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Department of Computer Science

Course support

Linear Programming

For students in the 3rd year – License in Computer Science

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This course material is intended for third-year License students. This module aims to raise student awareness of the practical importance linked optimization areas, to understand the underlying theory, and to be able to use these techniques in practical problems. This course is organized into five parts: The first part is a reminder on linear algebra. The second part is a state of the art on optimization. The third part is dedicated to linear programming. The fourth part presents simplex algorithm. The last part is dedicated to exercises.

LP

1. Mathematical Reminders (Linear Algebra)

1.1. The set N

It is the set of all-natural whole numbers. A natural number is a positive or zero number, allowing objects to be counted.

Examples: 0, 1, 2, 3, 4, 5, 6, etc.

1.2. The Z set

It is the set of relative integers. A relative integer is not only a natural integer, but also appears as a natural integer with a positive or negative sign.

Examples: …. -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, +6, +7, +8, etc.

1.3. The D set

It is the set of relative decimal numbers. A relative decimal number is not only a relative integer, but can also be a floating-point number, positive or negative.

A number is decimal if it can be written as $a/10^n$, a belonging to Z and n to N

Examples: …. -5, -4, -4.2, -3, -2, -1.5, -1, 0, +0.7, +1, +2, +2.4, +3, +4, +5, +6, +6.75 +7, +8, etc.

1.4. The Q set

It is the set of rational numbers. A rational number is not only a relative decimal number, but can also be a number that can be expressed with the quotient of two relative integers. The denominator being non-zero.

Examples: …. -5/4, -4, -4.2, -3, -2, -1.5, -1/2, 0, +0.7, +1, +2, +2.4, +3, +4/5, +5, +6, +6.75, +7/2, +8

1.5. The R set

It is the set of real numbers. A real number is not only a rational number, but can also be a number whose decimal expansion is infinite, and not periodic.

Examples: …. -5/4, -4, -4.2, -3, -2, -1.524, -1/2, 0, +0.7, +1, +2, +2.41, +3, +4/5, +5, +6, +6.75, +7/2, +8…

1.6. Vector Space

A vector space is a set with a structure that allows linear combinations to be performed .

Given a body **K** , a vector space *E* over **K** is a commutative group (whose law is noted +) equipped with a "compatible" action of **K** (in the sense of the definition below). The elements of *E* are called vectors (or points), and the elements of **K** scalars .

We call K-vector space all non-empty together E equipped with an internal composition law denoted+

$$K \times E \to E$$
$$(\lambda, x) \to \lambda x$$

Such as

1. $(E, +)$ *is abelien group*
2. $\forall \lambda, \mu \in K, \forall x \in E,\ on\ a\ (\lambda + \mu)x = \lambda x + \mu x$
3. $\forall \lambda \in K,\ \forall x, y \in E,\ on\ a\ \lambda(x + y) = \lambda x + \lambda y$
4. $\forall \lambda, \mu \in K, \forall x \in E,\ on\ a\ \lambda(\mu x) = (\lambda\mu)x$
5. $\forall x \in E,\ on\ a\ 1x = x$

1.6.1. Trivial or null vector space

Simplest example of a vector space is the null space {0}, which contains only the null vector (see Axiom 3. of vector spaces). Vector addition and multiplication by a scalar are trivial. A basis of this vector space is the empty set , $\emptyset = \{\ \}$. Thus $\{0\}$ is the vector space of dimension 0 over K. Every vector space over K contains a vector subspace isomorphic to this one.

1.6.2. Matrix spaces

Given two fixed natural integers m and n , the set $M_{m,n}(K)$ of matrices with coefficients in K with m rows and n columns, equipped with the addition of the matrices and the multiplication by a scalar of the matrices (consisting of multiplying each coefficient by the same scalar) is a vector space over K . The zero vector is none other than the matrix null .

1.7. Group

A group is the data of a set $G$and an internal composition law noted $*$as follows:

$$G \times G \to G$$
$$(x, y) \to x * y$$

Such that G verifies the following three properties:

1. (Neutral element) It exists $e \in G\ tel\ que\ \forall x \in G,\ e * x = x * e = x$
2. (Associativity) For all $x, y, z \in G,\ (x * y) * z = x * (y * z)$
3. (Inverse element) For all $x \in G,\ it\ exists\ x' \in G\ tel\ que\ x * x' = x' * x = e$

If furthermore, $\forall x, y \in G,\ on\ a\ x * y = y * x$; we say that $*$is commutative and $(G,*)$is a commutative or abelian group.

Example:

$Z,\ Q,\ R$ are abelian groups: $0$is the neutral element, the inverse of $x$is $-x$.

N is not a group because the inverse element condition is not verified.

1.8. Combination linear

Definition 1

Let be $\{x_1, \dots, x_p\}$ a family of vectors of a vector space E. Any vector of E of the form $a_1x_1 + \cdots + a_px_p = \sum_{k=1}^{p} a_kx_k$ *où les* $a_k \in R$ is called a linear combination of vectors $x_k,\ k = 1, \dots, p$.

Definition 2

If $\overrightarrow{u_1}, \overrightarrow{u_2}, \dots, \overrightarrow{u_n}$ are vectors, and if $\alpha_1, \alpha_2, \dots, \alpha_n$ are scalars, then we say that the vector

$$\vec{v} = \alpha_1\overrightarrow{u_1} + \alpha_2\overrightarrow{u_2} + \cdots + \alpha_n\overrightarrow{u_n} = \sum_{i=1}^{n} \alpha_i\overrightarrow{u_i}$$

is a linear combination of the vectors $\overrightarrow{u_1}, \overrightarrow{u_2}, \dots, \overrightarrow{u_n}$

Definition

The vector space generated by the family is called $\mathcal{F} = (e_i)_{1 \le i \le n}$ the vector subspace generated by the part $\{e_1, \dots, e_n\}$. We denote it by vect $\mathcal{F}, vect(e_i)_{1 \le i \le n}\ ou\ vect(e_1, \dots, e_n)$.

Theorem

If $(e_1, \dots, e_n)$ is a family of vectors of E then vect$(e_1, \dots, e_n)$ is the set of linear combinations of vectors $e_1, \dots, e_n$, that is:

$$vect(e_1, \dots, e_n) = \{\sum_{i=1}^{n} \lambda_ie_i\ | \lambda_1, \dots, \lambda_n \in K\}$$

Definition

We say that the family $(e_1, \dots, e_n)$ of vectors E is free if it satisfies $\forall\lambda_1, \dots, \lambda_n \in K, \lambda_1e_1 + \cdots + \lambda_ne_n = 0 \ \rightarrow \lambda_1 = \cdots = \lambda_n = 0$. we say that the vectors $e_1, \dots, e_n$ are linearly independent.

of vectors of is E said to $(e_1, \dots, e_n)$ be linked if it is not free which means $\forall\lambda_1, \dots, \lambda_n \in K, \lambda_1e_1 + \cdots + \lambda_ne_n = 0 \ \rightarrow (\lambda_1, \dots, \lambda_n) \neq (0, \dots, 0)$. An equality $\lambda_1e_1 + \cdots + \lambda_ne_n = 0$ with $\lambda_1, \dots, \lambda_n$ not all zeros is called a linear relation on vectors $e_1, \dots, e_n$.

Linear application

Let be $(E,+)$ et $(F,\ +)$two K-vector spaces. We say that $f: E \rightarrow F$is linear if:

1. $\forall x, y \in E,\ on\ a\ f(x+y) = f(x) + f(y)$
2. $\forall \lambda \in K,\ \forall x \in E,\ on\ a\ f(\lambda x) = \lambda f(x)$

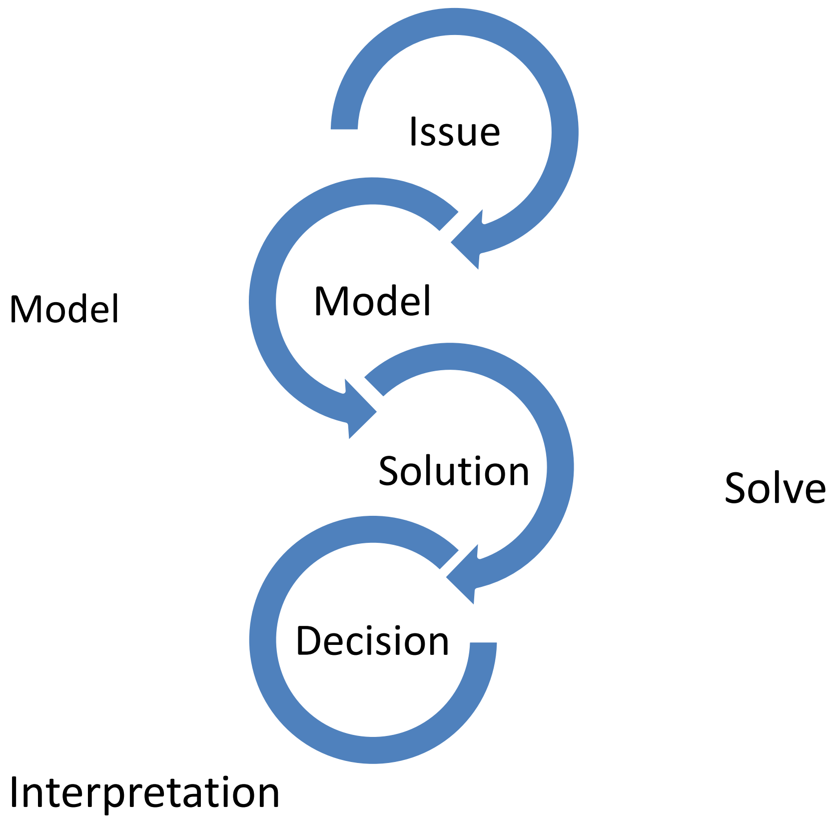
f is said to be linear if for all$\vec{u}, \vec{v}, \alpha, \beta$: $f(\alpha\vec{u} + \beta\vec{v}) = \alpha f(\vec{u}) + \beta f(\vec{v})$

2. Optimization

Optimization is a branch of mathematics and computer science that seeks to model, a analyze and to solve real problems which consists of determining which are the solution(s) satisfying an objective and respecting constraints. We can say :

- In other words, it comes down to solving an optimization problem.
- An optimization problem is a mathematical model of a real problem.
- We seek to minimize or maximize an objective function under constraints.

2.1. Modeling



Example

A farmer would like to grow two kinds of vegetables: broccoli and zucchini. He could plant all his land for this if necessary. He uses two kinds of fertilizers (A and B).

The yields of broccoli and zucchini are 4 kg/m2 and 5 kg/m2 respectively.

Its stocks are 8 liters of fertilizer A and 7 liters of fertilizer B.

The requirements for these materials are 2 L/m2 of fertilizer A and 1 L/m2 of fertilizer B for broccoli compared to 1 L/m2 of fertilizer A and 2 L/m2 of fertilizer B for zucchini.

The farmer wants to produce the maximum weight of vegetables.

Resolutions stage

1. Analysis and formulation of the problem

- Identify the different variables, their natures and their areas of study
- Define the objective of the problem
- Define possible constraints

2. Mathematical modeling of the problem

- Formulate a mathematical function, called an objective function, that describes the problem
- Write the constraints in mathematical form

3. Choosing a method for solving the modeled problem

- Choosing an optimization method appropriate to the type of problem
- Apply the optimization method to solve the modeled problem

2.2. Classification of optimization problems

Continuous Vs Discrete

Discrete optimization problems Decision variables take values from a discrete set. (integers, objects, etc.)

Continuous optimization problems Decision variables can take any real value.

With constraints Vs Without constraints

Deterministic Vs Stochastic

Problems optimization deterministic Model data are known precisely and do not assume any probability or uncertainty.

Stochastic optimization problems Models that have randomness

Linear Vs Non- Linear

Linear optimization problems Objective function and constraints are represented by linear relations.

Nonlinear optimization problems Objective function or constraints are represented by nonlinear relations.

Static Vs Dynamic

Problems optimization static Function objective and constraints do not change over time.

Problems optimization dynamics objective or constraints change over time.

3. Introduction and properties of linear programming

Mathematical programming is the translation of a real problem expressed verbally, into a mathematical problem. The goal will be to seek the optimal solution to the problem. This optimal solution minimizes or maximizes (Optimizes) a cost or a profit and it depends linearly on the variables to be determined. In linear programming, the term programming designates the organization of calculations and not the realization of a computer program.

Linear programming problems are optimization problems whose objective function and constraints are all linear.

Linear programming is a central area of optimization. Generally, linear programming problems are easy problems.

Solving a mathematical programming problem, or calculating a mathematical program, is to search for the maximum (or minimum) of an algebraic function of variables linked by equations or/and algebraic inequalities of any degree, called constraints. In the simple case where the function to be maximized (or minimized) and all the constraints are of the first degree, the problem is called linear programming. The latter is one of the most important optimization techniques used in operational research, this research is one of the decision-making processes that allows to find an optimal solution (if it exists) thanks to mathematical modeling.

Linear programming is a tool for operational research, it is also a branch of applied mathematics. It consists of searching for optimal solutions to a problem.

The general goal of linear programming is:

- To model an economic problem into a linear programming problem;
- To optimize a function under constraints;
- To interpret in economic terms the results provided by linear programming.

Any linear program is therefore made up of three main parts:

- The unknowns, called non-negative variables or activity variables.
- Equations or inequalities that serve as constraints and that verify the n activity variables; each of the equations or inequalities being a linear combination of the first degree with respect to the activity variables. These variables or unknowns can be accompanied by positive, negative or zero coefficients.
- The economic function or objective function to be maximized or minimized in which the coefficients can be positive, negative or zero.

3.1. Definition, canonical and standard form of a linear program

3.1.1. Definition

A linear program is a problem that consists of finding an optimum (maximum or minimum) of a linear function of several variables called an economic function.

The variables must satisfy a system of linear equations and/or inequalities called a system of constraints.

This program is written in the following mathematical form:

PL $\left\{ \begin{array}{l} \text{Objective Function} \\ \\ \\ \end{array} \right.$

Optimize $Z = \Sigma j = 1, n \; cjxj$

Under constraints $\Sigma \; j = 1, n \; aij \; xj \leq bi \; (1 \leq i \leq m)$

With $xj \geq 0 \; (1 \leq j \leq n)$

A choice of the variables **$(x_1, \ldots, x_n)$** is called a solution to the problem.

A solution is feasible if it satisfies the constraints.

Z is called the objective function. It associates a value with each solution. A solution is optimal if it is feasible and " maximizes or minimizes " the objective function.

cj : are economic coefficients.

bi : are resources .

aij : are technical coefficients.

Generally speaking, a mathematical programming problem involves four categories of elements:

- variables or activities,
- coefficients,
- resources,
- coefficients.

Activities are the decision variables of the problem studied.

The company must select the best activities $x = (x_1, \dots, x_n)$, i.e. the one which is most in line with its objectives.

Economic coefficients measure the degree of achievement of the company's objective, associated with a unit value of each of the variables. Each variable is thus associated with an economic coefficient c_j. The evaluation of the coefficients $c_j$depends on the type of objective pursued: depending on the case it will be a sales price, a gross margin, a unit variable cost, etc.

Resources can also be of a very diverse nature depending on the problem encountered.

In any case, these are the elements that limit the economic calculation of the company: limited production capacities, standards to be respected, sales potential, etc.

given vector of resources .$b = (b_1, \dots, b_m)$

By technical coefficient we will designate the degree of consumption of a resource by an activity.

The technical coefficient will correspond to $a_{ij}$the resource $i$and the activity j.

To the extent that the problem studied involves $n$activities and $m$resources, technical coefficients will have to be considered $m \, x \, n$.

3.1.2. Concept of canonical forms

When the set of constraints is presented in the form of inequalities (≥ or ≤) we speak of canonical form.

However, it is appropriate to distinguish between a type 1 canonical program and a type 2 canonical program:

- A type 1 canonical program is a program in which the inequality constraints are rotated in the "less than or equal" direction, the desired objective being the maximization of the criterion function or economic function;
- A canonical type 2 program has inequality constraints in the "greater than or equal" direction and the objective is a minimum.

Mathematically the canonical form is written as follows:

$$\text{Max or Min } Z = C_1 X_1 + C_2 X_2 + \dots\dots\dots + C_n$$

$$a_{11} X_1 + a_{12} X_2 + \dots\dots\dots + a_{1n} \leq b_1$$

$$a_{21} X_1 + a_{22} X_2 + \dots\dots\dots + a_{2n} X_n \leq b_2$$

$$\dots\dots\dots\dots\dots\dots \quad a_{m1} X_n \leq b_m$$

$$X_1 \geq 0 ; \quad X_2 \geq 0; \quad \dots\dots\dots; X_n \geq 0$$

3.1.3. Concept of standard form

All constraints represent equalities. The objective can be the maximum or the minimum.

The standard form is written as follows:

Max or Min Z = $C_1 X_1 + C_2 X_2 + \dots\dots\dots + C_n$ $a_{11} X_1 + a_{12} X_2 + \dots\dots\dots + a_{1n} X_n = b_1$

$a_{21} X_1 + a_{22} X_2 + \dots\dots\dots + a_{2n} X_n = b_2$

.... $a_{m1} X_1 + a_{m2} X_2 + \dots\dots\dots + a_{mn} x_n = b_m$

$x_1 \geq 0$; $X_2 \geq 0$; ....; $X_n \geq 0$

3.1.4. Mixed form concept

Sometimes the constraints are rotated one way, the other in the opposite direction, the objective being either a maximum or a minimum. But we can also have a mixture of equalities (=) or inequalities (≥ or ≤). Such a program is a mixed program. It is also said to be in mixed form.

4. Modeling a linear problem

Formalizing a program is a delicate but essential task because it determines the subsequent discovery of the right solution. It involves the same phases regardless of the techniques required later for processing (linear programming or nonlinear programming):

1. **Problem** detection and **variable** identification. These variables must correspond exactly to the concerns of the decision maker. In mathematical programming, the variables are decision variables.
2. The formulation of the **economic function** (or objective function) translating the preferences of the decision-maker expressed in the form of a function of the identified variables.
3. The formulation of **constraints.** It is very rare that a manager has complete freedom of action. Most often there are limits not to be exceeded which take the form of mathematical equations or inequalities.

The person responsible for a decision only has his or her competence to carry out a correct formalization of the problem posed because there is no method in this area.

Example

A factory produces two products (A) and (B) using raw materials I, II and III. The operation of the factory is as follows:

- 1 unit of product (A) requires 2 units of I and 1 unit of II.
- 1 unit of product (B) requires 1 unit of I, 2 units of II and 1 unit of III.

It is assumed that the factory has raw materials I, II and III in quantities of 8, 7 and 3 respectively. The profit due to the manufacture of one unit of product (A) (resp. (B)) is equal to 4 (resp. 5) Algerian Dinars (DA). The objective is to maximize the profit while respecting the constraints on the raw material.

Mathematical formulation

If we designate by x_1 and x_2 respectively the quantities sold of product (A) and (B), the total gain is worth

$$Z(x_1, x_2) = 4x_1 + 5x_2$$

On the other hand, the availability of raw materials amounts to requiring quantities, used for the manufacture of x_1 and x_2, which are lower than the quantities available. These constraints are expressed by the following inequalities:

$$\begin{cases} 2x_1 + x_2 & \leq 8 \\ x_1 + 2x_2 & \leq 7 \\ \quad x_2 & \leq 3 \end{cases}$$

Of course, the variables x_1 and x_2 must be positive. In conclusion, the profit maximization problem translates mathematically into a linear program that is written in the form:

$$\begin{cases} \max[Z(x_1, x_2) = & 4x_1 + 5x_2] \\ 2x_1 + x_2 & \leq 8 \\ x_1 + 2x_2 & \leq 7 \\ x_2 & \leq 3 \\ x_1 \geq 0,\ x_2 \geq 0 & \end{cases}$$

3.2. Resolution method

We will present techniques that allow solving linear programs:

- The graphical method: the use of this method is restricted to (PL) having a number of variables at most equal to 3.
- The simplex method: iterative algorithm developed by George Dantzig in 1951.

3.2.1. The graphic method

This method is only applicable in the case where there are only two variables. Its advantage is to be able to understand what the general Simplex method does, without going into the purely mathematical technique.

In this method; only activity variables or real variables will be used. There will therefore be no deviation variables or artificial variables. After translating the problem posed into a mathematical model, we will simply limit ourselves to:

- Graphically represent the limit lines (equations coming from the initial inequalities);

- Delimit the boundary of the polygonal envelope, that is to say construct the domain of acceptability;
- Successively replace the coordinates of each vertex of the polygon in the economic function in order to obtain the optimal combination sought (minimum or maximum).

In general, to find the minimum, we will choose the point closest to the origin, while for the maximum it will be the point furthest away. Instead of listing all the points of the acceptability polygon (method previously used), we can use the process of moving the line of the economic function parallel to its inclination to the origin and at each of the vertices of the acceptability domain. For the cost, we will choose the line closest to the origin and for the maximum (margin or profit), the furthest away.

4. Simplex algorithm

The Simplex method is an iterative procedure that allows improving the resolution of the objective function at each step. The process ends when you cannot continue to improve the value, that is, the optimal solution has been reached (the highest or lowest possible value, as the case may be).

From the base of the value of the objective function at any point, the method is to find another point that improves the previous value. As indicated in the Graphical method, these points are the vertices of the polygon (or polyhedron, if the number of variables is greater than two), which is the region determined by the constraints that the problem is subjected to (called feasible region). The search is carried out by moving the edges of the polygon, from the current vertex to an adjacent element that allows to improve the value of the objective function. Whenever the feasible region is present, since the number of vertices and edges is finite, it is possible to find the solution.

The Simplex method is based on the following property: if the objective function Z does not take its maximum value at vertex A, then there is an edge whose starting point is A and along which the value of Z increases.

It will be necessary to consider that the Simplex method only works with the constraints of the problem whose constraints are of the type "≤" (less than or equal) and its independent coefficients are greater than or equal to 0. Thus, the restrictions should be normalized to meet these requirements before starting the Simplex algorithm. In the case that the constraints of type "≥" (greater than or equal) or "=" (equal) appear after this process, or they cannot be modified, it will be necessary to use other resolution methods, being the Two Phases method the most common.

4.1. Adaptation of the model to the Simplex method

The standard form of the problem model consists of an objective function under specified constraints:

Objective function: $c_1 \cdot x_1 + c_2 \cdot x_2 + ... + c_n \cdot x_n$

under the constraints:

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 + ... + a_{1n} \cdot x_n = b_1$$
$$a_{21} \cdot x_1 + a_{22} \cdot x_2 + ... + a_{2n} \cdot x_n = b_2$$
$$...$$
$$a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + ... + a_{mn} \cdot x_n = b_m$$
$$x_1, ..., x_n \geq 0$$

The model must meet the following conditions:

1. The objective will be to maximize or minimize the value of the objective function (e.g., increase profits or reduce losses, respectively).

2. All constraints must be equations of equality (mathematical identities).

3. All variables (X_i) must be a positive or zero value (non-negativity condition).

4. The independent terms (b_i) of each equation must be non-negative.

4.2. Normalization of constraints

Another condition of the standard model of the problem is that all constraints are equations of equality (that is, equality constraints), it will be necessary to change the inequality constraints or inequalities into these mathematical identities.

The condition of non-negativity of the variables ($x_1, ..., x_n \geq 0$) is the only exception and remains identical.

4.2.1. Constraint of type "≤"

To normalize an inequality constraint of the type " ≤", we add a new variable, called **the gap variable** x_s (on the non-negativity condition : $x_s \geq 0$). This new variable appears with a zero coefficient in the objective function, and by adding in the corresponding equation (which will now be a mathematical identity or an equality equation).

$$a_{11} x_1 + a_{12} x_2 \leq b_1 \qquad a_{11} x_1 + a_{12} x_2 + \mathbf{1\, x_s} = b_1$$

4.2.2. Constraint of type "≥"

In the case of an inequality of type "≥", we must also add a new variable called **the excess variable** x_s (with the non-negativity condition : $x_s \geq 0$). This new variable appears with a zero coefficient in the objective function, and by subtracting the corresponding equation.

Now, a problem arises with the non-negativity condition of this new variable of the problem. The inequalities containing the inequality type "≥" would be:

$$a_{11} x_1 + a_{12} x_2 \geq b_1 \qquad a_{11} x_1 + a_{12} x_2 \mathbf{- 1\, x_s} = b_1$$

In the first iteration of the Simplex method, the base variables will not be at the base and will become zero. In this case, the new variable x_s , after converting x1 and x2 to zero, will take the value -b1 and will not fulfill the non-negativity condition. We must add another variable ***a***

, called **artificial variable** , which will also appear with a zero coefficient in the objective function and adding in the corresponding constraint. As follows:

a 11 x 1 + a 12 x 2 ≥ b 1 a 11 x 1 + a 12 x 2 **- 1 x s + 1 *a*** = b 1

4.2.3. Type constraint "="

Contrary to what one might think, for constraints of type "=" (despite being identities) we also need to add **artificial variables *a*** . As in the previous case, the coefficient will be zero in the objective function and it will add in the corresponding constraint.

a 11 x 1 + a 12 x 2 = b 1 a 11 x 1 + a 12 x 2 + **1 *a*** = b 1

In the latter case, it is clear that the artificial variables are in violation of the laws of algebra, so it will be necessary to ensure that these artificial variables have a value of 0 in the final solution. This is accomplished by the **Two Phase method** and so it will always be necessary to do it whenever we have this kind of variables.

The following table summarizes the type of variable that appears in the standardized equation and its sign according to the inequalities:

Form of inequality	**Form of the new variable**
≥	- excess + artificial
=	+ artificial
≤	+ gap

4.2.4. Development of the Simplex Method

Once the model is standardized, it may be necessary to apply the Simplex method.

The steps of each method are explained step by step below, specifying the aspects to take into consideration.

Simplex procedure

Algorithm Step

The first table

$z_j = c_j \times a_{ij}$

c_j		**30**	**50**	**0**	**0**	**0**	**b_i**	**bi /col(pivot)**
c_j	**v. base**	**x**	**y**	**e_1**	**e_2**	**e_3**		
0	**e_1**	**3**	**2**	**1**	**0**	**0**	**1800**	
0	**e_2**	**1**	**0**	**0**	**1**	**0**	**400**	
0	**e_3**	**0**	**1**	**0**	**0**	**1**	**600**	
z_j		**0**	**0**	**0**	**0**	**0**	**0**	
$c_j - z_j$		**30**	**50**	**0**	**0**	**0**		

<u>1st iteration:</u>

Choice of the variable entering the database:

Maximum of $c_j - z_j$ for max problems

Minimum of $c_j - z_j$ for min problems

Choice of outgoing variable:

Min(bi/Pivot column)

$$\frac{b_i}{a_{ik}}\bigg|\, a_{ik} > 0$$

c_j		30	50	0	0	0	b_i	bi /col(pivot)
c_j	v. base	x	y	e_1	e_2	e_3		
0	e_1	3	2	1	0	0	1800	1800/2 = 900
0	e_2	1	0	0	1	0	400	400/0 = ∞
0	e_3	0	1	0	0	1	600	600/1 = 600
z_j		0	0	0	0	0	0	
$c_j - z_j$		30	50	0	0	0		

We start by dividing the pivot line by the pivot number.

c_j		30	50	0	0	0	b_i	bi /col(pivot)
c_j	v. base	x	y	e_1	e_2	e_3		
0	e_1							1800/2 = 900
0	e_2							400/0 = ∞
50	y	0	1	0	0	1	600	600/1 = 600
z_j								
$c_j - z_j$								

We continue with the identity matrix for the basic variables. We write 1 at the intersection of each variable and 0 elsewhere.

c_j		30	50	0	0	0	b_i	bi /col(pivot)
c_j	v. base	x	y	e_1	e_2	e_3		
0	e_1		0	1	0			1800/2 = 900
0	e_2		0	0	1			400/0 = ∞
50	y	0	1	0	0	1	600	600/1 = 600
z_j								
c_j-z_j								

Calculate the values of the other lines

Eij = Eij – [Aij *Pivot/Pivot line]

c_j		30	50	0	0	0	b_i	bi /col(pivot)
c_j	v. base	x	y	e_1	e_2	e_3		
0	e_1	3	0	1	0	-2	600	1800/2 = 900
0	e_2	1	0	0	1	0	400	400/0 = ∞
50	y	0	1	0	0	1	600	600/1 = 600
z_j		0	50	0	0	50	30,000	
c_j-z_j		30	0	0	0	-50		

2nd iteration:

c_j		30	50	0	0	0	b_i	bi /col(pivot)
c_j	v. base	x	y	e_1	e_2	e_3		
0	e_1	3	0	1	0	-2	600	600/3=200
0	e_2	1	0	0	1	0	400	400/1=400
50	Y	0	1	0	0	1	600	600/0=∞
z_j		0	50	0	0	50	30,000	
c_j-z_j		30	0	0	0	-50		

We start by dividing the pivot line by the pivot number.

c_j		30	50	0	0	0	b_i	bi /col(pivot)
c_j	v. base	x	y	e_1	e_2	e_3		
30	X	1	0	1/3	0	-2/3	200	600/3=200
0	e_2							400/1=400
50	Y							600/0=∞
z_j								
c_j-z_j								

We continue with the identity matrix for the basic variables. We write 1 at the intersection of each variable and 0 elsewhere.

c_j		30	50	0	0	0	b_i	bi /col(pivot)
c_j	v. base	x	y	e_1	e_2	e_3		
30	X	1	0	1/3	0	-2/3	200	600/3=200
0	e_2	0	0		1			400/1=400
50	Y	0	1		0			600/0=∞
z_j								
c_j-z_j								

Calculate the values of the other lines

Eij = Eij – [Aij *Pivot/Pivot line]

c_j		30	50	0	0	0	b_i	bi /col(pivot)
c_j	v. base	x	y	e_1	e_2	e_3		
30	X	1	0	1/3	0	-2/3	200	600/3=200
0	e_2	0	0	1/3	1	2/3	200	400/1=400
50	Y	0	1	0	0	1	600	600/0=∞
z_j		30	50	10	0	30	36,000	
c_j-z_j		0	0	-10	0	-30		

X = 200

Y = 600

Z = 36,000

Noticed:

Calculations are always made from the table of the previous iteration.

We stop when we obtain the optimality criterion. The simplex algorithm stops when:

$$c_j - z_j \leq 0 \text{ for a max problem}$$

- $c_j - z_j \geq 0$ for a min problem

Choice of the variable which enters the base:

When a variable becomes basic, that is, it enters the base, it becomes part of the solution. Noting the reduction in costs in row Z, we decide that the variable in the column in which it is the lowest value (or the highest absolute value) among the negative values enters the base.

Choice of the variable that comes out of the base

Once the incoming variable is determined, we determine that the basic outgoing variable is located in the line whose quotient P 0 / P j is the smallest of the strictly positive (given that this operation will only be done when P j is greater than 0).

Pivot element

The pivot element of the table is marked by the intersection between the column of the incoming variable and the row of the outgoing variable.

Table update

The lines corresponding to the objective function and the titles will remain unchanged in the new table. All other values will have to be calculated as explained below:

In the pivot element row each new element is calculated as:

Pivot Line Element = Old Pivot Line Element / Pivot

In the remaining lines each element is calculated:

*New Row Element = Old Row Element - (Old Row Element in Pivot Column * New Pivot Row Element)*

In this way, we achieve that all elements of the column of the incoming variable are zero except the one in the row of the outgoing variable whose value is 1. (This is analogous to using the Gauss-Jordan method for solving systems of linear equations).

Stop condition

It meets the stopping condition when the indicator line does not contain any negative value among the reduced costs (when the objective is to maximize), that is, there is no possibility of improvement.

If the stopping condition is not met, it is necessary to perform one more iteration of the algorithm, that is, determining the variable that becomes basic and ceases to be so, finding the pivot element, updating the values of the table and checking if the stopping condition is met this time.

It is also possible to determine that the problem is restricted and its solution can always be improved. In this case, it is not necessary to continue the iteration indefinitely and the algorithm can be terminated. This situation occurs when in the column of the incoming variable in the base all the values are negative or zero.

5. Exercises

5.1. Exercise 1

A company manufacturing two products: chairs and tables. You have two types of limited resources: available labor time and available wood. Each chair requires 2 hours of labor and 1 unit of wood, while each table requires 3 hours of labor and 2 units of wood. The profit per chair is 30 DA and the profit per table is 50 DA. You want to maximize your profit by deciding how many chairs and tables to manufacture.

5.2. Exercise 2

A pet food production company. You produce two types of products: dog food and cat food. Each type of kibble requires the use of two main ingredients: meat and cereals. Each kilogram of dog food requires 0.5 kg of meat and 0.4 kg of cereals, while each kilogram of cat food requires 0.3 kg of meat and 0.6 kg of cereals. You have 100 kg of meat and 80 kg of cereals for production. The profit per kilogram of dog food is 4 DA, while the profit per kilogram of cat food is 5 DA. You want to maximize your profit by deciding how many kilograms of each type of kibble to produce.

5.3. Exercise 3

Suppose you are running a juice production business. You produce four types of juice: orange juice, apple juice, grape juice, and peach juice. Each type of juice requires the use of fresh fruit and water. Each liter of orange juice requires 0.5 liters of fresh fruit and 0.2 liters of water, each liter of apple juice requires 0.4 liters of fresh fruit and 0.3 liters of water, each liter of grape juice requires 0.3 liters of fresh fruit and 0.4 liters of water, and each liter of peach juice requires 0.6 liters of fresh fruit and 0.5 liters of water. You have 200 liters of fresh fruit and 100 liters of water for production. The profit per liter of orange juice is 30 DA, per liter of apple juice is 20 DA, per liter of grape juice is 40 DA, and per liter of peach juice is 50 DA. You want to maximize your profit by deciding how many liters of each type of juice to produce.

5.4. Exercise 4

A firm manufactures two products, A and B. The profit per unit for A is 10 DA, for B is 15 DA. The firm has 200 hours of labor per day and 50 units of raw material. Each unit of A requires 4 hours of labor and 2 units of raw material, while each unit of B requires 6 hours of labor and 3 units of raw material. Model this problem as a PL problem.

5.5. Exercise 5

A nutritionist wants to create a balanced diet from two foods, X and Y, while minimizing costs. The cost per unit of X is 20 DA, of Y is 30 DA. The minimum nutritional requirements are 60 units of protein and 50 units of fat. Each unit of X contains 20 units of protein and 10 units of fat, while each unit of Y contains 15 units of protein and 12 units of fat. Model this problem as a LP problem.

5.6. Exercise 6

A company has several factories, warehouses, and stores. The goal is to determine how to distribute products from factories to stores in a way that minimizes transportation costs while meeting production and demand constraints.

- There are Nfactories $(F_1, F_2, \ldots, F_N)$producing different products.
- There are M warehouses$(W_1, W_2, \ldots, W_M)$ to store the products.
- There are P stores $(S_1, S_2, \ldots, S_P)$with demand for products.

The cost of transporting a unit of product from the factory $i$to the warehouse $j$is given by C_{ij}, and the cost of transporting from the warehouse $j$to the store $k$is D_{jk}. Furthermore, each factory has a production limit A_i, each warehouse has a storage capacity B_j, and each store has a demand D_k. Model this problem as a LP problem.

5.7. Exercise 7

Write these linear problems in canonical and standard form.

$$\text{Max } Z = 2X_1 + 4X_2 - 5X_3$$
$$6X_1 - 2X_2 + 12X_3 \leq 26$$
$$X_1 + 9X_2 - 7X_3 \leq 11$$
$$5X_1 + 6X_2 - 3X_3 \leq 8$$
$$X_1 \geq 0$$
$$X_2 \geq 0$$
$$X_3 \leq 0$$

$$\text{Max } Z = 2X_1 + 4X_2 - 5X_3$$
$$6X_1 - 2X_2 + 12X_3 \leq 26$$
$$X_1 + 9X_2 - 7X_3 \leq 11$$
$$5X_1 + 6X_2 - 3X_3 \leq 8$$
$$X_1 \geq 0$$
$$X_2 \geq 0$$
$$X_3 \geq 0$$

$$\text{Max } Z = 2X_1 + 4X_2 - 5X_3$$
$$6X_1 - 2X_2 + 12X_3 \leq 26$$
$$X_1 + 9X_2 - 7X_3 \geq 11$$
$$5X_1 + 6X_2 - 3X_3 \leq 8$$
$$X_1 \geq 0$$
$$X_2 \geq 0$$
$$X_3 \geq 0$$

$$\text{Max } Z = 2X_1 + 4X_2 - 5X_3$$
$$6X_1 - 2X_2 + 12X_3 \leq 26$$
$$X_1 + 9X_2 - 7X_3 = 11$$
$$5X_1 + 6X_2 - 3X_3 \leq 8$$
$$X_1 \geq 0$$
$$X_2 \geq 0$$
$$X_3 \geq 0$$

5.8. Exercise 8

A factory produces two models of machines, one which we will call model A requires 2 kg of raw material and 30 hours of manufacturing and gives a profit of 7 DA. The other which we will call B requires 4 kg of raw material and 15 hours of manufacturing and gives a profit of 6 DA. We have 200 kg of raw material and 1200 hours of work.

What production should we have to obtain maximum profit?

5.9. Exercise 9

For the following linear programs:

1. Graphically represent the feasible set from the constraints.
2. Determine all variables.
3. Identify the optimal solution.

$$\max Z = 2x_1 + 3x_2$$
$$\begin{cases} 0.25x_1 + 0.5x_2 \leq 40 \\ 0.4x_1 + 0.2x_2 \leq 40 \\ 0.8x_2 \leq 40 \\ x_1, x_2 \geq 0 \end{cases}$$

$$\max Z = -2x_1 + 3x_2$$
$$\begin{cases} x_1 \leq 5 \\ 2x_1 - 3x_2 \leq 6 \\ x_1, x_2 \geq 0 \end{cases}$$

$$\max Z = x_1 + 3x_2$$
$$\begin{cases} 2x_1 + 6x_2 \leq 30 \\ x_1 \leq 10 \\ x_2 \leq 4 \\ x_1, x_2 \geq 0 \end{cases}$$

$$\max Z = x_1 + x_2$$
$$\begin{cases} 3x_1 + 2x_2 \leq 40 \\ x_1 \leq 10 \\ x_2 \leq 5 \\ x_1, x_2 \geq 0 \end{cases}$$

$$\max Z = 3x_1 + 2x_2$$
$$\begin{cases} x_1 + 2x_2 \leq 2 \\ 2x_1 + 4x_2 \geq 8 \\ x_1, x_2 \geq 0 \end{cases}$$

$$\max Z = 900x_1 + 1000x_2$$
$$\begin{cases} 11x_1 + 9x_2 \leq 9900 \\ 7x_1 + 12x_2 \leq 8400 \\ 6x_1 + 16x_2 \leq 9600 \\ x_1, x_2 \geq 0 \end{cases}$$

5.10. Exercise 10

A craftsman makes objects A and objects B.

The production of an object A requires 30 DA of raw material and 125 DA of labor.

The production of an object B requires 70 DA of raw material and 75 DA of labor.

The profits made are 54 DA per object A and 45 DA per object B.

Let x be the number of objects A manufactured and y the number of objects B manufactured in one day.

The daily expenditure on raw materials must not exceed 560 DA.

The daily labor expenditure must not exceed 1250 DA.

1. Translate these two hypotheses into inequalities.
2. The plane is reported to an orthonormal reference frame (O: i, j). Graphically represent the set of points M(x,y) whose coordinates verify the constraints.
3. Express the daily profit as a function of x and y.
4. Determining the production of objects A and B would ensure this maximum profit.

5.11. Exercise 11

A company manufactures black and white record players and television sets. 140 workers work in manufacturing. The cost price, parts and labor, of a TV set is 400 DA. It is only 300 DA for a record player. The company's accounting department gives instructions not to exceed the sum of 240,000 DA per week, parts and labor. Each worker works 40 hours per week.

The department heads estimate that it takes 10 hours of labor to manufacture a record player and only 5 hours to manufacture a TV set. The sales department cannot sell more than 480 TV sets and 480 record players per week. The sales prices are such that the company, all expenses paid, makes a profit of 240 DA per TV set and 160 DA per record player.

1. Let x *be* the number of record players and y the number of television sets manufactured per week . Write the constraint system .
2. a: Draw in the plane the polygon of points M (x , y) for which manufacturing is possible.

   b: Determine the profit b per week as a function of x and y

   c: Determine the manufacturing that ensures maximum profit.

5.12. Exercise 12

An oil refinery processes two types of crude to produce finished products with the following yields:

	Brut 1	**Brut 2**
Essence	25%	35%
Diesel	30%	30%
Fuel	45%	35%

Production quotas require the production of at most 825 thousand m3 of gasoline and 750 thousand m3 of gas. oil and 1065 thousand m3 of fuel. The profit margin left by the treatment of crude 1 is 3 thousand euros per thousand m3 and that of crude 2 is 4 thousand euros per thousand m3.

Calculate, using the simplex method, what quantities of each oil must be processed to obtain maximum profit. Give a graphical interpretation.

5.13. Exercise 13

Solve the following programs using the simplex method.

$$\begin{cases} \max(x_1 + 2x_2) \\ x_1 + 3x_2 \leq 21 \\ -x_1 + 3x_2 \leq 18 \\ x_1 - x_2 \leq 5 \\ x_1, x_2 \geq 0 \end{cases} \qquad \begin{cases} \min(x_1 - 3x_2) \\ 3x_1 - 2x_2 \leq 7 \\ -x_1 + 4x_2 \leq 9 \\ -2x_1 + 3x_2 \leq 6 \\ x_1, x_2 \geq 0 \end{cases}$$

6. References

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