiplication, and $1 = I_n$ the identity matrix.

Example B.1.4

The vector space of polynomial $\mathbb{F}[x]$ is a $\mathbb{F}$-algebra, with polynomial multiplication. Thus, if

$$f(x) = \sum_{i=1}^{r} a_i x^i, \quad g(x) = \sum_{j=1}^{s} b_i x^j,$$

then fg is the polynomial

$$(fg)(x) = f(x)g(x) = \sum_{k=1}^{r+s} c_k x^k$$

where

$$c_k = \sum_{i+j=k} a_i b_j.$$

Definition B.1.5 Homomorphism of associative algebras

Let R and S be two associative algebras over a field $\mathbb{F}$. A linear mapping f from R to S of $\mathbb{F}$-vector spaces is called homomorphism of associative algebras if $f(1_R) = 1_S$. and

$$f(r \cdot r') = f(r) \cdot f(r') \quad \text{for all} \quad r, r' \in R.$$

Definition B.1.6 Subalgebra

A nonempty subset A of an associative algebras R over a field $\mathbb{F}$ is called subalgebra of R if it is a vector subspace of R and $a \cdot a' \in A$ for all $a, a' \in A$.

Example B.1.7

The set

$$A = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{F}) \mid a, b, c, d \in \mathbb{F} \right\}$$

is a subalgebra of $\mathcal{M}_3(\mathbb{F})$.

Proposition B.1.8

The intersection of a family of sublagebras $(A_i)_{i\in I}$ of an algebra R is also a sublagebra of R.

Proof. We know that, the intersection of a family of subspaces is a subspace. Let $a, b \in \bigcap_{i \in I} A_i$, then $a, b \in A_i$ for all $i \in I$. Since A_i is a subalgebra, we get $ab \in A_i$ for all $i \in I$. Hence $ab \in \bigcap_{i \in I} A_i$. □

Definition B.1.9 Subalgebra generated by a set

Suppose that S is a nonempty subset of an algebra R. The subalgebra generated by the set S is denoted by $\mathrm{Alg}(S)$ and is defined to be the smallest subalgebra of R that contains the set S. In terms of S alone,

$$\mathrm{Alg}(S) = \mathrm{span}\{s_1 \cdots s_m \mid m \in \mathbb{N}, s_1, ..., s_m \in S\}.$$

Example B.1.10

Let V be a vector space over a filed $\mathbb{F}$, and $\{v_1, \ldots, v_n\}$ a basis for V. Then

$$S = \{1\} \cup \{v_1, \ldots, v_n\}$$

S is a set of generators for the tensor algebra $T(V)$:

$$T(V) = \mathrm{Alg}(\{1\} \cup \{v_1, \ldots, v_n\}).$$

Example B.1.11

$$\mathbb{F}[x] = \mathrm{Alg}(\{1, x\}).$$

and

$$\mathbb{F}[x, y] = \mathrm{Alg}(\{1, x, y\}).$$

More general, we have

$$\mathbb{F}[x_1, \ldots, x_n] = \mathrm{Alg}(\{1, x_1, \ldots, x_n\}).$$

Definition B.1.12 Ideal

A subalgebra $\mathfrak{a}$ of R is called an ideal of R if, for all $a \in \mathfrak{a}$, $r \in R$, we have

$$r \cdot a \in \mathfrak{a} \text{ and } a \cdot r \in \mathfrak{a}$$

Example B.1.13

The set $A = \{P \in \mathbb{F}[x] \mid P(0) = 0\}$ is an ideal of $\mathbb{F}[x]$.

Proposition B.1.14

The intersection of a family of ideals $(\mathfrak{a}_i)_{i \in I}$ of an algebra R is also an ideal.

Proof. From Proposition B.1.8, $\bigcap_{i\in I} \mathfrak{a}_i$ is a subalgebra of R. Let $a \in \bigcap_{i\in I} \mathfrak{a}_i$ and $r \in R$, so $a \in \mathfrak{a}_i$ for all $i \in I$. Since $\mathfrak{a}_i$ is ideal, we obtain $ra \in \mathfrak{a}_i$ and $ar \in \mathfrak{a}_i$ for all $i \in I$. Therefore ra and ar are in $\bigcap_{i\in I} \mathfrak{a}_i$. □

Definition B.1.15 Factor algebra (or quotient algebra)

For any ideal $\mathfrak{a}$ of R, we can define an equivalence relation on R by declaring x to be equivalent to y if and only if $x - y \in \mathfrak{a}$. We denote the set of equivalence classes of elements of R by

$$R/\mathfrak{a} = \{\overline{x}, \mid \quad x \in R\}$$

and the equivalence class (or coset) of every element $x \in R$ is indicated by $\overline{x}$; thus,

$$\overline{x} = \{y \in R \mid y - x \in \mathfrak{a}\} = x + \mathfrak{a}.$$

Consider the factor space $R/\mathfrak{a}$ defined by

$$R/\mathfrak{a} = \{\overline{r} = r + \mathfrak{a} \quad \mid \quad r \in R\}$$

This set is a $\mathbb{F}$-vector space with the following addition and scalar multiplication:

$$\overline{r_1} + \overline{r_2} = \overline{r_1 + r_2}$$

and

$$\alpha(\overline{r}) = \overline{\alpha r}$$

In addition, the following multiplication

$$\cdot \quad : R/\mathfrak{a} \times R/\mathfrak{a} \longrightarrow R/\mathfrak{a}$$

$$(\overline{r_1}, \overline{r_2}) \longmapsto \overline{r_1 \cdot r_2}.$$

is well defined (independent of the choice of representative for ($\overline{r_1}$ and $\overline{r_2}$) and bilinear. Hence $R/\mathfrak{a}$ is an associative algebra, and it's called the factor algebra of R by $\mathfrak{a}$.

B.2 Graded vector spaces

Definition B.2.1 Direct product / Direct sum of vector spaces

Let $\left(V_i\right)_{i=0}^{\infty}$ be infinitely collection of $\mathbb{F}$-vector spaces.

- A direct product $\prod_{i=0}^{\infty} V_i$ is the set of all sequences $(v_1, v_2, ...)$ where each $v_i \in V_i$ with usual pointwise addition

$$(v_0, v_1, v_2, \dots) + (w_0, w_1, w_2, \dots) = (v_0 + w_0, v_1 + w_1, v_2 + w_2, \dots),$$

and scalar multiplication

$$\lambda(v_0, v_1, v_2, \dots) = (\lambda v_0, \lambda v_1, \lambda v_2, \dots)$$

- The direct sum $\bigoplus_{i=0}^{\infty} V_i$ is the set of all sequences $(v_0, , v_1, v_2, ...)$ where each $v_i \in V_i$ such that

$$\{i \mid v_i \neq 0\} \quad \text{is finite}$$

with usual pointwise addition and scalar multiplication.

If we identify

$$v_i \in V_i \quad \longleftrightarrow \quad (0, \ldots, 0, v_i, 0, \ldots) \in \bigoplus_{i=0}^{\infty} V_i$$

$$\uparrow$$

ith term

then V_i can be considered as a subset of $\bigoplus_{i=0}^{\infty} V_i$.

If $v = (v_0, v_1, v_2, ...) \in \bigoplus_{i=1}^{\infty} V_i$, there exists an integer i_0 such that $v_i = 0$ for all $i > i_0$. Thus we can write the element v as

$$v = \sum_{i=0}^{i_0} v_i.$$

Definition B.2.2 Graded vector space

- The direct sum presented in the previous definition $\bigoplus_{i=0}^{\infty} V_i$ is called a graded vector space.
- Every element in $v_i \in V_i$ is called homogeneous element of degree i.
- Moreover, if $w \in \bigoplus_{i=0}^{\infty} V_i$ such that $w = \sum_{i=0}^{i_0} w_i$, where $w_i \in V_i$, then w_i is called the homogeneous component of w of degree i.

Definition B.2.3 Graded associative algebra

If a graded vector space $R = \bigoplus_{i=0}^{\infty} R_i$ is an associative algebra such that for all $x_i \in R_i$ and $x_i \in R_j$, we have $x_i x_j \in R_{i+j}$, then R is called a graded associative algebra.

Example B.2.4

The tensor algebra $T(V) = \bigoplus_{p=0}^{\infty} T^p(V)$ is a graded associative algebra.

$\mathbb{F}[x] = \bigoplus_{p=0}^{\infty} \mathbb{F}x^i = \mathbb{F} \oplus \ Fx \oplus \ Fx^2 \oplus \ Fx^3 \oplus \cdots$ is a graded associative algebra, where $\mathbb{F}x^i = \text{span}(x^i)$.

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