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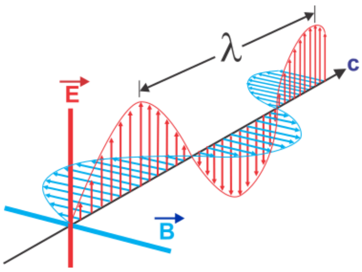
وزارة التعليم العالي و البحث العلمي

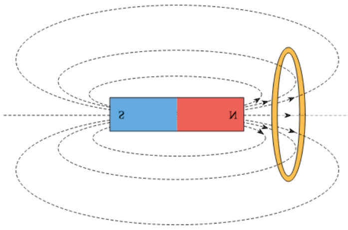
جامعة غليزان

DEMOCRATIC AND POPULAR REPUBLIC OF algeria
Ministry of Higher Education and Scientific Research
University of Relizane

Faculty of Exact, Natural and Life Sciences
Department of Physics

ELCTROMAGNETISM





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Destined for second-year LMD-SM students

Destiné aux étudiants de deuxième année LMD-SM
Option: Materials physics

- University year 2023/2024

CONTENTS

CHAPTER III: ELECTROMAGNETIC WAVES PROPAGATION

Preface

This work is designed for second-year LMD students, option: materials physics. Physics department.

. The content of this course corresponds to the official program of electromagnetism taught in the second year.

This course of electromagnetism is constituted of about four chapters, its objective introduces students to understand the interaction between electricity and magnetism, thus to understand all the physical phenomena, which fall into the field of electromagnetism physics:

- In the first chapter, we introduced some basic concepts: Dirac distribution (definition and properties), vector analysis relations (Gradient, divergence, Rotational and Laplacian) in Cartesian, cylindrical and spherical coordinates.
- The second chapter is devoted to stationary phenomena, for which magnetic and electric effects are decoupled. (magnetostatic electrostatics)) as well as the fundamental laws of electromagnetism which are described by Maxwell's equations in vacuum, dielectric, conductor, magnetic and plasma.
- The third chapter is dedicated to the propagation of electromagnetic waves in vacuum and in different materials: dielectrics, conductors and ionized gases (plasma).
- The fourth chapter deals with the propagation of electromagnetic waves in the different types of waveguide: rectangular waveguide, cylindrical waveguide and coaxial line

CHAPTER I: MATHEMATICAL TOOLS

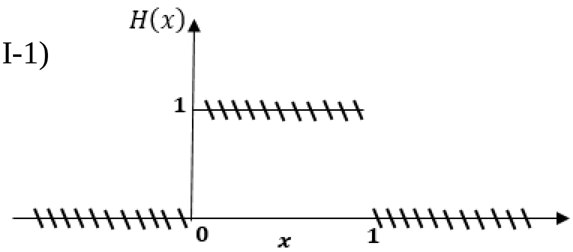
- **Dirac distribution**
  - **Heaviside function H(x)**
  - **Door fonction $\pi(x)$**
  - **Dirac equation formula$\delta(t)$**
- **Local laws**
  - **Divergence of a vector field**
  - **Rotatel $\overrightarrow{rot}\vec{V}$**
  - **Gradient $\overrightarrow{grad}f$**
  - **Laplacian**
  - **Stokes-Ampère theorem**
  - **Green-Ostrogradsky theorem**

In this chapter, we introduce students to the various mathematical tools they will need to study the different phenomena of electromagnetism: the Dirac distribution, divergence, gradient, scalar and vector Laplacian and rotational, as well as the electrostatic field and magnetostatic field.

I-1 Dirac distribution :

I-1.1 Heaviside function H(x): is a function defined from R* to the interval [0,1]

$$\begin{cases} H(x) = 0 \quad si \quad x < 0, x > 1 \\ H(x) = 1 \quad si \quad 0 < x < 1 \end{cases} \qquad \text{(I-1)}$$



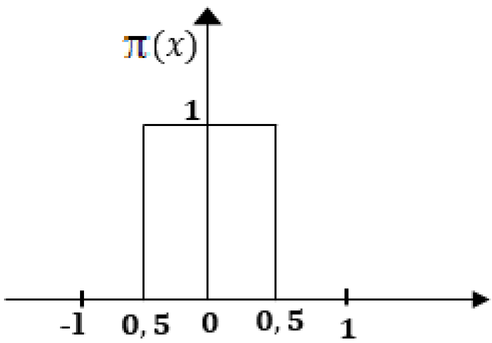
FigureI-1: Heaviside function H(x)

Where the Echelon function is not defined in 0

I-1.2 Door fonction $\boldsymbol{\pi(x)}$:

The gate π(x) is a discontinuous function defined by pieces:

$$\begin{cases} \pi(x) = 0 \quad si \qquad |x| < 0 \\ \pi(x) = 1 \quad si \ non \ x \notin \left]-\frac{1}{2},\frac{1}{2}\right[\end{cases} \qquad \text{(I-2)}$$



FigureI-2: Door fonction $\pi(x)$

This function is not defined on both edges

$x = \pm\frac{1}{2}$, this function is said to be of width 1

(non-zero on an interval of width 1).

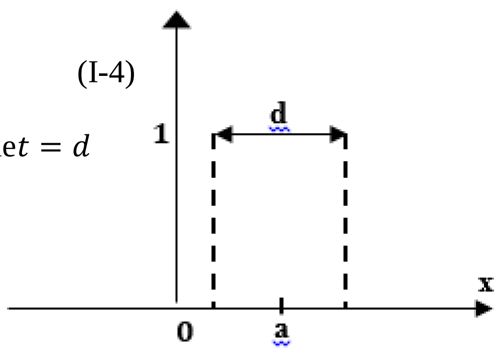
(x) is related to the Heaviside function by :

$$\pi(x) = H\left(x + \frac{1}{2}\right) + H(x - \frac{1}{2}) \qquad \text{(I-3)}$$

In physics, the gate function is used to define signals of finite duration.

$$f(t) = \pi\left(\frac{t-d}{a}\right) \qquad \text{(I-4)}$$

Corresponds to a door of width a and e centered on the value$t = d$



FigureI-3: charge width density sphere

Example 1: charge width density e sphere

of diameter D and uniform loadρ,

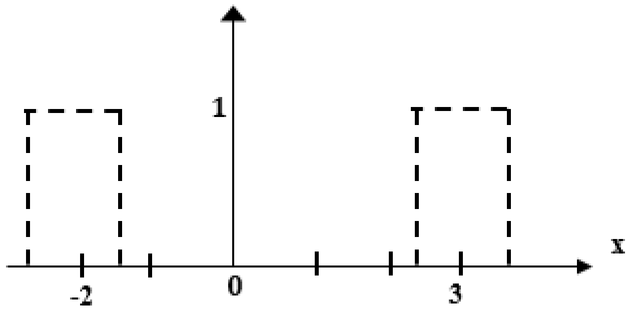
$$\rho(r) = \rho_0 \pi\left(\frac{r}{D}\right)$$

From which: r: the center of the sphere

D : Door width

Door summation:

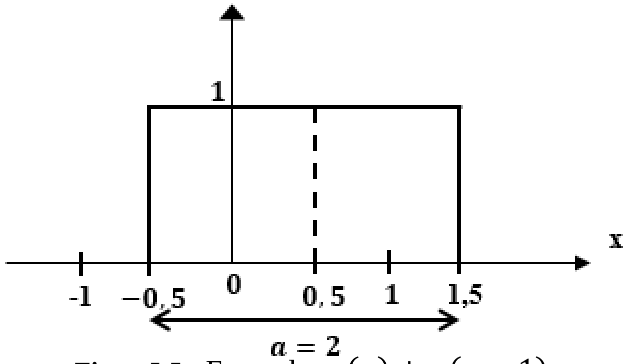
$\pi(x-3)+\pi(x+2)$



FigureI-4: Door summation

Example 2

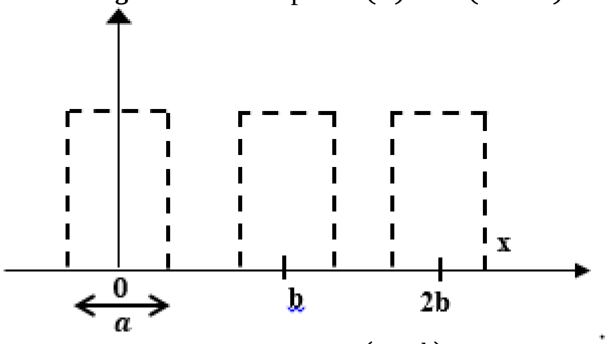
$= \pi\left(\frac{x-0{,}5}{a}\right), \quad a = 2$



FigureI-5: Example: $\pi(x)+\pi(x-1)$

Example 3

$\sum \pi\left(\frac{x+nb}{a}\right), \quad a$: la largeur



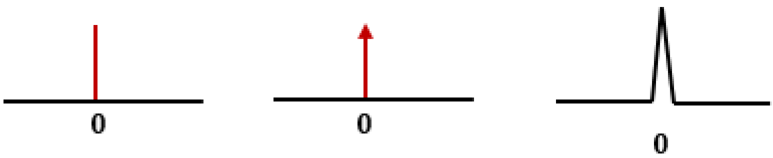
FigureI-6: Example: $\sum \pi\left(\frac{x+nb}{a}\right)$

I-1.3 Dirac equation formula$\boldsymbol{\delta(t)}$:

Consider the function$g_e(x) = \frac{1}{e}\pi\left(\frac{x}{e}\right)$

e: width,

$\frac{1}{e}$: is the height



FigureI-7: Dirac equation formula$\delta(t)$

$\int_{-\infty}^{+\infty} g_e(x)dx = 1$
(I-5)

Example :

$$g_t = \frac{1}{2t} = \begin{cases} \frac{1}{2t} & -t < x < t \\ 0 & if\ no \end{cases}$$

so

$$\int_{-\infty}^{+\infty} g_t(x)dx = \int_{-t}^{t} \frac{1}{2t} dx = area$$

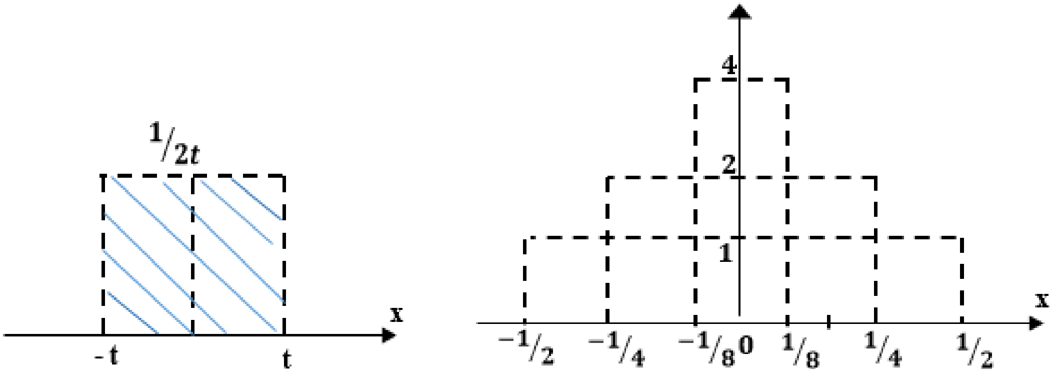
For the rectangle

$$\int_{-\infty}^{+\infty} g_t(x)dx = 2t\frac{1}{2t} = 1$$

$$\int_{-\infty}^{+\infty} \frac{1}{2t} dx = \left[\frac{x}{2t}\right]_{-t}^{t} = \frac{t}{2t} - \frac{t}{2t} = 1$$

$$\lim_{t\to 0} g_t = \delta(t),$$

$$\lim_{t\to 0} \int_{-\infty}^{+\infty} g_t\,(x)dx = 1 \qquad \text{(I-6)}$$



FigureI-8: Dirac equation example $g_t = \frac{1}{2t}$

When x→0 the function g_t (x)has a width

that tends to 0 and a height that tends to infinity

When x→0 the function g_t (x)has a width

which tends towards 0 and a height which tends

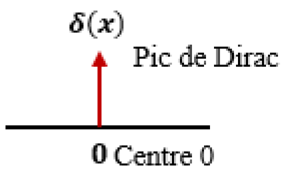
towards infinity,

but its integral is always equal to 1

in which case we call the distribution of Dirac $\delta(t)$.

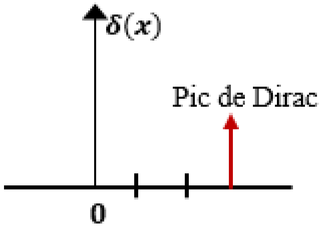
$$\delta(t) = \lim_{t\to 0} \frac{1}{t}\pi\left(\frac{x}{t}\right) \qquad \text{(I-7)}$$

Note: the function δ has width O, infinite height and integral 1



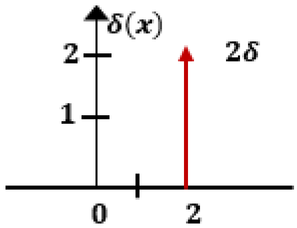
FigureI-9: wwwwwww

For $\delta(x-3)$



FigureI-9: Dirac equation example $\delta(x-3)$

For $2\delta(x-2)$



FigureI-10: Dirac equation example $2\delta(x-2)$

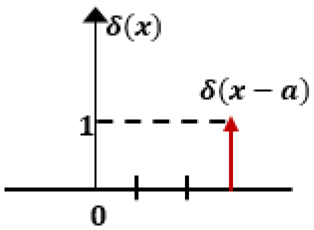
Note:

for $N\delta(x) \Rightarrow \int N\delta(x) = N.1$ N : height

But if $N = 0 \Rightarrow \delta(x) = 0$

Change of origin

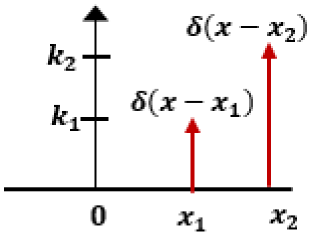
$\delta(x-a)$ vaut O par tout sauf en $x = a$



FigureI-11: Dirac equation $\delta(x-a)$

Somation :

$K_1\delta(x-x_1) + K_2\delta(x-x_2)$



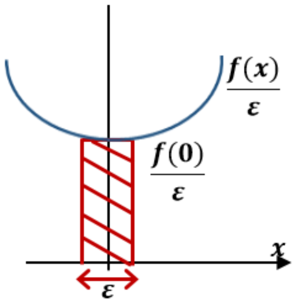
FigureI-12: Somation : $K_1\delta(x-x_1) + K_2\delta(x-x_2)$

I-1.4 Properties :

$\int f(x)g_e dx$ equals the area of the rectangle of width e and height $\frac{f(0)}{e}$ this area sells $f(O)$

$f(x).\delta(x) = 0$ except en $x = 0$

$f(x).\delta(x) = f(O)\delta(x)$

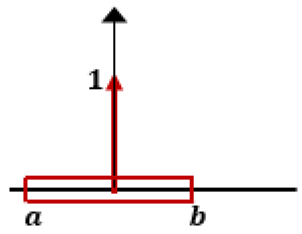


FigureI-13: the area of the rectangle $\int f(x)g_e dx$

General case :

$f(x).\delta(x-a) = f(a)\delta(x-a)$

$\int_a^b \delta(x)dx = 1$



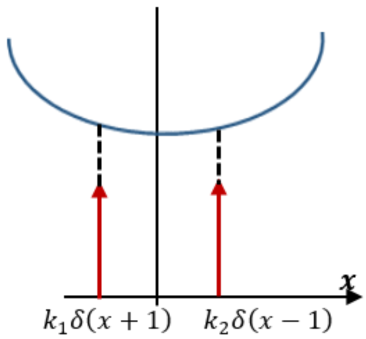
FigureI-14: the area of $\int_a^b \delta(x)dx = 1$

Example

Or $f(x) = x^2 - 1$, $f(x) = 0 \;\; si\; x = 1\; ou\; x = -1$

$$\delta(f(x)) \begin{cases} 0 & si \;\; x \neq \pm 1 \\ \neq 0 & si \;\; x = \pm 1 \end{cases}$$

$$\delta\big(f(x)\big) = k_1\delta(x+1) + k_2\delta(x-1)$$



FigureI-15: example $\delta\big(f(x)\big) = k_1\delta(x+1) + k_2\delta(x-1)$

Note :

$f(x).\delta(x)$ is zero everywhere except in $x = 0$

So

$$\int f(x).\delta(x) = \int f(0).\delta(x) \quad \text{(I-8)}$$

In general cases : $f(x).\delta(x-a) = f(a).\delta(x-a)$ (I-9)

Fundamental property :

Let f be a function of $\mathbb{R} \to \mathbb{R}$

It is assumed that $I = \int_{-\infty}^{+\infty} f(x).\delta(x)dx$

To reconcile the integral I, we use the g_e

$$\delta(x) = \lim_{e\to 0} g_e(x)$$

$$I = \int_{-\infty}^{+\infty} f(x) \lim_{e\to 0} g_e(x)\, dx$$

$$I = \lim_{e\to 0} \int_{-\infty}^{+\infty} f(x) g_e(x) dx$$

$$\int_{-\infty}^{+\infty} f(x).\delta(x)dx = f(0)$$

$$f(x).\delta(x) = f(0).\delta(x)$$

In general :

$$f(x).\delta(x-a) = f(0).\delta(x-a)$$

Scale change :

$$\delta(ax) = \frac{1}{|a|}\delta(x) \qquad \text{(I-10)}$$

Graphically, when $e \to 0$, we interpret the integral $\int_{-\infty}^{+\infty} f(x)g_e(x)dx$ is the area of the rectangle of width *e and height* $\frac{f(0)}{e}$, this area is $f(0)$.

I-2 Local laws :

I-2.1 Divergence of a vector field:

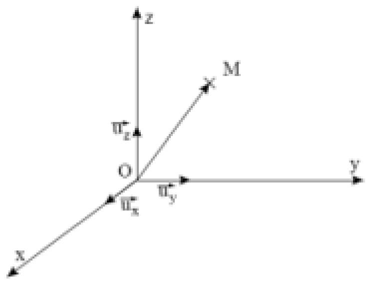
The divergence is a scalar, which takes as argument a vector,

Divergence in Cartesian coordinates :

$$div = \vec{\nabla}\bullet, \quad \vec{\nabla} = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$$

Let a vector $\vec{A} = \overrightarrow{A_x} + \overrightarrow{A_y} + \overrightarrow{A_z}$

$$div\vec{A} = \vec{\nabla}\bullet\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad \text{(I-11)}$$



FigureI-16: Cartesian coordinates

Example 1 :

Let a vector $\vec{A} = 3xy\vec{\imath} + 2xz\vec{\jmath} + zy^2\vec{k}$

$$div\vec{A} = \vec{\nabla}\bullet\vec{A} = \frac{\partial(3xy)}{\partial x} + \frac{\partial(2xz)}{\partial y} + \frac{\partial(zy^2)}{\partial z}$$

$$div\vec{A} = \vec{\nabla}\bullet\vec{A} = 3y + y^2$$

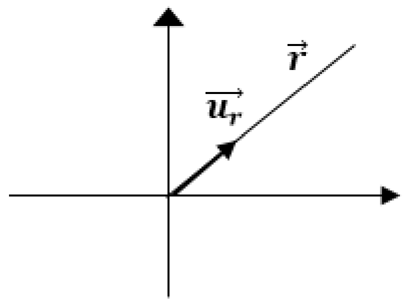
Example 2 : calculated

$$div\vec{V}, \quad \vec{V} = \frac{\overrightarrow{u_r}}{r^2}$$

$\overrightarrow{u_r}$: unit vector $\overrightarrow{u_r} = \frac{\vec{r}}{r}$, $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$,

$$\|\vec{r}\| = \sqrt{x^2 + y^2 + z^2}$$

$$\vec{V} = \frac{\overrightarrow{u_r}}{r^2}, \Rightarrow \quad \vec{V}\begin{pmatrix} \frac{x}{(x^2+y^2+z^2)^{3/2}} \\ \frac{y}{(x^2+y^2+z^2)^{3/2}} \\ \frac{z}{(x^2+y^2+z^2)^{3/2}} \end{pmatrix}$$



FigureI-17: Unit vector $\overrightarrow{u_r}$

Calculation $div\vec{V} = \vec{\nabla} \bullet \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

$$\begin{cases} \frac{\partial V_x}{\partial x} = r^{-3} - 3x^2 r^{-5} \\ \frac{\partial V_y}{\partial y} = r^{-3} - 3y^2 r^{-5} \\ \frac{\partial V_z}{\partial z} = r^{-3} - 3z^2 r^{-5} \end{cases} \Rightarrow \quad div\vec{V} = 3r^{-3} - 3(x^2 + y^2 + z^2)r^{-5} = 3r^{-3} - 3r^{-3} = 0$$

On a $\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{\overrightarrow{u_r}}{r^2} \quad \Rightarrow \quad \vec{E} \propto \frac{\overrightarrow{u_r}}{r^2} \quad \Rightarrow \quad div\vec{E} = 0$: for a point load

For an electrostatic field $div\vec{E} = 0$

General case (Gauss equation):

$$div\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(I-12)}$$

Where.

ρ : is the volumetric load.

Note :

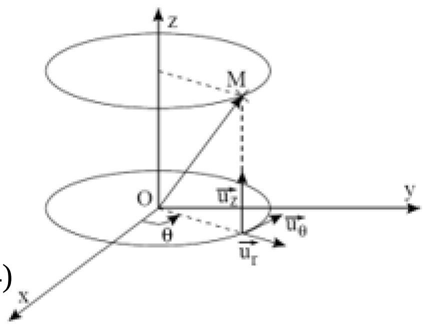
$$div\vec{V} = \vec{\nabla} \bullet \vec{V} \Rightarrow div\vec{V} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \bullet \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \qquad \text{(I-13)}$$

- - Divergence in cylindrical coordinates:

Vector $\vec{V}$ in cylindrical coordinates : $\vec{V}\begin{pmatrix} V_r \\ V_\theta \\ V_z \end{pmatrix}$

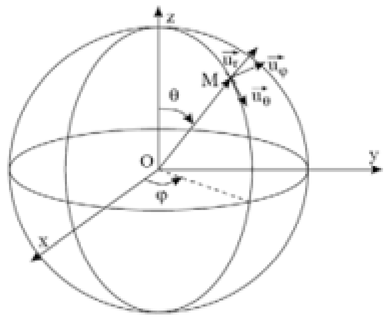
$$V = V_r\vec{u}_r + V_\theta\vec{u}_\theta + V_z\vec{u}_z$$

$$div\vec{V} = \vec{\nabla} \bullet \vec{V} = \frac{1}{r}\frac{\partial rV_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} \qquad \text{(I-14)}$$



FigureI-18: Cylindrical coordinates

- - Divergence in spherical coordinates :



Vector $\vec{V}$in spherical coordinates: $\vec{V}\begin{pmatrix} V_r \\ V_\emptyset \\ V_\theta \end{pmatrix}$

$$V = V_r\vec{u}_r + V_\emptyset\vec{u}_\emptyset + V_\theta\vec{u}_\theta$$

$$div\vec{V} = \vec{\nabla}\bullet\vec{V} = \frac{1}{r^2}\frac{\partial r^2 V_r}{\partial r} + \frac{1}{rsin(\emptyset)}\frac{\partial \sin(\emptyset)V_\theta}{\partial \emptyset} + \frac{\partial V_z}{rsin(\emptyset)\partial z} \qquad \text{(I-15)}$$

FigureI-19 : Spherical coordinates

I-2.2 Rotatel $\overrightarrow{rot}\vec{V}$:

The rotational is a vector which takes as argument a vector:

The rotational of $\vec{V}$ is the vector product of nabla $\vec{\nabla}$ and the vector $\vec{V}$.

- Rotational in Cartesian coordinates :

$$\overrightarrow{rot}\vec{V} = \vec{\nabla}\Lambda\vec{V} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \Lambda \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{vmatrix} \vec{I} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \qquad \text{(I-16)}$$

$$\overrightarrow{rot}\vec{V} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_y & V_z \end{vmatrix}\vec{\imath} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_x & V_z \end{vmatrix}\vec{\jmath} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_x & V_y \end{vmatrix}\vec{k} \qquad \text{(I-17)}$$

Example1 : soit $\vec{V} = (xy + z)\vec{\imath} + (z^3 - 2y)\vec{\jmath} + xyz\vec{k}$

$$\overrightarrow{rot}\vec{V} = (xz - 3z^2)\vec{\imath} - (x - yz)\vec{\jmath} - x\vec{k}$$

Example 2 :

$$\overrightarrow{rot}\vec{V}, \quad \vec{V} = \frac{\vec{u_r}}{r^2}, \quad \overrightarrow{rot}\frac{\vec{u_r}}{r^2} = 0 \qquad \Rightarrow \quad \overrightarrow{rot}\vec{E} = \vec{0}$$

In electrostatics : $\overrightarrow{rot}\vec{E} = \vec{0}$

- Rotational in cylindrical coordinates :

$$\overrightarrow{rot}\vec{V} = \begin{pmatrix} \frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z} \\ \frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} \\ \frac{1}{r}(\frac{\partial r V_\theta}{\partial r} - \frac{\partial V_r}{\partial \theta}) \end{pmatrix} \qquad \text{(I-18)}$$

- Rotational in spherical coordinates :

$$\overrightarrow{rot}\vec{V} = \begin{pmatrix} \frac{1}{rsin(\emptyset)}\left(\frac{\partial \sin(\emptyset)V_\theta}{\partial \emptyset} - \frac{\partial V_\theta}{\partial \theta}\right) \\ \frac{1}{rsin(\emptyset)}\left(\frac{\partial V_r}{\partial \theta} - \frac{\partial (rsi(\emptyset)V_\theta}{\partial r}\right) \\ \frac{1}{r}\left(\frac{\partial r V_\emptyset}{\partial r} - \frac{\partial V_r}{\partial \emptyset}\right) \end{pmatrix} \qquad \text{(I-19)}$$

I-2-.3 Gradient $\overrightarrow{grad} f$:

The gradient is a vector that takes a scalar f as its argument (unlike divergence). :

Let *f* be a scalar function

- Gradient in Cartesian coordinates:

$$\overrightarrow{grad} f = \vec{\nabla}.f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k} \qquad \text{(I-20)}$$

- Gradient in cylindrical coordinates :

$$\overrightarrow{grad} f = \vec{\nabla}.f = \begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r}\frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial z} \end{pmatrix} \qquad \text{(I-21)}$$

- Gradient in spherical coordinates :

$$\overrightarrow{grad} f = \begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r}\frac{\partial f}{\partial \emptyset} \\ \frac{1}{rsin(\emptyset)}\frac{\partial f}{\partial \theta} \end{pmatrix} \qquad \text{(I-22)}$$

Example :

Soit $f = xy + 2x^2 z \quad \overrightarrow{grad} f = \vec{\nabla}.f = (y - 2xz)\vec{\imath} - (x)\vec{\jmath} - 2x^2\vec{k}$

I-2.4 Laplacian : Δ

I-2.4.1 Scalar Laplacian:

The scalar Laplacian is a scalar, which takes as argument a scalar.

- Scalar Laplacian in Cartesian coordinates :

$$\Delta = \vec{\nabla} \bullet \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\Delta f = \vec{\nabla} \bullet \vec{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \qquad \text{(I-23)}$$

- Scalar Laplacian in cylindrical coordinates :

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \qquad \text{(I-24)}$$

- Scalar Laplacian in spherical coordinates:

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin(\emptyset)}\frac{\partial}{\partial \emptyset}\left(\sin(\emptyset)\frac{\partial f}{\partial \emptyset}\right) + \frac{1}{r^2 sin^2(\emptyset)}\frac{\partial^2 f}{\partial \theta^2} \qquad \text{(I-25)}$$

I-2.4- Poisson's equation:

$$\Delta V = div\overrightarrow{grad}V\ ,\quad \vec{E} = \overrightarrow{grad}V \quad \Longrightarrow \Delta V = div\vec{E}$$

$$\Delta V + div\vec{E} = 0 \qquad \Rightarrow \qquad \Delta V + \frac{\rho}{\varepsilon_0} = 0 \qquad \text{(I-26)}$$

V: $potential$, $\vec{E}$: electric field.

I-2.4 .2 Laplacian vector $\vec{\Delta}$:

The Laplacian vector is a vector that takes a vector as argument. .

- Vector Laplacian in Cartesian coordinates :

$$\vec{\Delta} = \vec{\nabla}\Lambda\vec{\nabla} = \begin{vmatrix} \vec{I} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \qquad \text{(I-27)}$$

$$\vec{\Delta}\vec{V} = \vec{\nabla}\Lambda\vec{\nabla}V = \begin{vmatrix} \vec{I} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V_x}{\partial x} & \frac{\partial V_y}{\partial y} & \frac{\partial V_z}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \\ \frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2} \\ \frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} \end{vmatrix} \qquad \text{(I-28)}$$

- Laplacian vector in cylindrical coordinates :

$$\vec{\Delta}\vec{V} = \begin{cases} \left(\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2} + \frac{\partial^2 V_r}{\partial z^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} - \frac{V_r}{r^2}\right)\vec{u}_r \\ \left(\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{\partial^2 V_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r^2}\right)\vec{u}_\theta \\ \left(\frac{\partial^2 V_z}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2} + \frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r}\right)\vec{u}_z \end{cases} \qquad \text{(I-29)}$$

$$\vec{\Delta}\vec{V} = \begin{pmatrix} \frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2} + \frac{\partial^2 V_r}{\partial z^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} - \frac{V_r}{r^2} \\ \frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{\partial^2 V_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r^2} \\ \frac{\partial^2 V_z}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2} + \frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r} \end{pmatrix} \qquad \text{(I-30)}$$

- Laplacian vector in spherical coordinates :

$$\vec{\Delta}\vec{V} = \begin{cases} \left(\frac{1}{r}\frac{\partial^2 rV_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2} + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 V_r}{\partial \emptyset^2} + \frac{\cot(\theta)}{r^2}\frac{\partial V_r}{\partial \theta} - \frac{2V_r}{r^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} - \frac{2\cot(\theta)V_\theta}{r^2} - \frac{2}{r^2\sin(\theta)}\frac{\partial V_\emptyset}{\partial \emptyset}\right)\vec{u}_r \\ \left(\frac{2}{r^2}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r^2\sin^2(\theta)} + \frac{1}{r}\frac{\partial^2 (rV_\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 V_\theta}{\partial \emptyset^2} + \frac{\cot(\theta)}{r^2}\frac{\partial^2 V_\theta}{\partial \theta} - \frac{2}{r^2}\frac{\cot(\theta)}{\sin(\theta)}\frac{\partial V_\emptyset}{\partial \emptyset}\right)\vec{u}_\theta \\ \left(\frac{2}{r^2\sin(\theta)}\frac{\partial V_r}{\partial \emptyset} + \frac{2}{r^2}\frac{\cot(\theta)}{\sin(\theta)}\frac{\partial V_\emptyset}{\partial \emptyset} + \frac{1}{r}\frac{\partial^2 (rV_\emptyset)}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_\emptyset}{\partial \theta^2} + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 V_\emptyset}{\partial \emptyset^2} + \frac{\cot(\theta)}{r^2}\frac{\partial^2 V_\emptyset}{\partial \theta} - \frac{V_\emptyset}{r^2 sin^2(\theta)}\right)\vec{u}_\emptyset \end{cases}$$

(I-31)

I-2.5 Properties :

Let f be a scalar field and let $\vec{A}$, $\vec{B}$ be vector fields. :

- div(f$\vec{A}$)=fdiv$\vec{A}$ + $\vec{A}\overrightarrow{grad}$(f) (I-32)
- $\overrightarrow{rot}$(f$\vec{A}$)= $\overrightarrow{grad}$(f)∧$\vec{A}$ + f$\overrightarrow{rot}$($\vec{A}$) (I-33)
- $\overrightarrow{rot}$($\overrightarrow{grad}$ (f))=0 et div(rot($\vec{A}$))=0 (I-34)
- $\overrightarrow{rot}$($\overrightarrow{rot}$($\vec{A}$))= $\overrightarrow{grad}$(div($\vec{A}$)) - Δ$\vec{A}$ (I-35)
- div($\vec{A}$∧$\vec{B}$)=$\vec{B}$•$\overrightarrow{rot}$($\vec{A}$)- $\vec{A}$•$\overrightarrow{rot}$($\vec{B}$) (I-36)

I-2.6 Stokes-Ampère theorem:

The circulation of a vector field $\vec{A}$ along a closed contour:

$$\oint_c \vec{A} \bullet \overrightarrow{dl} = \iint_s \overrightarrow{rotA} \bullet \overrightarrow{ds} \qquad \text{(I-37)}$$

$$\mathcal{C} = \oint_c \vec{E} \bullet \overrightarrow{dl} = \iint_s \overrightarrow{rotE} \bullet \overrightarrow{ds} = 0 \qquad \text{(I-38)}$$

The circulation of an electric field E is zero in a contour $\mathcal{C}$

I-2.7 Green-Ostrogradsky theorem :

$$\iiint div\vec{A} = \oiint_{\Sigma} \vec{A} \bullet \overrightarrow{ds} \qquad \text{(I-39)}$$

$\vec{A}$: vector field
∑: closed surface
According to Green-Ostrogradsky's theorem :

$$\iiint div\vec{E} = \oiint_{\Sigma} \vec{E} \bullet \overrightarrow{ds} \qquad \text{(I-40)}$$

We have $div\vec{E} = \frac{\rho}{\varepsilon_0}$

Internal load :

$$Q_{int} = \iiint \rho dv \qquad \text{(I-41)}$$

$$Q_{int} = \varepsilon_0 \oiint \vec{E} \bullet \overrightarrow{ds} \qquad \text{(I-42)}$$

$$\begin{cases} \iiint div\vec{E}\, dv = \oiint_{\Sigma} \vec{E} \bullet \overrightarrow{ds} \\ \frac{Q}{\varepsilon_0} = \oiint \vec{E} \bullet \overrightarrow{ds} \\ Q_{int} = \iiint \rho dv \end{cases} \Rightarrow \iiint div\vec{E} dv = \frac{1}{\varepsilon_0} \iiint \rho dv \Rightarrow div\vec{E} = \frac{\rho}{\varepsilon_0}$$

CHAPTER II : MAXWELL'S EQUATIONS

- **Electrostatics**
- **Magnetostatic field**
- **Law of charge conservation**
- **Maxwell's equations in vacuum**
- **Maxwell's equations in media:**
  - **Conductors**
  - **Magnetic medium**
  - **Plasma (ionized gas)**

Maxwell's equations mathematically model the interactions between electric charges, electric currents, electric fields and magnetic fields.. Maxwell's equations are fundamental laws of physics, and there are four of them (Maxwell-Gauss, Maxwell-flux, Maxwell-Faraday, Maxwell-Ampere). These equations describe electrical, magnetic and luminous phenomena.

II-1 Electrostatics :

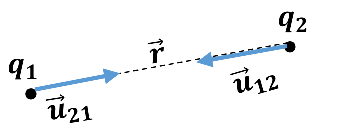
II-1.1 Colombian force (Coulomb's law) :

Or two charges a$q_1$and q_2

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\vec{u}_{12} \qquad \text{(II-1)}$$

$$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\vec{u}_{21} \qquad \text{(II-2)}$$

$$\|\vec{F}_{21}\| = \|\vec{F}_{12}\| \qquad \text{(II-3)}$$



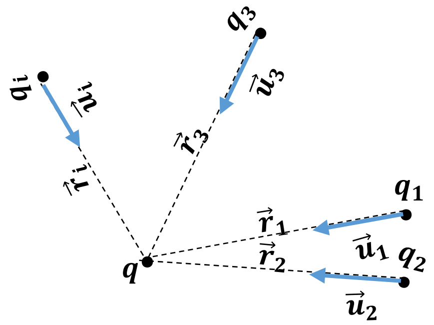
FigureII-1 : two charges aq_1,q_2

For multiple charges nq

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots \ldots \ldots \vec{F}_n$$

$$\vec{F} = \frac{q}{4\pi\varepsilon_0}\sum_{i=1}^{i=n}\frac{q_i}{r_i}\overrightarrow{u_i} \qquad \text{(II-4)}$$

u_r: unit vector $\overrightarrow{u_r} = \frac{\overrightarrow{OM}}{OM}$, $\overrightarrow{OM} = \vec{r}$



FigureII-2: multiple charges nq

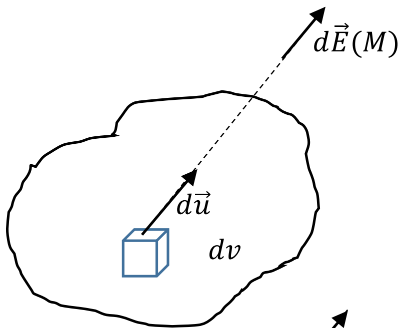
II-1.2 Electrostatic field:

An electrostatic field given by :

$$\vec{E} = K\frac{q}{r^2}\overrightarrow{u_r} \quad , \qquad \text{(II-5)}$$

- for a volume distribution: $dq = \rho dv$
  ρ: volume density
  $d\vec{E} = \frac{dq}{4\pi\varepsilon_0 r^2}\overrightarrow{u_r}$

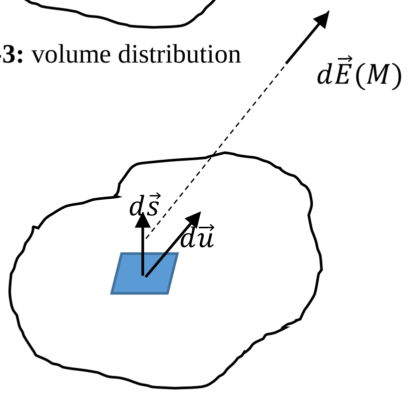
$$\vec{E}(M) = \frac{1}{4\pi\varepsilon_0}\iiint\frac{\rho}{r^2}dv\overrightarrow{u_r} \qquad \text{(II-6)}$$



FigureII-3: volume distribution

- For surface distribution: $dq = \sigma ds$
  σ: surface density

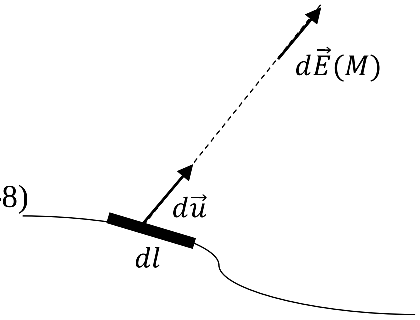
$$\vec{E}(M) = \frac{1}{4\pi\varepsilon_0}\iiint\frac{\sigma}{r^2}d\vec{s}\bullet\overrightarrow{u_r} \qquad \text{(II-7)}$$



FigureII-4: surface distribution

- For a linear distribution : $dq = \lambda dl$
  λ: linear distribution

$$\vec{E}(M) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\lambda}{r^2} d\vec{l} \bullet \overrightarrow{u_r} \quad \text{(II-8)}$$



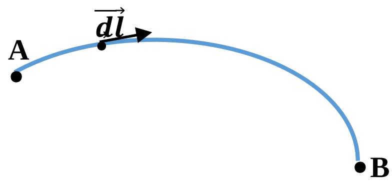
FigureII-5: linear distribution

II-1.3 Electrostatic field circulation:

$$\mathcal{C} = \vec{E} \bullet \overrightarrow{dl}$$

Field circulation $\vec{E}$ between A and B along (c)

$$\mathcal{C} = \int_A^B \vec{E} \bullet \overrightarrow{dl} \quad \text{(II-9)}$$



FigureII-6: Electrostatic field circulation

On a closed curve :

$$\mathcal{C} = \oint \vec{E} \bullet \overrightarrow{dl} = 0 \quad \text{(II-10)}$$

II-1.4 Electrostatic potential:

The potential created by a charge q at a distance r :

$$V(r) = \frac{q}{4\pi\varepsilon_0} \quad \text{(II-11)}$$

$$\Delta V = -\int \vec{E} \bullet \overrightarrow{dl} \quad \text{(II-12)}$$

- For a volume distribution : $dq = \rho dv$

$$V = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho}{r} dv \quad \text{(II-13)}$$

- For surface distribution : $dq = \sigma ds$

$$V = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\sigma}{r} ds \quad \text{(II-14)}$$

- For a linear distribution : $dq = \lambda dl$

$$V = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\lambda}{r} dl \quad \text{(II-15)}$$

II-4.5 Gauss's theorem:

$\emptyset$: electrostatic field flow through a closed surface (S)

$$\begin{cases} \emptyset = \oiint \vec{E} \bullet \overrightarrow{ds} \\ \emptyset = \frac{Q_{int}}{\varepsilon_0} \end{cases} \Rightarrow \oiint \vec{E} \bullet \overrightarrow{ds} = \frac{Q_{int}}{\varepsilon_0} \quad \text{(II-16)}$$

II-1.6 Passage relations:

the transition relation for an electrostatic field :

$$\overrightarrow{E_1} - \overrightarrow{E_2} = \frac{\sigma}{\varepsilon_0}\vec{n}_{1\to 2} \qquad \text{(II-17)}$$

$\vec{n}_{1\to 2}$: unit vector from 1 to 2

$$\vec{E}_1 = \vec{E}_{1n} + \vec{E}_{1t} \qquad \text{(II-18)}$$

$$\vec{E}_2 = \vec{E}_{2n} + \vec{E}_{2t} \qquad \text{(II-19)}$$

$\vec{E}_{1n}, \vec{E}_{2n}$: Normal components.

$\vec{E}_{1t}, \vec{E}_{2t}$: Tangential components.

$$\vec{E}_2 - \vec{E}_1 = \vec{E}_{2n} - \vec{E}_{1n} = \frac{\sigma}{\varepsilon_0}\vec{n}_{1\to 2}$$

$$\vec{E}_{2t} - \vec{E}_{1t} = 0 \qquad \text{(II-20)}$$

II-2 Magnetostatic field:

II-2.1 Magnetostatic field created by a current:

II-2.1.a Lorentz's law:

$$\vec{F} = q\vec{\vartheta}\Lambda\vec{B} \qquad \text{(II-21)}$$

$\vec{F}$: Lorentz force

$\vec{\vartheta}$: The velocity of the charge

$\vec{B}$: Magnetic field

Magnetic field unit is Tesla (T)

II-2.1.b Biot Savart Law:

$d\vec{c} = Id\vec{l}$: elementary current vector

$d\vec{B}$: Elementary magnetic field

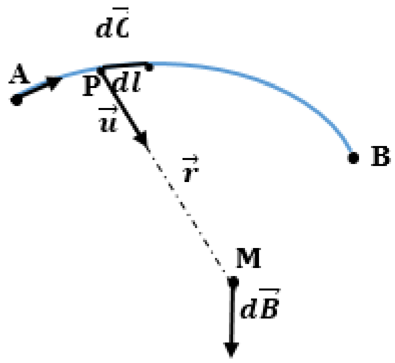
$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}\,\Lambda\,\overrightarrow{PM}}{PM^3} = \frac{\mu_0}{4\pi}\frac{d\vec{c}}{r^2}\,\Lambda\overrightarrow{u} \qquad \text{(II-22)}$$

$\left(d\vec{B}, d\vec{l}, \vec{r}\right)$: is a direct triad.

Total magnetic field :

$$\vec{B}(M) = \int_A^B d\vec{B}(M) \qquad \text{(II-23)}$$

$$\vec{B}(M) = \int \frac{\mu_0}{4\pi} d\vec{C}\,\Lambda\frac{\overrightarrow{PM}}{PM^3} \qquad \text{(II-24)}$$



FigureII-7: magnetic field

II-2.1.c Current element:

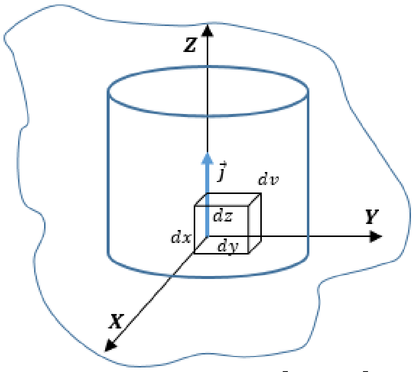
- Volume distribution :

Volume current element : $dv = dx.dy.dz$

Elementary current vector :

$$d\vec{c} = Id\vec{l} = \vec{j} \bullet \vec{s} \bullet d\vec{l} = \vec{j}.dv \quad \text{(II-25)}$$

$$\vec{B}(M) = \frac{\mu_0}{4\pi} \int \vec{j} dv \wedge \frac{\overrightarrow{PM}}{PM^3} \quad \text{(II-26)}$$



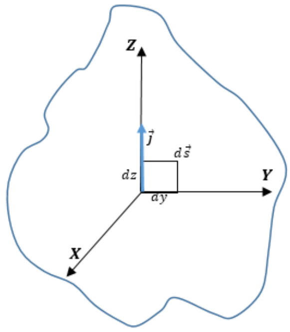
FigureII-9: Volume distribution

- Surface distribution :

Surface current element :$dx \to 0$

$$d\vec{c} = Id\vec{l} = \vec{j} \bullet ds \quad \text{(II-27)}$$

$$\vec{B}(M) = \frac{\mu_0}{4\pi} \int \vec{J_s} ds \wedge \frac{\overrightarrow{PM}}{PM^3} \quad \text{(II-28)}$$



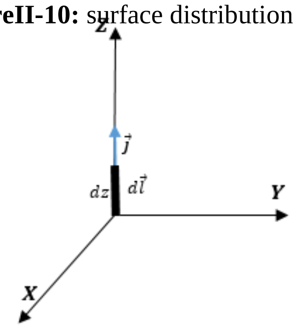
FigureII-10: surface distribution

- Linear distribution:

Linear current element :$dx \to 0 , dy \to 0$

$$d\vec{c} = Id\vec{l} \quad \text{(II-29)}$$

$$\vec{B}(M) = \frac{\mu_0}{4\pi} \int Id\vec{l} \wedge \frac{\overrightarrow{PM}}{PM^3} \quad \text{(II-30)}$$



FigureII-11: linear distribution

II-2.2 Magnetic flux:

The magnetic field flux through a surface (S) is given by :

$$\emptyset = \int \vec{B} \bullet \overrightarrow{ds} \quad \text{(II-31)}$$

Flow unit $\emptyset$ (Weber) $1\text{Wb} = \text{T.m}^2$

Magnetic flux through any closed surface is zero

$$\oiint \vec{B} \bullet \overrightarrow{ds} = 0 \quad \text{(II-32)}$$

According to Ostrogradski's theorem :

$$\oiint \vec{B} \bullet \overrightarrow{ds} = \iiint divB\, dv = 0$$

$$\Rightarrow divB = 0 \quad \text{(II-33)}$$

II-2.3 Ampère's theorem:

The circulation of the magnetic field along a contour (C) is given as follows:

$$\mathcal{C} = \int_C \vec{B} \bullet d\vec{l} = \mu_0 I \quad \text{(II-34)}$$

$\mathcal{C}$: the circulation of a magnetic field :

$$\mathcal{C} = \int_C \vec{B} \bullet d\vec{l} \qquad \text{(II-35)}$$

Another Ampère theorem formula:
According to Stokes' theorem :

$$\int_C \vec{B} \bullet d\vec{l} = \int_\Sigma \overrightarrow{rot}\,\vec{B} \bullet d\vec{s}$$

$$\begin{cases} \mathcal{C} = \int_C \vec{B} \bullet d\vec{l} = \int_\Sigma \overrightarrow{rot}\,\vec{B} \bullet d\vec{s} \\ \int_C \vec{B} \bullet d\vec{l} = \mu_0 I \\ I = \iint \vec{j} \bullet d\vec{s} \end{cases}$$

$$\Rightarrow \overrightarrow{rot}\,\vec{B} = \mu_0 \vec{j} \qquad \text{(II-36)}$$

II-2.4 Potential vector $\vec{A}$:

The potential vector $\vec{A}$ is defined by :

$$\vec{B} = \overrightarrow{rot}\,\vec{A} \quad , \qquad \text{(II-37)}$$

$$div(\overrightarrow{rot}\,\vec{A}) = 0 \qquad \text{(II-38)}$$

$div\vec{B} = 0$: there exists a vector $\vec{A}$, such that $\vec{B} = \overrightarrow{rot}\,\vec{A}$

Poisson equation:

$$\Delta V + \frac{\rho}{\varepsilon_0} = 0$$

By analogy :

$$\Delta\vec{A} + \mu_0 \vec{j} = 0 \qquad \text{(II-39)}$$

V: potential
$\vec{A}$: vector potential
ρ: charge density
$\vec{j}$: current density

The solution to Poisson's equation :

$$\vec{A}(M) = \frac{\mu_0}{4\pi} \iiint_v \frac{\vec{j}(P)}{PM} dv \qquad \text{(II-40)}$$

II-3 Law of charge conservation:

Given a volume v containing a global charge Q at time t

The global charge Q s'écrite :

$$Q = \iiint_v \rho(\vec{r}, t) dv \qquad \text{(II-41)}$$

$\rho(\vec{r}, t)$: Volumetric charge density.

$$dQ = \iiint \frac{\partial\rho}{\partial t} dt dv \qquad \text{(II-42)}$$

The total charge lost through flow across the surface f (s) :

$$dQ' = \iint \rho\vec{\vartheta}\, d\vec{s} dt \qquad \text{(II-43)}$$

Since the increase in charge q dQ must be offset by the loss of charge $dQ'$to satisfy the principle of charge conservation.

$$dQ + dQ' = 0 \qquad \text{(II-44)}$$

$$\iiint \frac{\partial \rho}{\partial t} dtdv + \iint \rho\vec{\vartheta}\, dtd\vec{s} = 0 \qquad \text{(II-45)}$$

We have : $\vec{j} = \rho\vec{\vartheta}$

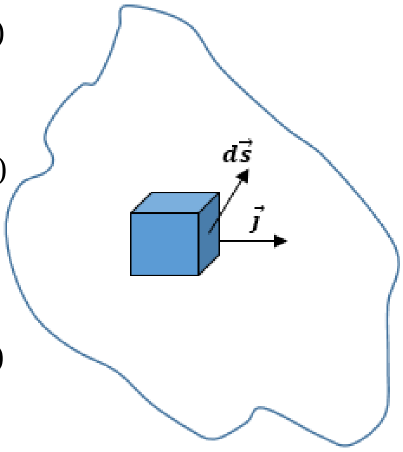
$$\Rightarrow \iiint \frac{\partial \rho}{\partial t} dv + \iint \vec{j}\, d\vec{s} = 0 \qquad \text{(II-46)}$$

$\iint \vec{j}\, d\vec{s} = \iiint div\vec{j}\, dv$ (Green-Ostrograski theorem)

So :

$$\iiint [\frac{\partial \rho}{\partial t} + div\vec{j}]dv = 0 \qquad \text{(II-47)}$$

$\frac{\partial \rho}{\partial t} + div\vec{j}$=0 (Law of charge conservation)



FigureII-12: elementary surfaces

In steady state : $\frac{\partial \rho}{\partial t} = 0 \;\Rightarrow\; div\vec{j} = 0$

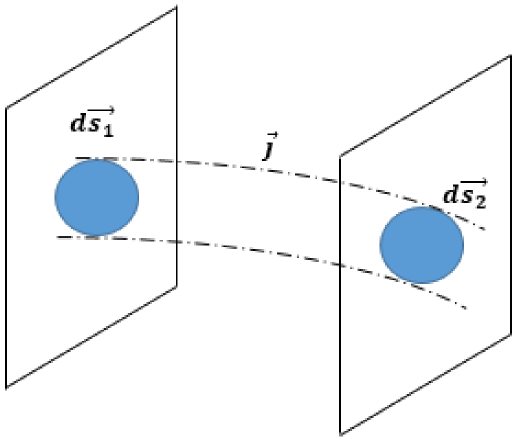
If $div\vec{j} = 0$: the vector $\vec{j}$ is said to be flux conservative

Consider a current tube bounded by two elementary surfaces $d\vec{s}_1, d\vec{s}_2$.

The total flow out of this current tube is zero

$$\vec{j}_1 d\vec{s}_1 + \vec{j}_2\, d\vec{s}_2 = 0 \;\Rightarrow\; \vec{j}_1 d\vec{s}_1 = -\vec{j}_2\, d\vec{s}_2 \qquad \text{(II-48)}$$

Outgoing flow is equal to incoming flow (flow conservation).

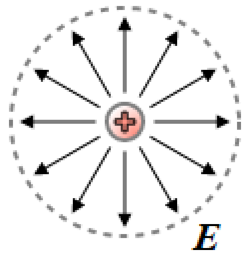


FigureII-13: current flow.

II-4 Maxwell-Gauss equation:

The divergence of the electric field is proportional to the distribution of electric charges..

$$div\,\vec{E} = \frac{\rho}{\varepsilon_O}\,\varepsilon \qquad \text{(II-49)} \; :$$



FigureII-14: electric field

$\vec{E}$: electric field (V/m)

ρ: charge density (charge distribution) (C/m^3)

ε_0: permittivity of free space ($s^4\,A^2/m^3kg$)

$$\frac{1}{4\pi\varepsilon_0} = 9.10^9$$

II-5 Maxwell-flux equation:

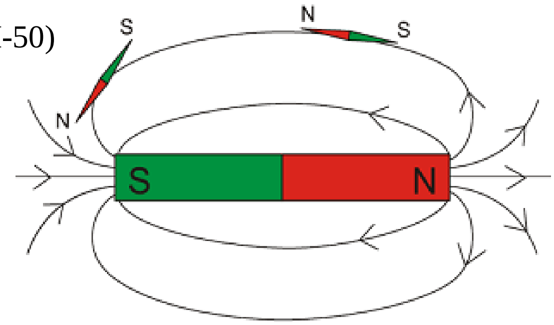
Magnetic field divergence is zero.

$$div\vec{B} = \vec{\nabla} \bullet \vec{B} = 0 \qquad \text{(II-50)}$$

The magnetic field flux through any closed surface S is zero:

the magnetic field is flux conservative,

The field lines emerge from one pole and move the other.



FigureII-15: magnetic field

II-6 Faraday's Law:

Let's consider an electrical circuit consisting simply of a conductor (copper) forming a loop..

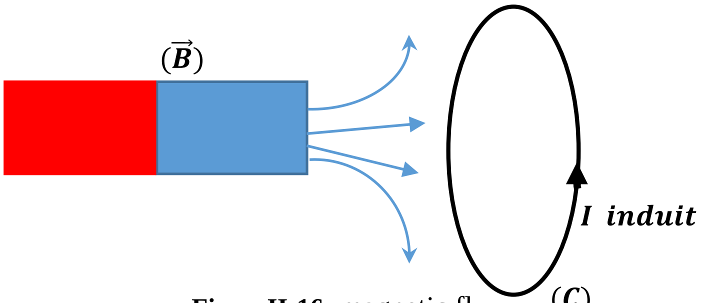
This circuit is immersed in a magnetic field $\vec{B}$, which can be achieved using a magnet.

If the magnet is moved, an electric current (induced current) is generated for the duration of the movement.

II-6.1 Faraday expression:

The flux of the $\vec{B}$ field passes through a surface (s) :

$$\emptyset = \oiint \vec{B} \bullet \overrightarrow{ds} \qquad \text{(II-51)}$$



FigureII-16: magnetic flux

If the flux $\emptyset$ varies with time, local Faraday's law applies.:

$$e = -\frac{d\emptyset}{dt} \qquad \text{(II-52)}$$

e : induced voltage or induced electromotive force in a circuit

Φ : magnetic flux (Weber)

The circulation of the electric field $\vec{E}$ along the closed circuit :

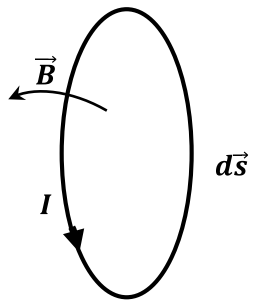
$$e = \oint \vec{E} \bullet \overrightarrow{dl} \qquad \text{(II-53)}$$

e : Volt, $\vec{E}: V.m^{-1}$

The circulation of the electric field on a closed contour (C) is the opposite of the variation of the magnetic flux through any surface entwined and oriented by this contour.

II-6.2 Maxwell-Faraday equation:

We have found that :



$$e = \oint \vec{E} \bullet \overrightarrow{dl}, \quad e = -\frac{d\emptyset}{dt}, \quad \emptyset = \oiint \vec{B} \bullet \overrightarrow{ds}$$

So:

$$e = -\frac{d}{dt}\oiint \vec{B} \bullet \overrightarrow{ds} \qquad \text{(II-54)}$$

$$e = \oint \vec{E} \bullet \overrightarrow{dl} = -\oiint \frac{d\vec{B}}{dt} \bullet \overrightarrow{ds} \qquad \text{(II-55)}$$

Using Stokes' theorem :

$$\oint \vec{E} \bullet \overrightarrow{dl} = \iint \overrightarrow{rotE} \bullet d\vec{s} \qquad \text{(II-56)}$$

From equation(1) and equation(2) we find that:

Maxwell-Faraday equation:

$$\overrightarrow{rotE} = \frac{d\vec{B}}{dt} \qquad \text{(II-57)}$$

The rotation of the electric field is proportional to the variation of the magnetic field over time..

Faraday's law shows that a variable magnetic field produces an electric field with a non-zero rotational field (unlike the electrostatic field)..

II-7 Ampère's theorem:

Local expression of Ampère's theorem :

$$\oint \vec{B} \bullet \overrightarrow{dl} = \mu_0 I \qquad \text{(II-58)}$$

I : current intensity.

$\vec{dl}$: displacement element vector along contour (C)

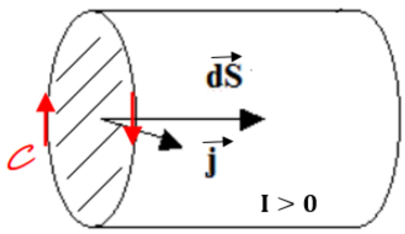
μ_0 : permeability of free space. $\mu_0 = 4\pi.10^{-7}kgmA^{-2}s^{-2}$

The current intensity I :

$$I = \iint_S \vec{j} \bullet \vec{dS} \qquad \text{(II-59)}$$

$\vec{j}$: vector current density (em^{-2}).

$$\oint \vec{B} \bullet \vec{dl} = \mu_0 \iint_S \vec{j} \bullet \vec{dS} \qquad \text{(II-60)}$$



FigureII-17: Conductor cross-section S

According to Stokes' theory :

$$\oint \vec{B} \bullet \vec{dl} = \iint_S \overrightarrow{rot\vec{B}} \bullet \vec{dS} \qquad \text{(II-61)}$$

So :

$$\overrightarrow{rot\vec{B}} = \mu_0 \vec{j} \qquad \text{(II-62)}$$

II-8 Compatibility of Ampère's theorem and charge conservation:

II-8.1 In steady state:

In the case of a stationary system : $\rho = cte \Rightarrow \frac{\partial \rho}{\partial t} = 0$

- Charge conservation :

$$div\vec{j} = -\frac{\partial \rho}{\partial t} \Rightarrow div\vec{j} = 0 \qquad \text{(II-63)}$$

- Ampère's theorem :

$$\overrightarrow{rot\vec{B}} = \mu_0 \vec{j} \Rightarrow \vec{j} = \frac{1}{\mu_0}\overrightarrow{rot\vec{B}}$$

Remarks:

$$div(\overrightarrow{rot\vec{B}}) = 0 \Rightarrow div\vec{j} = div\left(\frac{1}{\mu_0}\overrightarrow{rot\vec{B}}\right)$$

$$\Rightarrow div\vec{j} = \frac{1}{\mu_0} div\left(\overrightarrow{rot\vec{B}}\right) \qquad \text{(II-64)}$$

We find : $div\vec{j} = 0$

So: At steady state, the two equations are clearly compatible.

II-8.2 In variable speed: $\rho \neq cte \Rightarrow \frac{\partial \rho}{\partial t} \neq 0$

- Charge conservation :

$$\frac{\partial \rho}{\partial t} \neq 0 \Rightarrow div\vec{j} = \frac{\partial \rho}{\partial t} \neq 0 \qquad \text{(II-65)}$$

- Ampère's theorem :

$$\overrightarrow{rot\vec{B}} = \mu_0 \vec{j} \Rightarrow \vec{j} = \frac{1}{\mu_0}\overrightarrow{rot\vec{B}}$$

$$div\vec{j} = div\left(\frac{1}{\mu_0}\overrightarrow{rot\vec{B}}\right) \Rightarrow div\vec{j} = \frac{1}{\mu_0} div\left(\overrightarrow{rot\vec{B}}\right)$$

$$\Rightarrow div\vec{j} = 0$$

So : In variable speed operation, the two equations are clearly incompatible.

Maxwell's solution to this problem: From Gauss's theorem in variable regime :

$$\forall(M,t) \quad div\vec{E}(M,t) = \frac{\rho(M,t)}{\varepsilon_0}$$

This allows us to deduce :

$$\rho(M,t) = \varepsilon_0 div\vec{E}(M,t) \qquad \text{(II-66)}$$

$$\Rightarrow \quad \rho(M,t) = div\varepsilon_0\vec{E}(M,t)$$

$$\frac{\partial\rho(M,t)}{\partial t} = div\varepsilon_0\frac{\partial\vec{E}(M,t)}{\partial t} \qquad \text{(II-67)}$$

By introducing this relationship into the $div\vec{j}$

$$div\vec{j} = -\frac{\partial\rho}{\partial t}$$

We find : $div\vec{j} = -div\varepsilon_0\frac{\partial\vec{E}(M,t)}{\partial t}$

$$div(\vec{j} + \varepsilon_0\frac{\partial\vec{E}(M,t)}{\partial t}) = 0 \qquad \text{(II-68)}$$

$\vec{j} + \varepsilon_0\frac{\partial\vec{E}(M,t)}{\partial t}$: conservative flux

Maxwell proposed to modify Ampère's theorem by replacing the current density $\vec{j}$ by $\vec{j}$:

$$\vec{j} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

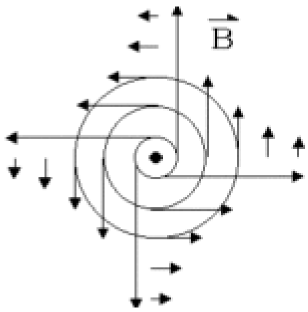
Maxwell-Ampere equation:

$$\overrightarrow{rotB} = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad \text{(II-69)}$$

$\mu_0\vec{j}$: Local Ampère theorem.

Displacement current :

$$j_d = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad \text{(II-70)}$$



FigureII-18 : Ling of magnetic field

The magnetic field rotational is the sum of the electric $\vec{j}$ and displacement $\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ (A) currents, entwined by this contour, and multiplied by μ_0.

Remarks :

The Maxwell-Faraday and Maxwell-Ampère equations both show that the two electric and magnetic fields are coupled, and that the variation of one is proportional to the intensity of the other.

II-9 Maxwell's equations in vacuum:

- **In steady state :**

Electrostatics	Electrostatic and magnetostatic fields are completely decoupled	Magnetostatic
$\frac{\partial\rho}{\partial t}=0$		$div\vec{j}=0$
$\overrightarrow{rot}\vec{E}=\vec{0}$		$\overrightarrow{rot}\vec{E}=\mu_0\vec{j}$
$div\vec{E}=\frac{\rho}{\varepsilon_0}$		$div\vec{B}=0$

- **In variable speed :** $\frac{\partial\rho}{\partial t}\neq 0,\quad \frac{\partial\rho}{\partial t}\neq 0$

Maxwell-Gauss equation: $div\vec{E}=\frac{\rho}{\varepsilon_0}$ (II-71)

Maxwell-magnetic flux equation: $div\vec{B}=0$ (II-72)

Maxwell-Faraday equation: $\overrightarrow{rot}\vec{E}=-\frac{d\vec{B}}{dt}$ (II-73)

Maxwell-Ampère equation: $\overrightarrow{rot}\vec{B}=\mu_0\vec{j}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ (II-74)

Remarks:

- In electrostatics: the electric field $\vec{E}$ is due to the presence of electric charges
- En électrostatique : le champ électrique $\vec{E}$ est dû à la présence de charges électriques, (no electric charge, no electric field).
- In magnetostatics: the magnetic field $\vec{B}$is due to the presence of electric current (no electric current, no magnetic field).
- $\overrightarrow{rot}\vec{E}=\frac{d\vec{B}}{dt}$: if the magnetic field $\vec{B}$ depends on time, we can have an electric field.
- $\overrightarrow{rot}\vec{B}=\mu_0\vec{j}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$: if the time-dependent electric field $\vec{E}$ creates a magnetic field.

II-10 Physical significance of Maxwell's equations:

Maxwell's equations are a priori valid in all environments

- **The 1st Maxwell equation:**

Maxwell-Gauss equation$div\vec{E}=\frac{\rho}{\varepsilon_0}$

- This equation expresses the fact that the electric field across a firm surface is related to the electric charge contained within that surface. On the other hand, it expresses the way in which electric charges are at the origin of the electric field.

- **The 2nd Maxwell equation :**

Maxwell's magnetic flux equation $div\vec{B} = 0 \Rightarrow \oiint \vec{B}\overrightarrow{ds} = 0$ This equation states that the magnetic flux through a closed surface is zero.

- **The 3rd Maxwell equation :**

Maxwell-Faraday equation $\overrightarrow{rot\ E} = -\frac{\partial \vec{B}}{\partial t}$

This equation describes the phenomenon of variable magnetic induction that generates an electric field.

- **The 4th Maxwell equation:**

Maxwell-Ampère equation $\overrightarrow{rot\ B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

- In stationary cases $\overrightarrow{rot\ B} = \mu_0 \vec{J}$ (Ampère's theorem)
- In variable cases $\overrightarrow{rot\ B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

This equation expresses how an electric current is the origin of a magnetic current. Note that a time-dependent electric field creates a magnetic field $\vec{B}$.

II-11 Maxwell's equations in medium:

In medium, we distinguish between bound charges (which can move over a microscopic distance) and free charges (which can move over a macroscopic distance).

Charges and current lies comprise polarization charges n_i volume charge density ρ_p, surface density σ_p, volume current $\vec{J}_p$.

$$\rho = \rho_l + \rho_p \qquad \text{(II-75)}$$

$$\vec{J} = \vec{J}_l + \vec{J}_p \qquad \text{(II-76)}$$

In Maxwell's vacuum equations ρ : is the free charge density and $\vec{J}$: is the current density

II-11.1 Dielectric medium:

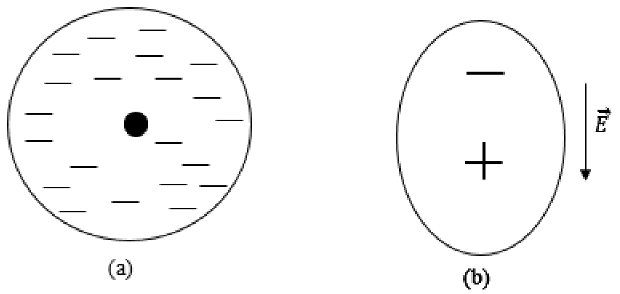
Dielectric medium is non-conductive material media (charge carriers cannot move freely over long distances macroscopically).

II-11.1-1 Some solid Dielectric mediums:

Glass, polypropylene, ceramics, most plastics …..

II-11.1-2 Polarization:

The atom is an electrically neutral object, consisting of a nucleus and a cloud of electrons.



FigureII-19 :(a) atom electrically neutral (b) **polarized atom**

(a) : in the absence of an external electric field; the atom retains its symmetry.

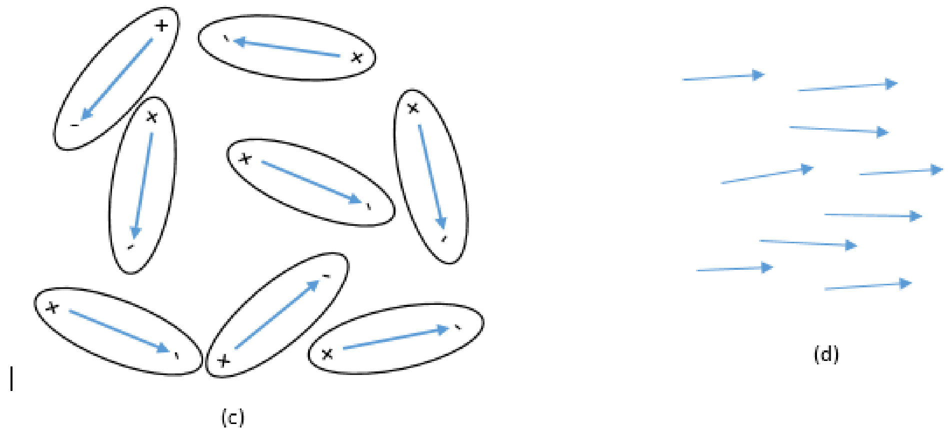
(b) : in the presence of an external electric field: the electric force moves the centers of mass in opposite directions, creating an internal dipole.

Each atom placed in an electric field $\vec{E}$carries a dipole moment. $\vec{P} = \alpha\varepsilon_0\vec{E}$,

$\alpha = 4\pi r_0^3$: electric polarization.

r_0 : atomic radius.

II-11.1-3 Orientation of polar molecules:



FigureII-20 : Orientation of polar molecules

(c) : in the absence of an external electric field, the distribution of dipole moments is random. The sum of dipole moments is zero.

A polar molecule has a dipole moment even in the absence of an external electric field.

(d) : The application of an electric field; polar molecules tend to orient their dipole moment parallel to the electric field.

Each of these dipoles contributes to the creation of an electric field of polarization $\vec{E}_p$. The number of dipoles per unit volume defines the polarization vector $\vec{P}$.

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \qquad \text{(II-77)}$$

The polarization electric field :

$$\vec{E}_P = \frac{1}{\varepsilon_0}\vec{P} = \chi_e \vec{E} \qquad \text{(II-78)}$$

Gauss's theorem:

$$div(\vec{E} + \vec{E}_P) = \frac{\rho}{\varepsilon_0} \qquad \text{(II-79)}$$

$$div[\vec{E}(1+\chi_e)] = \frac{\rho}{\varepsilon_0}$$

Relative permittivity :

$$\varepsilon_r = 1 + \chi_e \qquad \text{(II-80)}$$

Absolute permittivity of medium :

$$\varepsilon = \varepsilon_0 \varepsilon_r \qquad \text{(II-81)}$$

Gauss's theorem becomes :

$$div(\vec{E}\varepsilon_0) = \rho \qquad \text{(II-82)}$$

Vector of electrical excitation $\vec{D}$:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0(1+\chi_e)\vec{E} = \varepsilon_0 \varepsilon_r \vec{E}$$

$$\Rightarrow \quad \vec{D} = \varepsilon \vec{E} \qquad \text{(II-83)}$$

Gauss's theorem :

$$div\vec{D} = \rho \qquad \text{(II-84)}$$

II-11.1-4 Maxwell's equations in a dielectric medium:

Maxwell-Gauss equation: $div\vec{D} = \rho$ (II-85)

Maxwell-magnetic flux equation: $div\vec{B} = 0$ (II-86)

Maxwell-Faraday equation : $\overrightarrow{rotD} = -\varepsilon \frac{d\vec{B}}{dt}$ (II-87)

Maxwell-Ampère equation: $\overrightarrow{rotB} = \mu_0 \vec{j} + \mu_0 \frac{\partial \vec{D}}{\partial t}$ (II-88)

II-11.2 Conducting medium:

In conducting media, charges move freely.There is a relationship between the current density vector $\vec{j}$ and the total electric field $\vec{E}$

.Loi d'Ohm : $\vec{j} = \sigma\vec{E}$ (II-89)

σ : Conductivity of the conductor medium.

II-11.2-1 Relaxation time:

Charge conservation law

$$div\vec{j} + \frac{\partial\rho}{\partial t} = 0 \quad \Rightarrow \quad \sigma div\vec{E} + \frac{\partial\rho}{\partial t} = 0$$

We have : $div\vec{E} = \frac{\rho}{\varepsilon_0}$

So:

$$\sigma\frac{\rho}{\varepsilon_0} + \frac{\partial\rho}{\partial t} = 0$$

$$\frac{\partial\rho}{\partial t} + \sigma\frac{\rho}{\varepsilon_0} = 0 \qquad \text{(II-90)}$$

The solution to this equation is :

$$\rho = \rho_0 e^{-\frac{\sigma}{\varepsilon_0}} \qquad \text{(II-91)}$$

$$\rho = \rho_0 e^{-\frac{t}{T}} \qquad \text{(II-92)}$$

$T = \frac{\varepsilon_0}{\sigma}$ is the relaxation time of the medium

The relaxation time of a conductor gives a measure of the time it takes for the charge to disappear in the non-perfectly insulating medium..

For a good conductor $T \approx 10^{-20}s$

For a bad conductor $T \approx 10^{-10}s$

II-11.2-2 Maxwell's equations in conducting medium:

Maxwell-Gauss equation: $div\vec{E} = \frac{\rho}{\varepsilon_0}$ (II-93)

Maxwell-magnetic flux equation: $div\vec{B} = 0$ (II-94)

Maxwell-Faraday equation :: $\overrightarrow{rot\vec{E}} = -\frac{d\vec{B}}{dt}$ (II-95)

Maxwell-Ampère equation: $\overrightarrow{rot\vec{B}} = \mu_0\sigma\vec{E} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ (II-96)

According to Ohm's law: $\left|\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right| \ll \left|\sigma\vec{E}\right|$

So :

Maxwell-Ampère equation: $\overrightarrow{rot\vec{B}} = \mu_0\sigma\vec{E}$ (II-97)

II-11.3 Magnetic medium:

Let a magnetic medium occupy volume v. The magnetic dipole moment is given by $\vec{m}$

$\vec{m} = \iiint \vec{M}(r,t)dv$ (II-98)

$\vec{M}(r,t)$: magnetization vector (Am^{-1})

II-11.9.3-1 Magnetization current:

$\vec{J}_m = \overrightarrow{rot}(\vec{M}(r,t))$ (II-99)

Maxwell-Ampère equation :

Total current density in a magnetic and dielectric medium:

$\vec{J}_t = \vec{J}_{free} + \vec{J}_P + \vec{J}_m$ (II-100)

$\overrightarrow{rot}\vec{B} = \mu_0\vec{J}_t + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ (II-101)

$\vec{J}_P = \frac{\partial\vec{P}}{\partial t}, \quad \vec{J}_m = \overrightarrow{rot}\vec{M}$

$\overrightarrow{rot}\vec{B} = \mu_0(\vec{J}_{free} + \vec{J}_P + \vec{J}_m) + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

$\overrightarrow{rot}\vec{B} = \mu_0(\vec{J}_{free} + \frac{\partial\vec{P}}{\partial t} + \overrightarrow{rot}\vec{M}) + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

$\overrightarrow{rot}\vec{B} - \mu_0\overrightarrow{rot}\vec{M} = \mu_0\vec{J}_{free} + \mu_0(\frac{\partial\vec{P}}{\partial t} + \varepsilon_0\frac{\partial\vec{E}}{\partial t})$

$\overrightarrow{rot}(\vec{B} - \mu_0\vec{M}) = \mu_0\vec{J}_{free} + \mu_0\frac{\partial}{\partial t}(\vec{P} + \varepsilon_0\vec{E})$

$\overrightarrow{rot}(\frac{\vec{B}}{\mu_0} - \vec{M}) = \vec{J}_{free} + \frac{\partial}{\partial t}(\vec{P} + \varepsilon_0\vec{E})$ (II-102)

We have:

$\vec{P} + \varepsilon_0\vec{E} = \vec{D}$ (II-103)

$\overrightarrow{rot}(\frac{\vec{B}}{\mu_0} - \vec{M}) = \vec{J}_{free} + \frac{\partial}{\partial t}\vec{D}$ (II-104)

Magnetic excitation vector :

$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ (II-105)

Maxwell-Ampère equation becomes :

$\overrightarrow{rot}(\vec{H}) = \vec{J}_{free} + \frac{\partial}{\partial t}\vec{D}$ (II-106)

Magnetization vector $\vec{M}(r,t)$ is linked to the magnetic excitation vector $\vec{H}$.

$\vec{M}(r,t) = \chi_m\vec{H}$ (II-107)

$\vec{B} = \mu_0(\vec{H} + \vec{M}(r,t))$ (II-108)

$\vec{B} = \mu_0(\vec{H} + \chi_m\vec{H})$

$\vec{B} = \mu_0(1 + \chi_m)\vec{H} = \mu_0\mu_r\vec{H}$

$\vec{B} = \mu\vec{H}$ (II-109)

μ_0 : Permeability of free space

μ_r : Relative permeability

μ : Material permeability.

II-11.3-2 Maxwell's equations in a magnetic medium:

Maxwell-Gauss equation: $div\vec{E} = \frac{\rho}{\varepsilon_0}$ (II-110)

Maxwell-magnetic flux equation: $div\vec{H} = 0$ (II-111)

Maxwell-Faraday equation: $\overrightarrow{rot\vec{E}} = -\mu\frac{d\vec{H}}{dt}$ (II-112)

Maxwell-Ampère equation: $\overrightarrow{rot\vec{H}} = \vec{j} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}$ (II-113)

II-11.4 Plasma medium (ionized gas):

A plasma medium (or ionized gas) is a medium composed of ionized atoms or molecules that remain electrically neutral overall. A plasma is present as a fourth plasma state.

$$\text{solid} \underset{\Delta E}{\rightarrow} \text{liquid} \underset{\Delta E}{\rightarrow} \text{gas} \underset{\Delta E}{\rightarrow} \text{plasma}$$

In three states of matter (solid, liquid and gas), electrical interaction keeps negative electrons and positive atomic nuclei in proximity by the coulomb force. But in plasma, electric charges move independently.

Plasma medium are made up of atoms in which some or all electrons have been eliminated and positively or negatively charged nuclei, called ions, roam freely [S.Flugge,].

II-11.4-1 Plasma types:

There are two types of plasma :

- natural plasmas: stars, solar wind, lightning, sun, polar auroras[Arnold Hanslmeier].
- Artificial plasmas (laboratory plasmas): gas discharges, electric arcs, laser-generated plasma, glass neon plasma lamp,

current density vector $\vec{j}$ (loi d’Ohm) :

$\vec{j} = \sigma\vec{E}$

II-11.-4-2 Maxwell's equation in a plasma medium :

Maxwell-Gauss equation:: $div\vec{E} = \frac{\rho}{\varepsilon_0}$ (II-114)

Maxwell-magnetic flux equation: $div\vec{B} = 0$ (II-115)

Maxwell-Faraday equation: $\overrightarrow{rotE} = -\frac{d\vec{B}}{dt}$ (II-116)

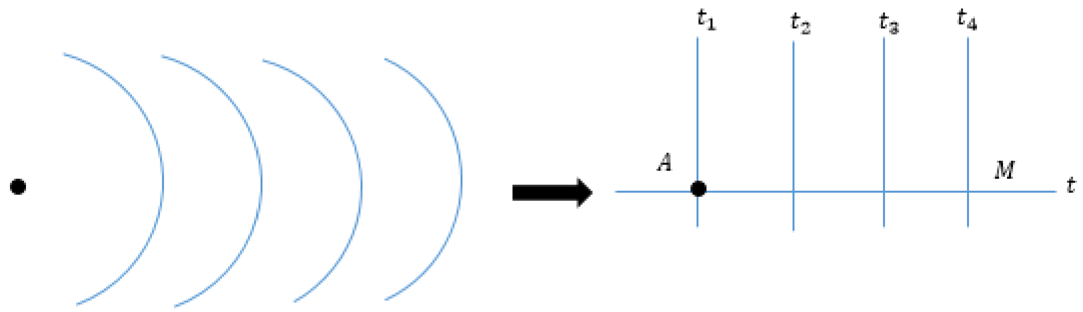
Maxwell-Ampère equation: $\overrightarrow{rotB} = \mu_0\sigma\vec{E} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ (II-117)

Maxwell's equations	
in a vacuum	**In Dielectric medium**
M- Gauss: $div\vec{E} = \frac{\rho}{\varepsilon_0}$ M-flux: $div\vec{B} = 0$ M-Faraday: $\overrightarrow{rotE} = -\varepsilon_0\frac{d\vec{B}}{dt}$ M-Ampère: $\overrightarrow{rotB} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{0}}{\partial t}$	M- Gauss: $div\vec{D} = \rho$ M-flux: $div\vec{B} = 0$ M-Faraday: $\overrightarrow{rotD} = -\varepsilon\frac{d\vec{B}}{dt}$ M-Ampère: $\overrightarrow{rotB} = \mu_0\vec{J} + \mu_0\frac{\partial\vec{D}}{\partial t}$
In conducting medium	**In Magnetic medium**
M- Gauss: $div\vec{E} = \frac{\rho}{\varepsilon_0}$ M-flux: $div\vec{B} = 0$ M-Faraday: $\overrightarrow{rotE} = -\varepsilon_0\frac{d\vec{B}}{dt}$ M-Ampère: $\overrightarrow{rotB} = \mu_0\sigma\vec{E}$	M- Gauss: $div\vec{E} = \frac{\rho}{\varepsilon_0}$ M-flux: $div\vec{H} = 0$ M-Faraday: $\overrightarrow{rotE} = -\mu\frac{d\vec{H}}{dt}$ M-Ampère: $\overrightarrow{rotH} = \vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}$
In plasma medium (ionized gas)	
M- Gauss: $div\vec{E} = \frac{\rho}{\varepsilon_0}$ M-flux: $div\vec{B} = 0$ M-Faraday: $\overrightarrow{rotE} = -\varepsilon_0\frac{d\vec{B}}{dt}$ M-Ampère: $\overrightarrow{rotB} = \mu_0\sigma\vec{E} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$	Ohm's law: $\left\|\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right\| \ll \left\|\sigma\vec{E}\right\|$ Magnetic excitation vector : $\vec{B} = \mu\vec{H}$ Electrical excitation vector: $\vec{D} = \varepsilon\vec{E}$ Absolute permitivity of the medium :$\varepsilon = \varepsilon_0\varepsilon_r$

CHAPTER III: ELECTROMAGNETIC WAVES PROPAGATION

- **Plane wave**
- **Electromagnetic wave**
- **Electromagnetic wave propagation**
- **Equation for electromagnetic wave propagation in a vacuum**
- **Propagation of electromagnetic waves in media**
  - **Electromagnetic wave propagation in a dielectric medium**
  - **Propagation of electromagnetic waves in conductors**
  - **Electromagnetic wave propagation in plasma**

If you throw a projectile into still water, from the point of impact to the projectile in the water. A series of small waves are propagated on the surface of the water.



In the ideal case, the signal propagates without distortion or attenuation.. The signal at the abscissa point at time t is therefore the same as the signal at O at time t $t-\frac{x}{\vartheta}$

$$s(x,t) = s(t-\frac{x}{\vartheta}) \qquad \text{(III-1)}$$

III-1 Plane wave:

An $s(x,t)$ wave is said to be plane if it depends on a single variable in Cartesian coordinates,

$$\forall y, \forall z; s(x,y,z,t) = s(x,t)$$

If we consider the same signal but propagating in the opposite direction, i.e. towards decreasing x:

$$s(x,t) = s(t+\frac{x}{\vartheta}) \qquad \text{(III-2)}$$

$\frac{x}{v}$: represents the time required for signal propagation from point o to point *M.*

ϑ : propagation speed depends on propagation medium.

Propagation equation :

Let a wave function s(x,t) be such that :

$$\frac{\partial^2 s(x,t)}{\partial x^2} - \frac{1}{\vartheta^2}\frac{\partial^2 s(x,t)}{\partial t^2} = 0 \qquad \text{(III-3)}$$

The general solution of the propagation equation is given by the following formula :

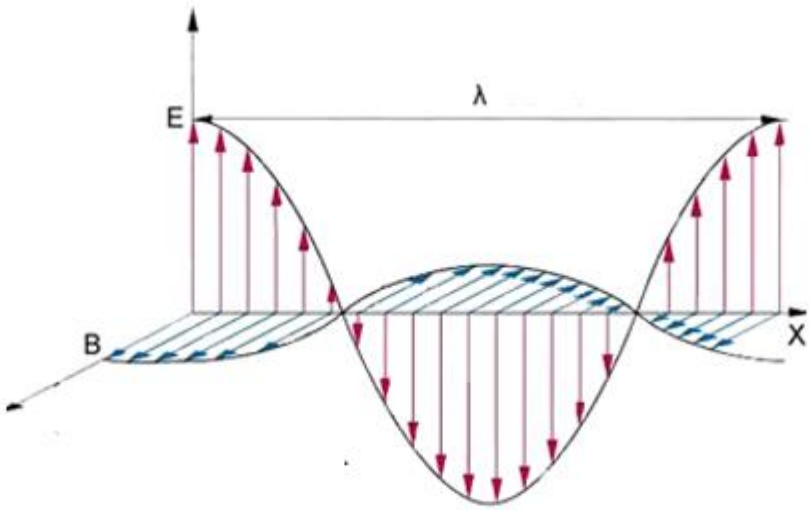
$$s(x,t) = Y^+\left(t-\frac{x}{\vartheta}\right) + Y^-\left(t+\frac{x}{\vartheta}\right) \qquad \text{(III-4)}$$

Y^+ : The wave function for propagation in the increasing direction of the (ox) axis

Y^- : The wave function for propagation in the decreasing direction of the (ox) axis.

III-2 Electromagnetic wave:

An electromagnetic wave is the combination of two disturbances, one electrical and the other magnetic. These two disturbances, oscillating at the same time but in two perpendicular planes, travel at the speed of light.



III-3 Electromagnetic wave propagation :

III-3.1 Electromagnetic wave propagation in a vacuum:

Vacuum is a medium characterized by a vacuum permittivity ε_0 and a magnetic permeability μ_0

Maxwell's equations in vacuum in the absence of current $\vec{j} = \vec{0}$ and charge density $\rho = 0$

Maxwell-Gauss equation: $div\vec{E} = 0$ (III-5)

Maxwell-magnetic flux equation: $div\vec{B} = 0$ (III-6)

Maxwell-Faraday equation: $\overrightarrow{rot}\vec{E} = -\frac{d\vec{B}}{dt}$ (III-7)

Maxwell-Ampere equation: $\overrightarrow{rot}\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ (III-8)

III-3.1-1. Electromagnetic field propagation equation:

III-3.1-1.a Electric field propagation equation $\vec{E}$:

Let's eliminate the magnetic field $\vec{B}$ from the equations (M-Faraday) and (M-Ampere) to obtain an equation as a function of the electric field $\vec{E}$.

Maxwell-Faraday equation: $\overrightarrow{rot}\,\overrightarrow{rotE} = -\frac{d}{dt}\overrightarrow{rot\ B}$

Remarks :

$$\overrightarrow{rot}\,\overrightarrow{rotE} = -\Delta\vec{E} + \overrightarrow{grad}(div\vec{E})\ , \qquad \text{(III-9)}$$

$$\overrightarrow{grad}\big(div\vec{E}\big) = 0 \qquad \text{(III-10)}$$

So:

$$\overrightarrow{rot}\,\overrightarrow{rotE} = -\Delta\vec{E} \qquad \text{(III-11)}$$

Maxwell-Ampère equation: $\frac{\partial}{\partial t}\overrightarrow{rotB} = \mu_0\varepsilon_0\frac{\partial}{\partial t}\frac{\partial\vec{E}}{\partial t}$

$$\overrightarrow{rot}\,\overrightarrow{rotE} = -\frac{d}{dt}\overrightarrow{rot\ B}$$
$$\frac{\partial}{\partial t}\overrightarrow{rotB} = \mu_0\varepsilon_0\frac{\partial}{\partial t}\frac{\partial\vec{E}}{\partial t}$$
$$\overrightarrow{rot}\,\overrightarrow{rotE} = -\Delta\vec{E}$$

So: $\Delta\vec{E} - \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = \vec{0}$ (III-12)

$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$: speed of light in a vacuum

$\varepsilon_0 = \frac{1}{36\pi} = 910^9(SI)$, $\mu_0 = 4\pi10^{-7}, c = 3.10^8\ m/s$

Electric field propagation equation $\vec{E}$:

$$\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \vec{0} \qquad \text{(III-13)}$$

III-3.1-1.b Magnetic field propagation equation $\vec{B}$:

Maxwell-Ampère equation: $\overrightarrow{rot}\,\overrightarrow{rotB} = \mu_0\varepsilon_0\frac{\partial}{\partial t}\overrightarrow{rotE}$

Maxwell-Faraday equation: $\frac{\partial}{\partial t}\ \overrightarrow{rotE} = -\frac{\partial}{\partial t}\frac{d}{dt}\vec{B}$

Remarks:

$$\overrightarrow{rot}\,\overrightarrow{rotB} = -\Delta\vec{B} \qquad \text{(III-14)}$$

$$\overrightarrow{rot}\,\overrightarrow{rotB} = \mu_0\varepsilon_0\frac{\partial}{\partial t}\overrightarrow{rotE}$$
$$\frac{\partial}{\partial t}\ \overrightarrow{rotE} = -\frac{\partial^2}{\partial t^2}\vec{B}$$
$$\overrightarrow{rot}\,\overrightarrow{rotB} = -\Delta\vec{B}$$

So : $\Delta\vec{B} - \mu_0\varepsilon_0 \frac{\partial^2\vec{B}}{\partial t^2} = \vec{0}$ (III-15)

Magnetic field propagation equation $\vec{B}$:

$$\Delta\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = \vec{0} \qquad \text{(III-16)}$$

Electromagnetic field propagation equations :

$$\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \vec{0}$$

$$\Delta\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = \vec{0}$$

Δ: Laplacian operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \text{(III-17)}$$

Plane wave in a single (ox) direction:

$$\frac{\partial^2\vec{E}}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \vec{0} \qquad \text{(III-18)}$$

$$\frac{\partial^2\vec{B}}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = \vec{0} \qquad \text{(III-19)}$$

Solution of the propagation equation :

$$\vec{E}(x,t) = \vec{E}^{+}\left(t - \frac{x}{c}\right) + \vec{E}^{-}\left(t + \frac{x}{c}\right) \qquad \text{(III-20)}$$

$$\vec{B}(x,t) = \vec{B}^{+}\left(t - \frac{x}{c}\right) + \vec{B}^{-}\left(t + \frac{x}{c}\right) \qquad \text{(III-21)}$$

III-3.1-1.3 Transversality of field:

An important property of the electromagnetic vectors E , B , taking into account the condition [A.F.Benhabib-A.Hajaj]:

$$div\vec{E} = 0 : \Rightarrow \frac{d\vec{E}_x}{dx} = 0$$

It can be seen that the vectors $\vec{E}, \vec{B}$ lie in a plane normal to the direction of propagation (ox)

The vectors $\vec{E}, \vec{B}$ are transverse, so the electromagnetic wave propagating in a vacuum is transverse.

$$div\vec{E} = 0 , div\vec{E} = \frac{d\vec{E}_x}{dx} + \frac{d\vec{E}_y}{dy} + \frac{d\vec{E}_z}{dz} = 0$$

$\Rightarrow \quad \frac{d\vec{E}_x}{dx} = 0$ (III-22)

So : $-\frac{1}{c^2}\frac{d^2\vec{E}_x}{dx^2} = 0$

$\Rightarrow \quad \vec{E}_x = 0$ (III-23)

$div\vec{B} = 0$, $div\vec{E} = \frac{d\vec{B}_x}{dx} + \frac{d\vec{B}_y}{dy} + \frac{d\vec{B}_z}{dz} = 0$

$\Rightarrow \quad \frac{d\vec{B}_x}{dx} = 0$ (III-24)

So : $-\frac{1}{c^2}\frac{d^2\vec{B}_x}{dx^2} = 0$

$\Rightarrow \quad \vec{B}_x = 0$ (III-25)

- The vectors $\vec{E}_x$ $\vec{B}_x$ are independent of space and time.

III-3.1.2 Complex notation:

Complex electromagnetic fields :

$\vec{\xi} = \vec{E}_0 e^{i(\omega t - \vec{k}\vec{r})}$ (III-26)

$\vec{\mathcal{B}} = \vec{B}_0 e^{i(\omega t - \vec{k}\vec{r})}$ (III-27)

III-3.1.2-1 Derivation by time:

$\frac{\partial\vec{\xi}}{\partial t} = i\omega\vec{\xi}$ (III-28)

$\frac{\partial^2\vec{\xi}}{\partial t^2} = -\omega^2\vec{\xi}$ (III-29)

So :

$\Rightarrow \quad \frac{\partial}{\partial t} = i\omega$ (III-30)

$\frac{\partial^2}{\partial t^2} = -\omega^2$ (III-31)

III-3.1.2-2 Divergence of an electric field:

$div\vec{\xi} = \frac{\partial \xi_x}{\partial x} + \frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_z}{\partial z}$ (III-32)

$div\vec{\xi} = -i(k_{0x} + k_{0y} + k_{0z})$

$\vec{k}_0\vec{r} = k_{0x}x + k_{0y}y + k_{0z}z$ (III-33)

$\vec{\xi}_x = \vec{E}_{0x} e^{i(\omega t - k_{0x}x)}$

$\vec{\xi}_y = \vec{E}_{0y} e^{i(\omega t - k_{0y}y)}$

$\vec{\xi}_z = \vec{E}_{0z} e^{i(\omega t - k_{0z}z)}$

$div\vec{\xi} = \vec{\nabla}\vec{\xi} = -i\vec{k}_0\vec{\xi}$ (III-34)

$\Rightarrow \vec{\nabla} = -i\vec{k}_0$ (III-35)

III-3.1.2-3 Rotational electric field:

$\overrightarrow{rot}\vec{\xi} = \vec{\nabla} \wedge \vec{\xi} = (-i\vec{k}_0) \wedge \vec{\xi},$ (III-36)

III-3.1.2-4 Electric field Laplacian:

$\Delta\xi = \vec{\nabla} \bullet \vec{\nabla} = (-i\vec{k}_0) \bullet (-i\vec{k}_0)$

$\Delta\xi = -k_0^2\xi$

$\Rightarrow \Delta = -k_0^2$ (III-37)

III-3.1.3 Equation for electromagnetic wave propagation in a vacuum:

Maxwell's equations in complex notation : ($\vec{j} = \vec{0}$ et $\rho = 0$)

- Maxwell- Gauss: $\vec{\nabla}\vec{\xi} = -i\vec{k}_0\vec{\xi} = 0$ (III-38)
- Maxwell-flux magnétique : $\vec{\nabla}\vec{\xi} = -i\vec{k}_0\vec{\xi} = 0$ (III-39)
- Maxwell-Faraday: $\vec{\nabla} \wedge \vec{\xi} = (-i\vec{k}_{0)} \wedge \vec{\xi} = -i\omega\vec{\mathcal{B}}$ (III-40)
- Maxwell-Ampère: $\vec{\nabla} \wedge \vec{\mathcal{B}} = (-i\vec{k}_0) \wedge \vec{\mathcal{B}} = i\omega\mu_0\varepsilon_0\vec{\xi}$ (III-41)

From equation (III-39) and equation (III-40):

$\vec{\mathcal{B}} = \frac{1}{\omega}\left(\vec{k}_0 \wedge \vec{\xi}\right)$ (III-42)

$\left|\vec{\mathcal{B}}\right| = \frac{k_O\xi}{\omega}$ (III-43)

III-3.1.3- Dispersion relation:

Let an electromagnetic wave propagate along the (ox) axis; $\vec{k} = k\vec{\imath}$

Propagation equation :

$\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \vec{0}$ (III-44)

$\Delta = \vec{\nabla} \wedge \vec{\nabla} = \left(i\vec{k}\right)^2 = -k^2,\ \frac{\partial^2}{\partial t^2} = -\omega^2$

$\Rightarrow\ -k^2\vec{E} + \frac{\omega^2}{c^2}\vec{E} = \vec{0}$ (III-45)

$k = \pm\frac{\omega}{c}$ (III-46)

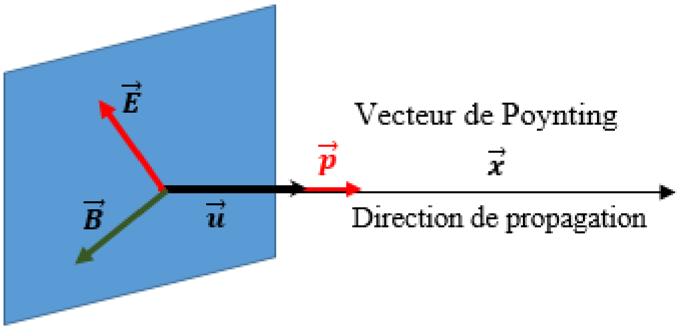
Remarks :

A medium is said to be dispersive if the dispersion relationship is non-linear.

So vacuum is a non-dispersive medium.

III-3.1.4 Poynting vector: $\vec{\pi}, \vec{R} ou\ \vec{P}$

This vector is carried by the direction of propagation of the electromagnetic wave, the flux of the Poynting vector across a surface is equal to the power carried by the wave across that surface.



Poynting vector module:

$$\vec{P} = \vec{E} \wedge \frac{\vec{B}}{\mu_0} \quad \text{(III-47)}$$

$$\vec{P} = \vec{E} \wedge \vec{H} \quad \text{(III-48)}$$

$$|\vec{P}| = \left|\vec{E} \wedge \frac{\vec{B}}{\mu_0}\right| \quad \text{(III-49)}$$

III-3.1.5 Energy conservation:

Electromagnetic energy density :

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0} \quad \text{(III-50)}$$

$$div\vec{P} = \frac{1}{\mu} div(\vec{E} \wedge \vec{B}) \quad \text{(III-51)}$$

$$= \frac{1}{\mu_0}\left(\vec{B}\overrightarrow{rot}(\vec{E}) - \vec{E}\overrightarrow{rot}(\vec{B})\right)$$

$$= \frac{1}{\mu_0}\left(\vec{B}\left(-\frac{\partial \vec{B}}{\partial t}\right) - \vec{E}\left(\varepsilon_0\mu_0\frac{\partial \vec{E}}{\partial t}\right)\right)$$

$$= -\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu} + \varepsilon_0\frac{E^2}{2}\right)$$

$$div\vec{P} = -\frac{\partial u}{\partial t} \quad \text{(III-52)}$$

So

Energy conservation equation :

$$div\vec{P} + \frac{\partial u}{\partial t} = 0 \quad \text{(III-53)}$$

III-3.1.6 Group speed:

The group velocity of a monochromatic traveling plane wave with pulsation ω and wave vector k :

$$v_g = \frac{d\omega}{dk} \quad \text{(III-54)}$$

III-3.2 Propagation of electromagnetic waves in media:

III-3.2-1 Electromagnetic wave propagation in a dielectric medium:

Maxwell's equations : ($\vec{J} = \vec{0}$ et $\rho = 0$)

Maxwell- Gauss: $\vec{\nabla}\vec{\xi} = -i\vec{k}_0\vec{\xi} = 0$ (III-55)

Maxwell-flux magnétique : $\vec{\nabla}\vec{\xi} = -i\vec{k}_0\vec{\xi} = 0$ (III-56)

Maxwell-Faraday: $\vec{\nabla} \wedge \vec{\xi} = (-i\vec{k}_0) \wedge \vec{\xi} = -i\omega\vec{\mathcal{B}}$ (III-57)

Maxwell-Ampère: $\vec{\nabla} \wedge \vec{\mathcal{B}} = (-i\vec{k}_0) \wedge \vec{\mathcal{B}} = i\omega\mu\varepsilon\vec{\xi}$ (III-58)

III-3.2-1.1 Equations of Electromagnetic Wave Propagation:

III-3.2-1.1.a Electric field wave propagation equations:

$$\Delta\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = \vec{0} \quad \text{(III-59)}$$

$$\Rightarrow \quad \Delta\vec{E} - \frac{1}{\vartheta^2}\frac{\partial^2\vec{E}}{\partial t^2} = \vec{0} \quad \text{(III-60)}$$

III-3.2-1.1.b Magnetic field wave propagation equations:

$$\Delta\vec{B} - \mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2} = \vec{0} \quad \text{(III-61)}$$

$$\Rightarrow \quad \Delta\vec{B} - \frac{1}{\vartheta^2}\frac{\partial^2\vec{B}}{\partial t^2} = \vec{0} \quad \text{(III-62)}$$

where:

ϑ: speed of propagation

$$\vartheta = \frac{1}{\sqrt{\mu\varepsilon}} \quad \text{(III-63)}$$

$$\vartheta = \frac{c}{\eta} \quad \text{(III-64)}$$

where:

η : refractive index of the medium η:

$$\eta = \sqrt{\mu_r\varepsilon_r} \quad \text{(III-65)}$$

III-3.2-1.2 Dispersion relation:

Propagation equation:

$$-k^2\vec{E}+\frac{\omega^2}{\vartheta^2}\vec{E}\ =\vec{0}$$

$$k(\omega)=\pm\frac{\omega}{\vartheta} \quad \text{(III-66)}$$

So; a vacuum is a non-dispersive medium

III-3.2-2 Propagation of electromagnetic waves in conductors:

Maxwell's equations: ($\vec{j}=\vec{0}$ et $\rho=0$)

Maxwell- Gauss: $\vec{\nabla}\vec{\xi}=-i\vec{k}_0\vec{\xi}=0$ (III-67)

Maxwell-flux magnétique : $\vec{\nabla}\vec{\xi}=-i\vec{k}_0\vec{\xi}=0$ (III-68)

Maxwell-Faraday: $\vec{\nabla}\wedge\vec{\xi}=(-i\vec{k}_0)\wedge\vec{\xi}=-i\omega\vec{\mathcal{B}}$ (III-69)

Maxwell-Ampère: $\vec{\nabla}\wedge\vec{\mathcal{B}}=(-i\vec{k}_0)\wedge\vec{\mathcal{B}}=i\omega\mu_0\sigma\vec{\xi}$ (III-70)

III-3.2-2.1 Equations of electromagnetic wave propagation:

III-3.2-2.1- Equations of electromagnetic wave propagation:

$$\Delta\vec{E}-\mu_0\sigma\frac{\partial\vec{E}}{\partial t}\ =\vec{0} \quad \text{(III-71)}$$

III-3.2-2.1- Equations of magnetic wave propagation:

$$\Delta\vec{B}-\mu_0\sigma\frac{\partial\vec{B}}{\partial t}\ =\vec{0} \quad \text{(III-72)}$$

III-3.2-2.2 Dispersion relation:

$$\Rightarrow -k^2\vec{E}-i\omega\mu_0\sigma\,\vec{E}=\vec{0} \quad \text{(III-73)}$$

$$\Rightarrow -k^2\vec{B}-i\omega\mu_0\sigma\,\vec{B}\ =\vec{0} \quad \text{(III-74)}$$

$$k^2=-i\omega\mu_0\sigma \quad \text{(III-75)}$$

- For a good driver : $\frac{\sigma}{\varepsilon\omega}\gg 1$
- For a dielectric : $\frac{\sigma}{\varepsilon\omega}\ll 1$

$$k^2=\omega\mu_0\sigma e^{-\frac{\pi}{2}i} \quad \text{(III-76)}$$

$$e^{-\frac{\pi}{2}i}=\cos\frac{\pi}{2}-i\sin\left(\frac{\pi}{2}\right)=-i \quad \text{(III-77)}$$

$$k=\pm\sqrt{\omega\mu_0\sigma}e^{-\frac{\pi}{4}i} \quad \text{(III-78)}$$

$$e^{-\frac{\pi}{4}i}\ =\cos\frac{\pi}{4}-i\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}(1-i) \quad \text{(III-79)}$$

So:

Dispersion equation :

$$k = (1 - i)\sqrt{\frac{\omega\mu_0\sigma}{2}} \qquad \text{(III-80)}$$

The dispersion relation $\omega(k)$ is not linear.

So; conductors are dispersive media.

- Electromagnetic wave :

Let's consider propagation along the (oz) axis

$$\vec{\xi}(z,t) = \vec{E}_0 e^{i(\omega t - (1-i)\sqrt{\frac{\omega\mu_0\sigma}{2}}z)} \qquad \text{(III-81)}$$

$$\vec{\xi}(z,t) = \vec{E}_0 e^{-\sqrt{\frac{\omega\mu_0\sigma}{2}}z} e^{i(\omega t - \sqrt{\frac{\omega\mu_0\sigma}{2}}z)}$$

$$\vec{\xi}(z,t) = \vec{E}_0 e^{-\frac{z}{\delta}} e^{i(\omega t - \frac{z}{\delta})} \qquad \text{(III-82)}$$

where :

δ : depth of wave penetration

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} \qquad \text{(III-83)}$$

the amplitude of the wave decreases exponentially as it propagates with distance δ (depth of wave penetration).

Metal	Conductivity $\sigma(Cm^{-1})$	Depth of Penetration $\delta(mm)$		
		$\omega = 50HZ$	$\omega = 1kZ$	$\omega = 1MZ$
Cu	5,8 10^7	9,3	2,1	0,066
Al	3,5 10^7	12,1	2,7	0,O85
Au	4 ,5 10^7	10,6	2,38	0 ,075
Ag	6,0 10^7	9 ,1	2,03	O ,O64

III-3.2-3 Electromagnetic wave propagation in plasma:

Maxwell's equations : ($\vec{j} = \vec{0}$ et $\rho = 0$)

Maxwell- Gauss: $\vec{\nabla}\vec{\xi} = -i\vec{k}_0\vec{\xi} = 0$ (III-84)

Maxwell-flux: $\vec{\nabla}\vec{\xi} = -i\vec{k}_0\vec{\xi} = 0$ (III-85)

Maxwell-Faraday: $\vec{\nabla} \wedge \vec{\xi} = (-i\vec{k}_{0)} \wedge \vec{\xi} = -i\omega\vec{\mathcal{B}}$ (III-86)

Maxwell-Ampère: $\vec{\nabla} \wedge \vec{\mathcal{B}} = \left(-i\vec{k}_0\right) \wedge \vec{\mathcal{B}} = \mu_0\sigma\vec{\xi} + i\omega\mu_0\varepsilon_0\vec{\xi}$ (III-87)

III-3.2-3.1 Equations of electromagnetic wave propagation:

III-3.2-3.1.a Equation of electric field wave propagation:

$$\frac{\partial^2\vec{E}}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \mu_0\sigma\frac{\partial\vec{E}}{\partial t} \quad \text{(III-88)}$$

III-3.2-3..b- Equation of magnetic field wave propagation:

$$\frac{\partial^2\vec{B}}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = \mu_0\sigma\frac{\partial\vec{B}}{\partial t} \quad \text{(III-89)}$$

III-3.2-3.2 Movement of ions and electrons:

If an external electric field $\vec{E}$ is applied to the plasma

$$\sum\vec{F} = m\vec{a} \quad \text{(III-90)}$$

$$q\vec{E} = m\vec{a} \quad \text{(III-91)}$$

- For electron: mass :m, charge : $q = -e$, volume density :n_e

$$\vec{E} = m\frac{d\vec{\vartheta}_e}{dt} \quad \text{(III-92)}$$

Electric field $\vec{E}$; $\vec{E} = E_0e^{i(\omega t-kz)}$

Propagation along the (oz) axis

Speed : $\vartheta = \vartheta_0e^{i(\omega t)}$ (III-93)

$$m\frac{d\vec{\vartheta}_e}{dt} = -e\vec{E}$$

$$\Rightarrow \ im\omega\vec{\vartheta}_e = -e\vec{E} \quad \text{(III-94)}$$

Electron speed:

$$\vec{\vartheta}_e = \frac{ie}{m\omega}\vec{E} \quad \text{(III-95)}$$

- For ion : mass :M, charge : $q = e$, volume density :n_i

$$e\vec{E} = M\vec{a} \quad \text{(III-96)}$$

$$\Rightarrow e\vec{E} = M\frac{d\vec{\vartheta}_i}{dt} \quad \text{(III-97)}$$

$$M\frac{d\vec{\vartheta}_i}{dt} = e\vec{E}$$

$$\Rightarrow \quad iM\omega\vec{\vartheta}_i = e\vec{E} \quad \text{(III-98)}$$

Ion speed:

$$\vec{\vartheta}_i = \frac{-ie}{M\omega}\vec{E} \quad \text{(III-99)}$$

We have :$\frac{m}{M} \ll 1 \quad \Rightarrow \quad \frac{\vartheta_e}{\vartheta_i} \ll 1$

The movement of ions can therefore be neglected in front of electrons..

III-3.2-3.3 Current density $\vec{j}$:

Total current density :

$$\vec{j} = nq\vartheta \qquad \text{(III-100)}$$

$$\vec{j} = \vec{j}_e + \vec{j}_i = n_e(-e)\vec{\vartheta}_e + n_i(e)\vec{\vartheta}_i \approx -ne\vec{\vartheta}_e \qquad \text{(III-101)}$$

At thermodynamic equilibrium: $n_e = n_i = n$

$$\vec{j} = -i\frac{ne^2}{m\omega}\vec{E} \qquad \text{(III-102)}$$

We have:

$$\vec{j} = \sigma\vec{E}$$

In comparison, we find:

$$\sigma = -i\frac{ne^2}{m\omega} \qquad \text{(III-103)}$$

Charge retention:

$$div\vec{j} + \frac{\partial\rho}{\partial t} = 0, \qquad \rho = \rho_0 e^{i(\omega t - kz)}$$

$$div\sigma\vec{E} + i\omega\rho = 0$$

$$\Rightarrow \sigma\, div\vec{E} + i\omega\rho = 0 \qquad \text{(III-104)}$$

We have :

$$div\vec{E} = \vec{\nabla}\vec{E} = -ikE = \frac{\rho}{\varepsilon_0}$$

$$\sigma\,\frac{\rho}{\varepsilon_0} + i\omega\rho = 0 \qquad \text{(III-105)}$$

$$\rho\left(\frac{\sigma}{\varepsilon_0} + i\omega\right) = 0 \quad \Rightarrow \quad \rho\left(-i\frac{ne^2}{m\omega\varepsilon_0} + i\omega\right) = 0$$

$$i\frac{\rho}{\omega}\left(\omega^2 - \frac{ne^2}{m\varepsilon_0}\right) = 0 \quad \Rightarrow \rho\left(\omega^2 - \frac{ne^2}{m\varepsilon_0}\right) = 0$$

$$\rho\left(\omega^2 - {\omega_p}^2\right) = 0 \qquad \text{(III-107)}$$

si $\omega^2 = {\omega_p}^2$, $\rho = 0$

where:

ω_p: plasma pulsation.

$$\omega_p = \sqrt{\frac{ne^2}{m\varepsilon_0}} \qquad \text{(III-108)}$$

- Electric field wave propagation equation:

$$\frac{\partial^2\vec{E}}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \mu_0\varepsilon_0\frac{ne^2}{m\varepsilon_0}\vec{E} \qquad \text{(III-109)}$$

$$\frac{\partial^2 \vec{E}}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{\omega_p^2}{c^2}\vec{E} \qquad \text{(III-110)}$$

- Magnetic field wave propagation equation :

$$\frac{\partial^2 \vec{B}}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = \mu_0 \varepsilon_0 \frac{ne^2}{m\varepsilon_0}\vec{B} \qquad \text{(III-111)}$$

$$\frac{\partial^2 \vec{B}}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = \frac{\omega_p^2}{c^2}\vec{B} \qquad \text{(III-112)}$$

III-3.2-3.4 Dispersion relation:

$$-k^2\vec{E} + \frac{\omega^2}{c^2}\vec{E} = \frac{\omega_p^2}{c^2}\vec{E} \qquad \text{(III-113)}$$

$$k^2 = \frac{\omega^2 - {\omega_p}^2}{c^2} \qquad \text{(III-114)}$$

$$k = \frac{1}{c}\sqrt{\omega^2 - {\omega_p}^2} \qquad \text{(III-115)}$$

For:

- $\omega > \omega_p$: propagation without attenuation, with dispersion.
- $\omega < \omega_p$: no propagation, evanescent wave, total reflection

The dispersion relationship is non-linear, so plasma is a dispersive medium.

CHAPTER IV: WAVEGUIDES

- Maxwell's equations in the guide
- Propagation equation
- Rectangular waveguide
- Circular guide
- Coaxial line

A waveguide is any portion of empty space (or a dielectric) bounded by conductors,

The waveguide is translationally invariant in the direction of propagation.

The role of a waveguide is to ensure the propagation of a signal over a long distance and without alteration. Waveguides are often used to transfer electromagnetic energy in TE (transverse electric) or TM (transverse magnetic) modes.

- Transverse electric mode TE the electric field is transverse to the direction of propagation, but the magnetic field has both transverse and longitudinal components.($E_z = 0, B_z \neq 0$).
- Transverse magnetic mode TM: the magnetic field is transverse to the direction of propagation, but the electric field has both transverse and longitudinal components.($E_z \neq 0, B_z = 0$).
- TEM (transverse electromagnetic) modes have no electric or magnetic fields in the direction of propagation.($E_z = 0, B_z = 0$).

Waveguides are used in :

- High-power transmitters
- Radar equipment
- Microwave ovens
- Low-noise converter blocks for TV reception antennas.

IV- 1 Maxwell's equations in the guide:

$div\vec{E} = 0$ (Maxwell- Gauss) (IV-1)

$div\vec{B} = 0$ (Maxwell-flux magnétique) (IV-2)

$\overrightarrow{rotE} = -\frac{d\vec{B}}{dt}$ (Maxwell-Faraday) (IV-3)

$\overrightarrow{rotB} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ (Maxwell-Ampère) (IV-4)

IV-2 Propagation equation:

Let $\vec{E}$ and $\vec{B}$ be electromagnetic fields

$$E(x, y, z, t) = E(x, y)e^{ik_g z}.e^{-i\omega t}$$

$$B(x, y, z, t) = B(x, y)e^{ik_g z}.e^{-i\omega t}$$

k_g: waveguide wave vector.

$$\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \vec{0} \Rightarrow \Delta\vec{E} + \frac{\omega^2}{c^2}\vec{E} = \vec{0} \qquad \text{(IV-5)}$$

$$\Delta\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = \vec{0} \Rightarrow \Delta\vec{B} + \frac{\omega^2}{c^2}\vec{B} = \vec{0} \qquad \text{(IV-6)}$$

IV-3 Differential equations:

$$div\vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \qquad \text{(IV-7)}$$

$$div\vec{B} = 0 \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \qquad \text{(IV-8)}$$

$$\overrightarrow{rot\vec{E}} = -\frac{d\vec{B}}{dt} \Rightarrow \begin{cases} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{d\vec{B}_x}{dt} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{d\vec{B}_y}{dt} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{d\vec{B}_z}{dt} \end{cases} \Rightarrow \begin{cases} \frac{\partial E_z}{\partial y} - ik_g E_y = -i\omega B_x \\ ik_g E_x - \frac{\partial E_z}{\partial x} = -i\omega B_y \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega B_z \end{cases} \qquad \text{(IV-9)}$$

$$\overrightarrow{rot\vec{B}} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \Rightarrow \begin{cases} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0\varepsilon_0\frac{d\vec{E}_x}{dt} \\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0\varepsilon_0\frac{d\vec{E}_y}{dt} \\ \frac{\partial E_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0\varepsilon_0\frac{d\vec{E}_z}{dt} \end{cases} \Rightarrow \begin{cases} \frac{\partial B_z}{\partial y} - ik_g B_y = i\frac{\omega}{c^2}E_x \\ ik_g B_x - \frac{\partial B_z}{\partial x} = i\frac{\omega}{c^2}E_y \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = i\frac{\omega}{c^2}E_z \end{cases} \qquad \text{(IV-10)}$$

Cross-sectional components (E_x, E_y, B_x, B_y) as a function of longitudinal components $(E_z, , B_z,)$

$$\begin{cases} E_x = \frac{i}{\frac{\omega^2}{c^2} - k_g^2}\left(k_g\frac{\partial E_z}{\partial x} - \omega\varepsilon_0\frac{\partial B_z}{\partial y}\right) \\ E_y = \frac{i}{\frac{\omega^2}{c^2} - k_g^2}\left(k_g\frac{\partial E_z}{\partial y} + \omega\varepsilon_0\frac{\partial B_z}{\partial x}\right) \\ B_x = \frac{i}{\frac{\omega^2}{c^2} - k_g^2}\left(k_g\frac{\partial B_z}{\partial x} - \omega\varepsilon_0\frac{\partial E_z}{\partial y}\right) \\ B_y = \frac{i}{\frac{\omega^2}{c^2} - k_g^2}\left(k_g\frac{\partial B_z}{\partial y} + \omega\varepsilon_0\frac{\partial E_z}{\partial x}\right) \end{cases} \qquad \text{(IV-11)}$$

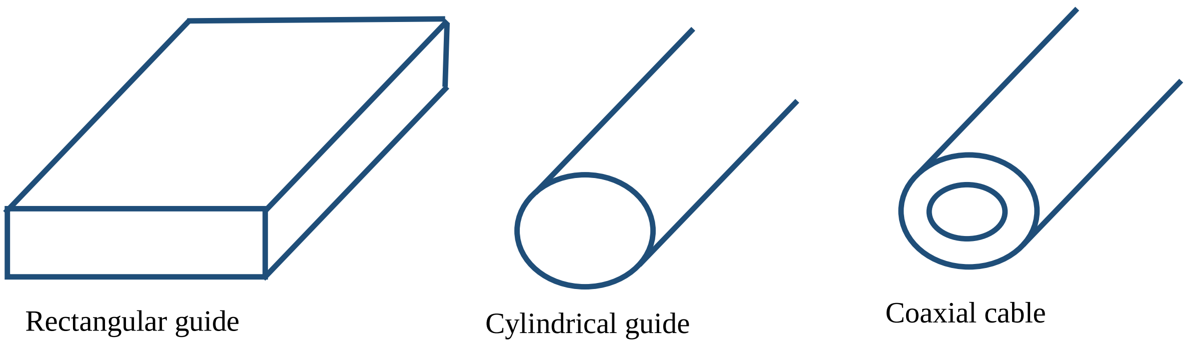


Figure IV-1: Examples of waveguides

IV-4 Rectangular waveguide:

Since the rectangular waveguide has a single conductor, it has a rectangular metal cross-section with width a along the $\overrightarrow{ox}$ axis and height b along the $\overrightarrow{oy}$ axis.

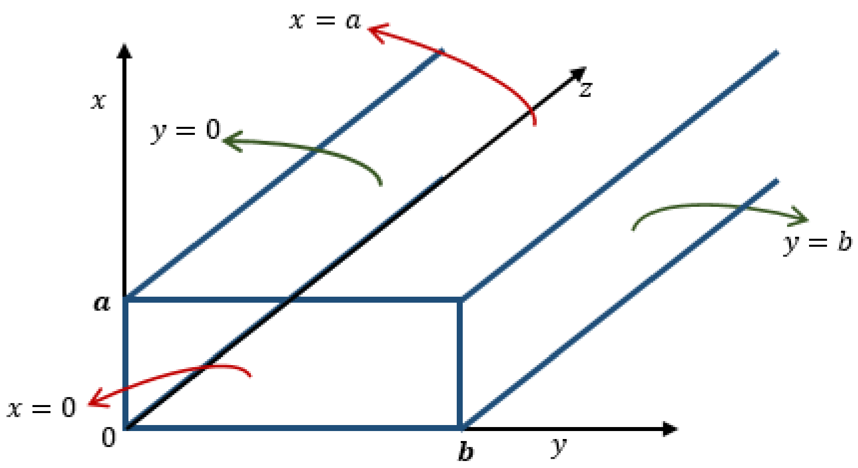


Figure IV-2: rectangular waveguide.

IV-4. 1 Boundary conditions:

The metallic waveguide limits the guide, so that the electromagnetic field inside the guide walls is zero.

The tangential component of $\vec{E}$ is zero at the guide terminals.

The transverse $\vec{E}$ field depends on the (x, y)coordinates.

$$\begin{cases} E(x,0) = 0 \\ E(x,b) = 0 \end{cases} \qquad \begin{matrix} E(0,y) = 0 \\ E(a,y) = 0 \end{matrix} \qquad \text{(IV-12).}$$

The transverse $\vec{B}$ field depends on the (x, y) coordinates

$$\frac{\partial \vec{B}}{\partial n} = \vec{0} \quad , \quad d\vec{n} = d\vec{x} + d\vec{y}$$

$$\begin{cases} \frac{\partial \vec{B}(0,y)}{\partial \vec{x}} = \vec{0} & \frac{\partial \vec{B}(x,0)}{\partial \vec{y}} = \vec{0} \\ \frac{\partial \vec{B}(a,y)}{\partial \vec{x}} = \vec{0} & \frac{\partial \vec{B}(x,b)}{\partial \vec{y}} = \vec{0} \end{cases} \qquad \text{(IV-13)}$$

IV-4.2 Mode study :

IV-4.2-1 TM modes (Transverse Magnetic):

Electric field propagation equation:

$$\Delta\vec{E} + \frac{\omega^2}{c^2}\vec{E} = \vec{0} \ , \ \frac{\omega^2}{c^2} = k_0^2$$

$$\frac{\partial^2 E(x,y,z,t)}{\partial x^2} + \frac{\partial^2 E(x,y,z,t)}{\partial y^2} + \frac{\partial^2 E(x,y,z,t)}{\partial z^2} + k_0^2 \vec{E}(x,y,z,t) = \vec{0}$$

$$\frac{\partial^2 E(x,y,z,t)}{\partial x^2} + \frac{\partial^2 E(x,y,z,t)}{\partial y^2} - k_g^2 \vec{E}(x,y,z,t) + k_0^2 \vec{E}(x,y,z,t) = \vec{0}$$

$$\frac{\partial^2 \vec{E}(x,y,z,t)}{\partial x^2} + \frac{\partial^2 \vec{E}(x,y,z,t)}{\partial y^2} + (k_0^2 - k_g^2)\vec{E}(x,y,z,t) = \vec{0} \quad , \quad k_0^2 - k_g^2 = k_c^2$$

$$\frac{\partial^2 \vec{E}(x,y)}{\partial x^2} + \frac{\partial^2 \vec{E}(x,y)}{\partial y^2} + k_c^2 \vec{E}(x,y) = \vec{0} \qquad \text{(IV-14)}$$

We assume : $E(x,y) = X(x); Y(x)$

Differential equation :

$$\ddot{X}Y + X\ddot{Y} + k_c^2 XY = 0$$

$$\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + k_c^2 = 0$$

We pose : $\begin{cases} \frac{\ddot{X}}{X} = -k_x^2 \\ \frac{\ddot{Y}}{Y} = -k_y^2 \end{cases} \quad , \quad k_x^2 + k_y^2 = k_c^2$ (IV-15)

Solution of the differential equation :

$$E(x,y) = (A cos k_x x + B sin k_x x)(C cos k_y y + D sin k_y y) \qquad \text{(IV-16)}$$

To determine the values of the constants (A,B,C,D,kx, ky), the boundary conditions are applied to the electric field components in the direction tangential to the waveguide wall.

Boundary conditions on walls :

$$\begin{cases} E(x,0) = 0 \\ E(x,b) = 0 \end{cases} \qquad \begin{matrix} E(0,y) = 0 \\ E(a,y) = 0 \end{matrix}$$

- For $y = 0$: $E(x,0) = 0 = (A cos k_x x + B sin k_x x).C$

$$\begin{cases} A cos k_x x + B sin k_x x \neq 0 \\ C = 0 \end{cases}$$

$\Rightarrow \quad E(x,y) = (A cos k_x x + B sin k_x x). D sin k_y y$

- For $y = b$: $E(x,b) = 0 = (A cos k_x x + B sin k_x x). D sin k_y b$

$$\begin{cases} A cos k_x x + B sin k_x x \neq 0 \\ D sin k_y b = 0 \\ D \neq 0 \end{cases}$$

$sin k_y b = 0 \quad \Rightarrow \quad k_y = \frac{n\pi}{b}$ pour n $= 0, 1, 2 \ldots$

$\Rightarrow \quad E(x,y) = (A cos k_x x + B sin k_x x). D sin \frac{n\pi}{b} y$

- For $x = 0$: $E(0,y) = 0 = A. D sin \frac{n\pi}{b} y$

$$\begin{cases} D sin \frac{n\pi}{b} y \neq 0 \\ A = 0 \end{cases}$$

$\Rightarrow \quad E(x,y) = Bsink_x x.Dsin\frac{n\pi}{b}y$

- For $x = a$: $\quad E(a,y) = 0 = Bsink_x a.Dsin\frac{n\pi}{b}y$

$$\begin{cases} Dsin\frac{n\pi}{b}y \neq 0 \\ Bsink_x a = 0 \\ B \neq 0 \end{cases}$$

$sink_x a = 0 \quad \Rightarrow \quad k_x = \frac{m\pi}{a}$, pour m = 0, 1, 2 ...

$\Rightarrow \quad E(x,y) = Bsin\frac{m\pi}{a}x.Dsin\frac{n\pi}{b}y$

So :

$$E(x,y,z,t) = E_0 sin\frac{m\pi}{a}x.sin\frac{n\pi}{b}y.e^{-i(\omega t - k_g z)} \quad \text{(IV-17)}$$

Where : $E_0 = BD$

IV-4.2-2 TE modes (Transverse Electric):

$$\begin{cases} \frac{\partial\vec{B}(0,y)}{\partial\vec{x}} = \vec{0} & \frac{\partial\vec{B}(x,0)}{\partial\vec{y}} = \vec{0} \\ \frac{\partial\vec{B}(a,y)}{\partial\vec{x}} = \vec{0} & \frac{\partial\vec{B}(x,b)}{\partial\vec{y}} = \vec{0} \end{cases}$$

Magnetic field propagation equation:

$$\Delta\vec{B} - \frac{\omega^2}{c^2}\vec{B} = \vec{0} \ , \frac{\omega^2}{c^2} = k_0^2 \ , k_g^2 - k_0^2 = k_c^2$$

$$\frac{\partial^2\vec{B}(x,y)}{\partial x^2} + \frac{\partial^2\vec{B}(x,y)}{\partial y^2} + k_c^2\vec{B}(x,y) = \vec{0}$$

We assume: $B(x,y) = X(x); Y(x)$

Differential equation :

$$\ddot{X}Y + X\ddot{Y} + k_c^2 XY = 0$$

$$\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + k_c^2 = 0$$

We pose : $\begin{cases} \frac{\ddot{X}}{X} = -k_x^2 \\ \frac{\ddot{Y}}{Y} = -k_y^2 \end{cases} \ , \quad k_x^2 + k_y^2 = k_c^2$

Solution of the differential equation :

$$B(x,y) = \left(\acute{A}cosk_x x + \acute{B}sink_x x\right)(\acute{C}cosk_y y + \acute{D}sink_y y) \quad \text{(IV-18)}$$

Boundary conditions on walls :

$$\begin{cases} \frac{\partial\vec{B}(x,0)}{\partial\vec{x}} = \vec{0} & \frac{\partial\vec{B}(0,y)}{\partial\vec{x}} = \vec{0} \\ \frac{\partial\vec{B}(x,b)}{\partial\vec{x}} = \vec{0} & \frac{\partial\vec{B}(a,y)}{\partial\vec{x}} = \vec{0} \end{cases}$$

$$\frac{\partial B(x,y)}{\partial x} = \left(-\acute{A}k_x sink_x x + \acute{B}k_x cosk_x x\right)(\acute{C}cosk_y y + \acute{D}sink_y y)$$

- For $x = 0$: $\frac{\partial B(0,y)}{\partial x} = 0 = \acute{B}(\acute{C}cosk_y y + \acute{D}sink_y y)$

$$\begin{cases} \acute{C}cosk_y y + \acute{D}sink_y y \neq 0 \\ \acute{B} = 0 \end{cases}$$

$$\Rightarrow \quad \frac{\partial B(x,y)}{\partial x} = \left(-\acute{A}k_x sink_x x\right)(\acute{C}cosk_y y + \acute{D}sink_y y)$$

- For $x = a$: $\frac{\partial B(a,y)}{\partial x} = 0 = \left(-\acute{A}k_x sink_x a\right)(\acute{C}cosk_y y + \acute{D}sink_y y)$

$$\begin{cases} \acute{C}cosk_y y + \acute{D}sink_y y \neq 0 \\ \acute{A}k_x sink_x a = 0 \\ \acute{A} \neq 0 \end{cases}$$

$$sink_x a = 0 \quad \Rightarrow \quad k_x = \frac{m\pi}{a}$$

$$\Rightarrow \quad \frac{\partial B(x,y)}{\partial x} = \left(-\acute{A}\frac{m\pi}{a} sin\frac{m\pi}{a}x\right)(\acute{C}cosk_y y + \acute{D}sink_y y)$$

$$\frac{\partial B(x,y)}{\partial y} = \left(\acute{A}cos\frac{m\pi}{a}x\right)(-\acute{C}k_y sink_y y + \acute{D}k_y cosk_y y)$$

- For $y = 0$: $\frac{\partial B(x,0)}{\partial y} = 0 = \left(\acute{A}cos\frac{m\pi}{a}x\right)(\acute{D}k_y cosk_y y)$

$$\begin{cases} \acute{A}cos\frac{m\pi}{a}x \neq 0 \\ \acute{C} = 0 \end{cases}$$

$$\Rightarrow \quad \frac{\partial B(x,y)}{\partial y} = \left(\acute{A}cos\frac{m\pi}{a}x\right)(-\acute{C}k_y sink_y y)$$

- for $y = b$: $\Rightarrow$ $\frac{\partial B(x,b)}{\partial y} = 0 = \left(\acute{A}cos\frac{m\pi}{a}x\right)(-\acute{C}k_y sink_y b)$

$$\begin{cases} \acute{A}cos\frac{m\pi}{a}x \neq 0 \\ \acute{C}k_y sink_y b = 0 \\ \acute{C} \neq 0 \end{cases}$$

$$sink_y b = 0 \quad \Rightarrow \quad k_y = \frac{n\pi}{b}$$

$$\Rightarrow \quad \frac{\partial B(x,y)}{\partial y} = \left(\acute{A}cos\frac{m\pi}{a}x\right)(-\acute{C}\frac{n\pi}{b} sin\frac{n\pi}{b}y)$$

$$B(x,y) = \left(\acute{A}cos\frac{m\pi}{a}x\right)(\acute{C}cos\frac{n\pi}{b}y)$$

So :

$$B(x,y,z,t) = B_0 cos\frac{m\pi}{a}x cos\frac{n\pi}{b}y.e^{-i(\omega t - k_g z)} \qquad \text{(IV-19)}$$

where: $B_0 = \acute{A}\acute{B}$

$$k_x^2 + k_y^2 = k_c^2 \quad \Rightarrow \quad k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \text{(IV-20)}$$

$$k_c^2 = \left(\frac{2\pi}{\lambda_c}\right)^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \Rightarrow \left(\frac{1}{\lambda_c}\right)^2 = \left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2 \qquad \text{(IV-21)}$$

IV-4.3 Cut-off wavelength:

The cut-off wavelength in a waveguide for a fundamental mode TEmn is given by the following relationship:

In TE_{nm} mode:

$$\left(\frac{1}{\lambda_c}\right)^2 = \left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2$$

$$\lambda_{cnm} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \quad \text{(IV-22)}$$

In TE_{10} mode:

$$\lambda_{c10} = \frac{2}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{0}{b}\right)^2}} = 2a \quad \text{(IV-23)}$$

IV-4.4 Cut-off frequency:

Each mode (for each value of m and n)has a cut-off frequency: this is a frequency ' below which these modes cannot propagate in the guide.

$$f_{cmn} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{c}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad \text{(IV-24)}$$

- For $TEmn$ mode : (on suppose que a $>$ b)

The dominant cut-off frequency (the lowest frequency) occurs in TE10 mode:

$$f_{c10} = \frac{1}{2\sqrt{\mu\varepsilon}} = \frac{c}{2\pi}\left(\frac{\pi}{a}\right) \quad \text{(IV-25)}$$

TE_{00} mode does not exist.

- For TMmn mode :

The dominant cut-off frequency (the lowest frequency) occurs in TM_{11} mode:

$$f_{c11} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = \frac{c}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2} \quad \text{(IV-26)}$$

modes TM_{00}, TM_{01} and TM_{10} is not possible, as the expression of $E(x, y, z, t)$ becomes zero.

IV-4.5 Group velocity:

The group velocity of a rectangular waveguide :

$$\vartheta = \frac{\omega}{k_g} = \frac{\omega}{\sqrt{k_0^2 - k_c^2}} = \frac{\omega}{\sqrt{k_0^2 - \left[(\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2\right]^2}} \quad \text{(IV-27)}$$

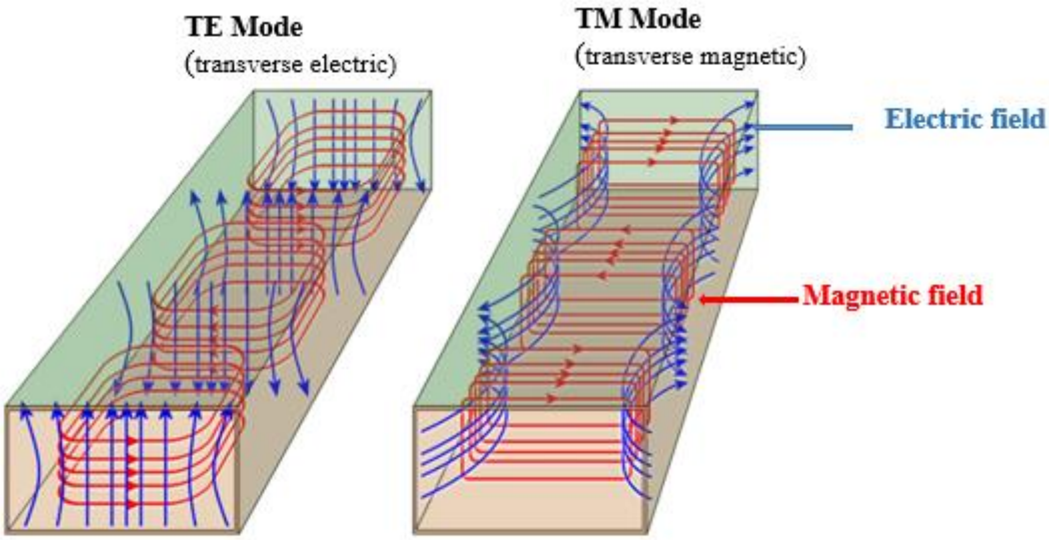


Figure IV-3: Transmission modes in a guide

IV-4.6 Electromagnetic wave in a rectangular waveguide:

An incident electromagnetic wave oblique at an angle Ψ to a conductive plane (Σ) **[Naimi Bouthaina Chergui Nadjette]**

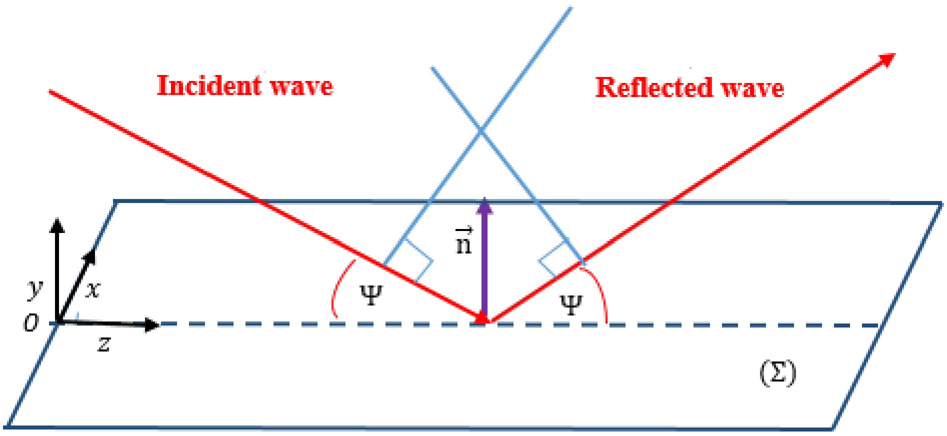


Figure IV-4 : Reflection under oblique incidence.

IV-4.7 Propagation conditions:

Continuity conditions at the surface: $E_y = 0$, 0 $E_z = 0$ et $B_x = 0$

If the incidence plane is parallel to (π) and (π́)' :

The distance b between planes (Σ) and $(\acute{\Sigma})$ is :$b = \frac{n\lambda}{2\sin\Psi}$

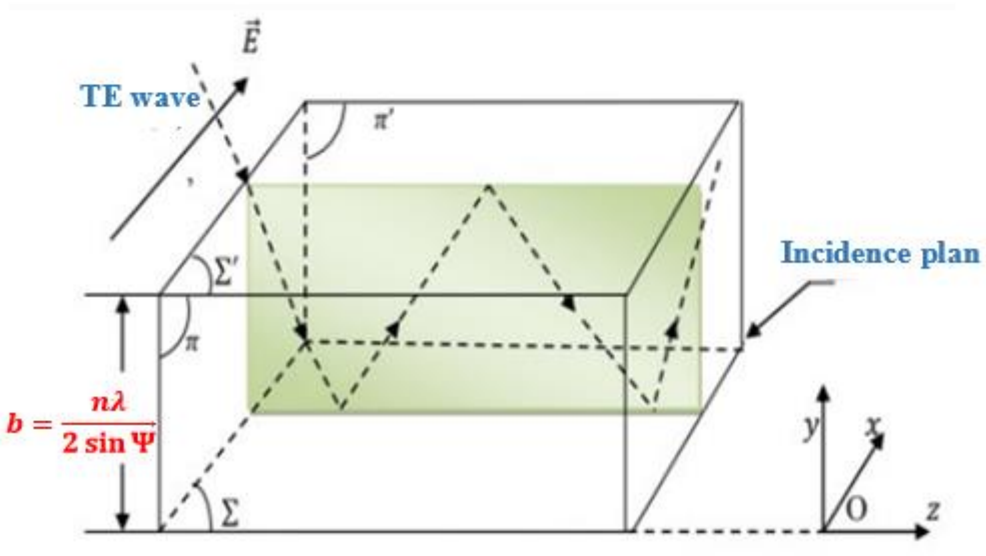


Figure IV-5 : Propagation in mode

If the plane of incidence is parallel to ∑and ∑' :

The distance a between planes (π) and (π́) is: $a = \frac{m\lambda}{2\sin\Psi}$

O.E.M -wave propagation inside a rectangular waveguide is possible if the electric field of the incident wave is perpendicular to the plane of incidence (TE wave).

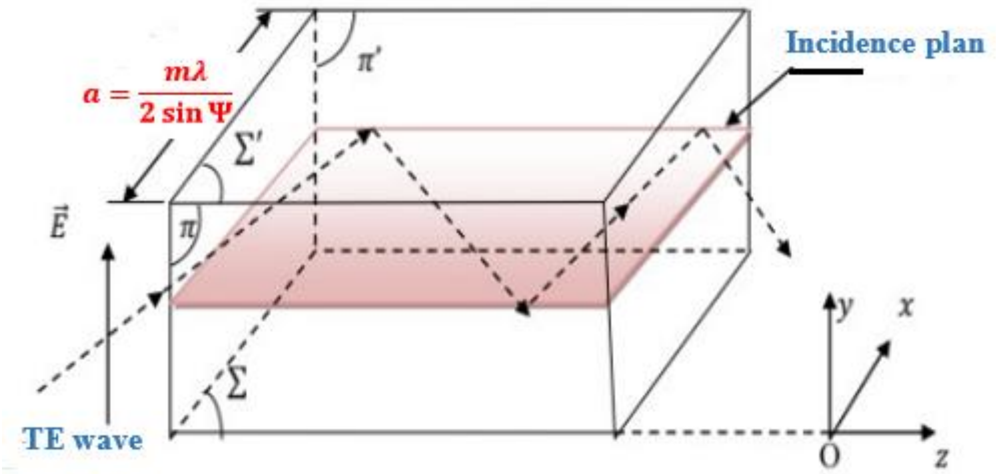


Figure IV-6 : Propagation in mode

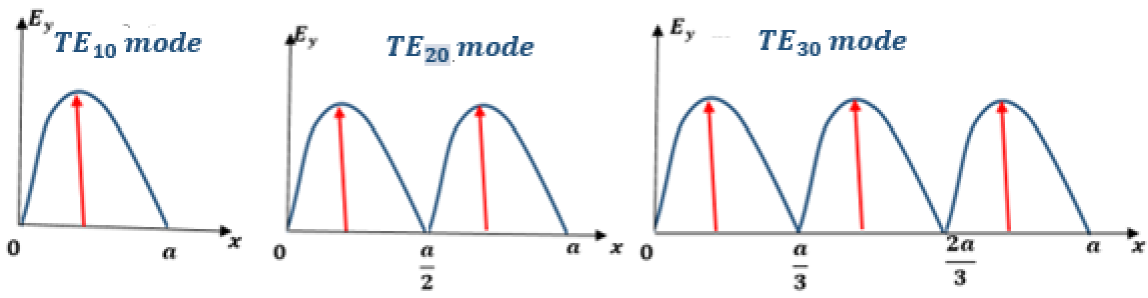


Figure IV-7 : Mode : TE_{10}, TE_{20}, TE_{30}

IV-5 Circular guide:

The circular waveguide is a hollow metal cylinder with radius a. The z axis is defined as the direction of propagation.

The circular waveguide has useful properties only if the circular waveguide is made very precisely, because the circular waveguide generally operates above the cutoff frequency of at least one of the higher-order modes,

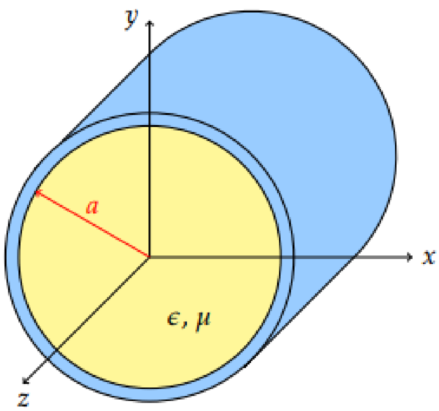


Figure IV-8 : Circular waveguide.

IV-5.1 Mode study:

We use cylindrical coordinates $(r, \phi\, et\, z)$

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \qquad \text{(IV-28)}$$

Propagation equation:

$$\Delta B_z + \frac{\omega^2}{c^2}B_z = 0$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]B_z + + \frac{\omega^2}{c^2}B_z = 0$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial r^2}\right]B_z+\frac{\omega^2}{c^2}B_z=0 \quad \text{(IV-29)}$$

We then notice an equation of the harmonic equation type, looking for a solution of the form (separation of variables):

$$B_z = R(r)\,\Phi(\theta)\,Z(z) \quad \text{(IV-30)}$$

Making this change and dividing by the product R Φ Z

$$\frac{R''}{R}+\frac{1}{r}\frac{R''}{R}+\frac{1}{r^2}\frac{\Phi"}{\Phi}+\frac{Z"}{Z}=0 \quad \text{(IV-31)}$$

It comes :

$$\frac{d^2R(r)}{dR^2}\Phi(\theta)\,Z(z)+\frac{1}{r}\frac{dR(r)}{\partial r}\Phi(\theta)\,Z(z)+\frac{1}{r^2}\frac{d^2\Phi(r)}{d\theta^2}\;R(r)Z(z)+\frac{\partial^2Z(z)}{\partial z^2}\Phi(\theta)\,R(r)+$$

$$\frac{\omega^2}{c^2}\Phi(\theta)\,R(r)Z(z)=0 \quad \text{(IV-32)}$$

We assume that :

$$\Phi(\theta)=e^{im\theta}\quad ,Z(z)=e^{ik_gz}\quad \frac{\omega^2}{c^2}=k_0^2\,,\;\;k_0^2-k_g^2=k_c^2$$

$$\frac{d^2R(r)}{dr^2}\;+\frac{1}{r}\frac{dR(r)}{\partial r}\;+\frac{1}{r^2}\frac{d^2\Phi(r)}{d\theta^2}\;R(r)-\;k_g^2R(r)+k_0^2\;R(r)=0 \quad \text{(IV-33)}$$

$$\frac{d^2R(r)}{dr^2}\;+\frac{1}{r}\frac{dR(r)}{\partial r}\;+\frac{1}{r^2}\frac{d^2\Phi(r)}{d\theta^2}\;R(r)+\;k_c^2\;R(r)=0$$

$$\frac{d^2R(r)}{dr^2}\;+\frac{1}{r}\frac{dR(r)}{\partial r}\;+\left(-\frac{m^2}{r^2}+\;k_c^2\right)R(r)=0$$

$$r^2\frac{d^2R(r)}{dr^2}\;+r\frac{dR(r)}{\partial r}\;+\left(r^2k_c^2-\frac{m^2}{r^2}+\;\right)R(r)=0 \quad \text{(IV-34)}$$

Equation (IV-34) is a second-order differential equation, there are two linearly independent solutions for each value of n. The general solution of this equation for integer R (r) is [NICOLE PEPIN] :

$$R\;(r)==\;AJ_n(k_cr)\;+\;BY_n(k_cr) \quad \text{(IV-35)}$$

where :

J_n : first order Bessel function of order m

Y_n : second order Bessel function of order m (Neumann function of order)

The figures below show the variation of J_and Y_n as a function of x $(x=k_cr)$.

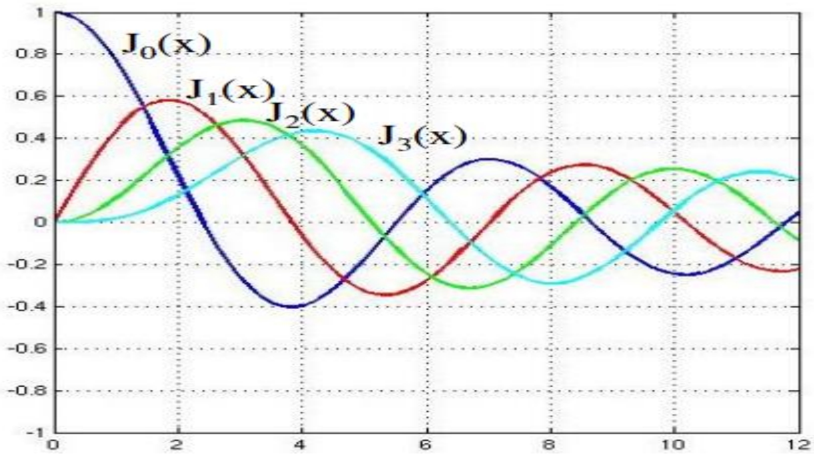


Figure IV-9: Variation of Bessel function First kind Jn(x).

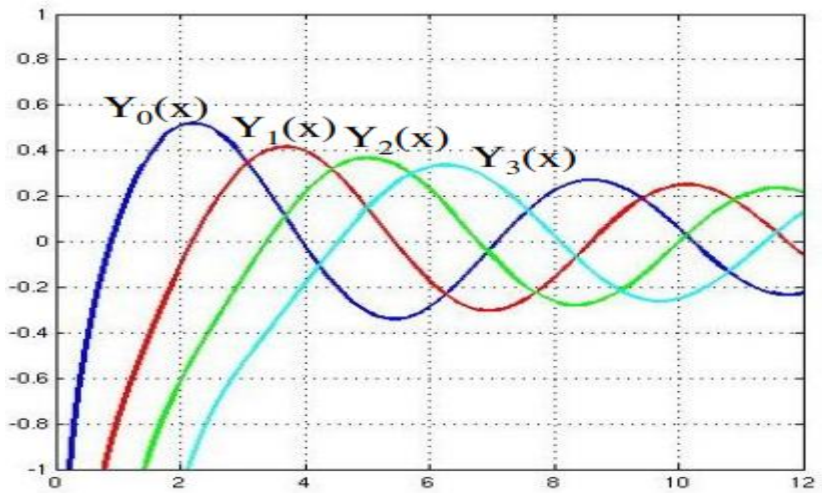


Figure IV-10: variation of Neumann function Second kind Yn(x).

Figure (1) shows that, in the vicinity of the origin of the first-order Bessel function of order m, the functions retain a finite value (at the guide center)

Figure (2) shows that the Neumann function tends towards infinity when the radius r tends towards zero (near the origin). The field cannot diverge at the guide center.

So ; For the solution to be finite everywhere, the function R(r) must be equal to the first order Bessel function of order m. [A.DJEBBARI D.BARKA]

IV-5.1-1 TE modes (tansverse electric):

$$B(r,\theta) = AJ_m(Kr)(Ccos(m\theta) + Dsin\ (m\theta)) \quad \text{(IV-36)}$$

For : m=0 ; $sin\ (m\theta) = 0$

General solution of the field component :

$$B(r,\theta) = ACJ_m(Kr)cos(m\theta) \quad \text{(IV-37)}$$

$$B(r,\theta,z,t) = B_0 J_m(Kr)cos(m\theta)e^{-i(\omega t - k_z z)} \quad \text{(IV-38)}$$

Propagation equation: TE mode :

$$\Delta B + \frac{\omega^2}{c^2}B = 0$$

On the waveguide surface we have:

$$E_\theta(a,\theta) = 0 \quad \text{(IV-39)}$$

$$E_\theta = \frac{i}{c^2K^2}\frac{\partial B_z}{\partial r},$$

$where$:

$$c: speed\ of\ light = \frac{1}{\sqrt{\varepsilon\mu_0}}$$

$$E_\theta = \frac{i}{c^2K^2}B_0\frac{\partial}{\partial r}(J_m(Kr)cos(m\theta)) \quad \text{(IV-40)}$$

En $r = a\,; J'_n(Ka) = 0$,

we obtain : $K = \frac{X\prime_{nm}}{a}$

X'_{nm} :is the m[th] root of the derivative of the Bessel function of the first derivative order m (solution of the wave equation)

The magnetic field component B of the TE mode is :

$$B(r,\theta,z,t) = B_0 J_m\left(\frac{X\prime_{nm}}{a}r\right)cos(m\theta)e^{-i(\omega t - k_z z)} \quad \text{(IV-41)}$$

IV-5.1-1.1 Dispersion relation:

$$\frac{\partial B^2}{\partial r^2} + \frac{\partial B}{\partial z} + \frac{\omega^2}{c^2}B = 0$$

$$\left(\frac{X\prime_{nm}}{a}\right)^2 + k_z^2 = \frac{\omega^2}{c^2}$$

$$\frac{\omega^2}{c^2} = \left(\frac{X\prime_{nm}}{a}\right)^2 + k_z^2 \quad \text{(IV-42)}$$

Dispersion relation :

$$\omega^2 = \varepsilon\mu_0(\left(\frac{X\prime_{nm}}{a}\right)^2 + k_z^2)$$

$$k_z^2 = \frac{\omega^2}{c^2} - \left(\frac{X\prime_{nm}}{a}\right)^2$$

$$k_z = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{X\prime_{nm}}{a}\right)^2} \quad \text{(IV-43)}$$

IV-5.1-1.2 Cut-off frequency:

The cut-off frequency frequency is given by the following expression :

$$f_{cmn} = \frac{X\prime_{nm}}{2\pi a\sqrt{\varepsilon\mu_0}} \quad \text{(IV-44)}$$

Cut-off pulse $(k_z = 0)$:

$$\omega_c = \frac{1}{\sqrt{\varepsilon\mu_0}}\left(\frac{X'_{nm}}{a}\right) \quad \text{(IV-45)}$$

Dispersion relation :

$$k_z = \sqrt{\varepsilon\mu_0(\omega^2 - (\omega_c)^{2)}} \quad \text{(IV-46)}$$

for $\omega < \omega_c$: k_z is imaginary. The mode is attenuated (no propagation).

for $\omega > \omega_c$: k_z is real. The mode propagates.

The table below shows the first values of X'_{mn} :

	n=1	n=2	n=3	n=4
m=0	3,832	7,016	10,173	13,324
m=1	1,841	5,331	8,536	11,706
m=2	3,054	6,076	9,969	13,170
m=3	4,201	8,015	11,346	14,58

Note that the TE_{11} mode has the lowest cutoff frequency (low value of $X'_{mn} = 1{,}841$).

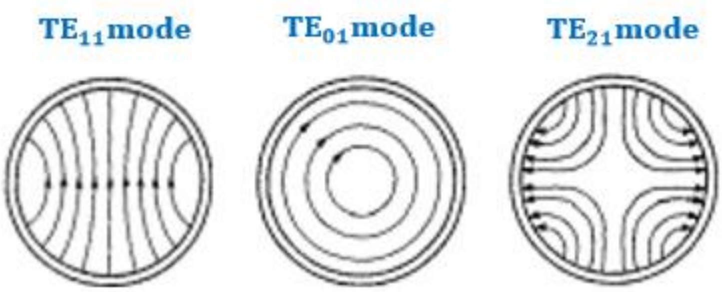


Figure IV-11 : TE modes of a circular guide: electric field lines.

IV-5.1-2 TM modes (tansverse magnetic):

The electric field component E_z is defined by

$$E(r,\theta,z,t) = E_0 J_m(Kr)cos(m\theta)e^{-i(\omega t - k_z z)} \quad \text{(IV-47)}$$

On the waveguide surface :

$$E_\theta(a,\theta) = 0 \quad \text{(IV-48)}$$

The solution For TM modes corresponding to the vicinity of the origin, the first-order Bessel function

for $r = a$; $J_m(Ka) = 0$,

we obtain : $K = \frac{X_{mn}}{a}$

X_{nm} : is the n[th] root of the derivative of the first-order Bessel function of order m (solution of the wave equation)

The electric field component E of the TE mode :

$$E(r,\theta,z,t) = E_0 J_m\left(\frac{X_{mn}}{a}r\right)cos(m\theta)e^{-i(\omega t-k_z z)} \qquad \text{(IV-49)}$$

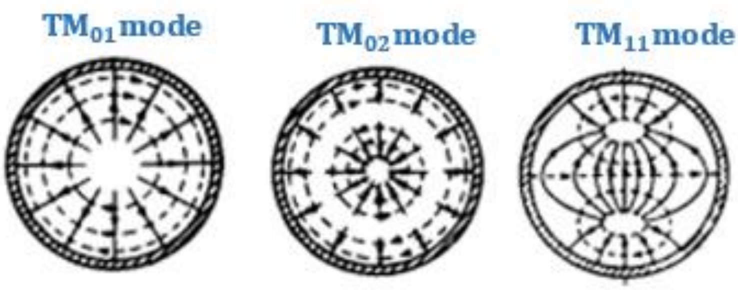


Figure IV-12 : TM modes of a circular guide: magnetic field lines[Bonnet-Ben Dhia]

.IV-5.1-2.1 Dispersion relation:

$$k_z^2 = \frac{\omega^2}{c^2} - \left(\frac{X_{nm}}{a}\right)^2 \qquad \text{(IV-50)}$$

$$k_z = \sqrt{\varepsilon\mu_0(\omega^2 - (\omega_c)^{2)}} \qquad \text{(IV-51)}$$

Forr $\omega < \omega_c$: k_z is imaginary. The mode is attenuated (no propagation).

For $\omega > \omega_c$: $k_z$is real. The mode propagates.

IV-5.1-2.2 Cut-off frequency:

$$f_{cmn} = \frac{X_{nm}}{2\pi a\sqrt{\varepsilon\mu_0}} \qquad \text{(IV-52)}$$

$X_{nm}$is m[th] root m[ième] racine of the Bessel function (solution to the wave equation)

The table below shows the first values of X_{mn} (j_m):

	n=1	n=2	n=3	n=4
m=0	2,402	5,520	8,654	11 ,792
m=1	3,832	7,016	10,173	13,324
m=2	5,136	8,417	11,620	14,796
m=3	6,380	9,761	13,015	16,223

Note that the TM_{01} mode has the lowest cut-off frequency (low value of $X_{mn} = 2{,}402$).

IV-5.2 Wavelength of a circular guide:

When a wave propagates in a waveguide. The same phase must then be found every λ_g $((\lambda_g k_c = 2\pi)$

$$\lambda_g = \frac{2\pi}{k_z} = \frac{1}{\omega\sqrt{\varepsilon\mu_0}} \frac{1}{\sqrt{1-\frac{\omega_c^2}{\omega^2}}} \quad \text{(IV-53)}$$

IV-6 Coaxial line:

Coaxial line is a cable with two concentric conductors separated by insulation. The central conductor is called the core, the other the braid. It is used to transmit signals at low or high frequencies. This type of waveguide is widely used in industry.

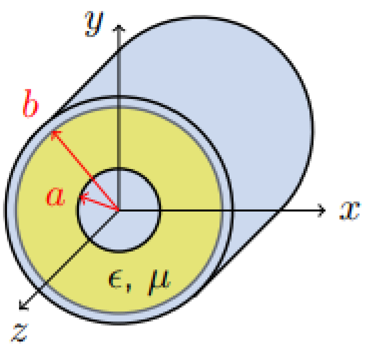


Figure IV-13 : Coaxial waveguide.

- The electric field starts at the center and builds up between the two conductors.
- The magnetic field forms concentric field lines around the core.
- The two fields are perpendicular to each other $(X, Y)$and move along the (OZ)axis.

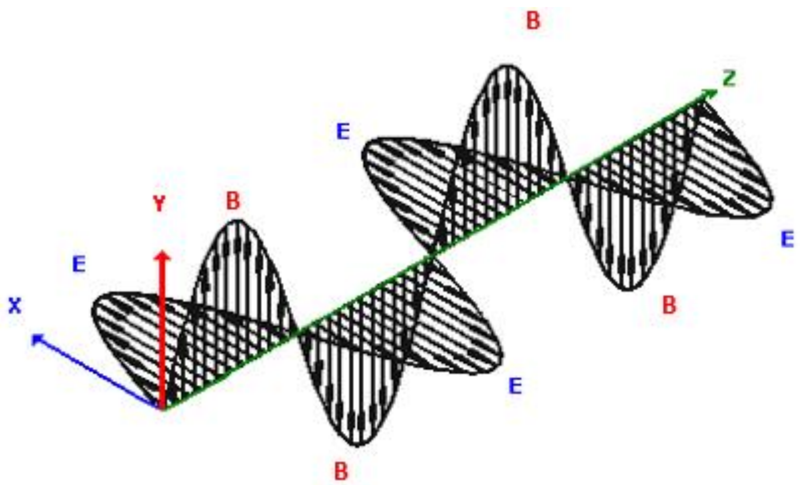


Figure IV-14 : Electromagnetic fields in a coaxial line.

IV-6.1 TEM modes study (transverse electric-magnetic):

From the maxwell -flux equation:

$$div\vec{B} = 0$$

Divergence in cylindrical coordinates :

$$div\vec{B} = \frac{1}{r}\frac{\partial rB_r}{\partial r} + \frac{1}{r}\frac{B_\theta}{\partial\theta} + \frac{\partial B_z}{\partial z} \qquad \text{(IV-54)}$$

Propagation along the (OZ) axis,$\vec{B} \perp \overrightarrow{OZ}$: $\frac{\partial B_z}{\partial z} = 0$

therefore ;

$$div\vec{B} = \frac{1}{r}\frac{\partial rB_r}{\partial r} + \frac{1}{r}\frac{\partial B_\theta}{\partial\theta} = 0 \qquad \text{(IV-55)}$$

The magnetic field B does not depend on θ (cylindrical symmetry)

$$\frac{\partial rB_r}{\partial r} = 0 \qquad \text{(IV-56)}$$

$$B_r = \frac{cste}{r} = \frac{A}{r} \qquad \text{(IV-57)}$$

By analogy

Maxwell-Gauss equation: $div\vec{E} = 0$

$$E_r = \frac{cste}{r} = \frac{A\prime}{r} \qquad \text{(IV-58)}$$

Maxwell-Faraday equation :

$$\overrightarrow{rot}\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Rotation in cylindrical coordinates :

$$\overrightarrow{rot}\vec{E} = \begin{pmatrix} \frac{1}{r}\frac{\partial E_z}{\partial\theta} - \frac{\partial E_\theta}{\partial z} \\ \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} \\ \frac{1}{r}\left(\frac{\partial rE_\theta}{\partial r} - \frac{\partial E_r}{\partial\theta}\right) \end{pmatrix} \qquad \text{(IV-59)}$$

$$\overrightarrow{rot}\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \Rightarrow \quad \begin{Bmatrix} \frac{1}{r}\frac{\partial E_z}{\partial\theta} - \frac{\partial E_\theta}{\partial z} = -\frac{\partial B_r}{\partial t} \\ \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{\partial B_\theta}{\partial t} \\ \frac{1}{r}\left(\frac{\partial rE_\theta}{\partial r} - \frac{\partial E_r}{\partial\theta}\right) = -\frac{\partial B_z}{\partial t} \end{Bmatrix} \qquad \text{(IV-60)}$$

Expressions of the electric field $\vec{E}$ and magnetic field $\vec{B}$:

$$\vec{E} = \vec{E}(r).e^{-i(\omega t - kz)}$$

$$\vec{B} = \vec{B}(r).e^{-i(\omega t - kz)}$$

$$\overrightarrow{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad \begin{cases} -ikE_\theta = i\omega B_r \\ -ikE_r = i\omega B_\theta \\ \frac{\partial rE_\theta}{\partial r} = 0 \end{cases} \Rightarrow \begin{cases} -E_\theta = \frac{\omega}{k}B_r \\ E_r = \frac{\omega}{k}B_\theta \\ E_\theta = \frac{cte}{r} = \frac{A"}{r} \end{cases} \qquad \text{(IV-61)}$$

$$\begin{cases} B_r = \frac{A}{r} \\ E_r = \frac{A\prime}{r} \\ E_r = \frac{\omega}{k}B_\theta \end{cases} \quad \Rightarrow \quad \frac{A\prime}{r} = \frac{\omega}{k}B_\theta \quad \Rightarrow \quad B_\theta = \frac{k}{\omega}\frac{A\prime}{r} \qquad \text{(IV-62)}$$

All components of electromagnetic fields are defined (electric field E and magnetic field B) ;

$$\Rightarrow \begin{cases} E_r = \frac{A\prime}{r} \\ E_\theta = \frac{A"}{r} \\ B_r = \frac{A}{r} \\ B_\theta = \frac{k}{\omega}\frac{A\prime}{r} \end{cases} \quad \vec{E}\begin{pmatrix} E_r = \frac{A\prime}{r} \\ E_\theta = \frac{A"}{r} \\ E_z = 0 \end{pmatrix}, \ \vec{B}\begin{pmatrix} B_r = \frac{A}{r} \\ B_\theta = \frac{k}{\omega}\frac{A\prime}{r} \\ B_O = 0 \end{pmatrix} \qquad \text{(IV-63)}$$

,

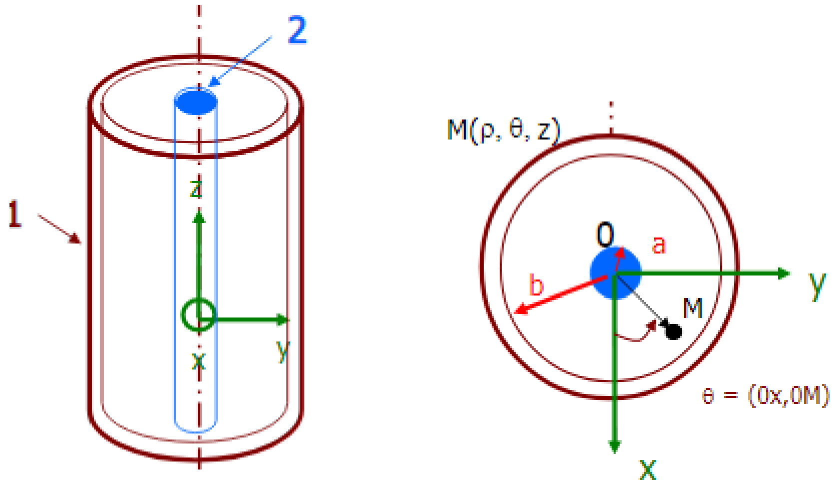


Figure IV-15: Coaxial line.

IV-6.2 Boundary conditions:

$$\vec{E}_{T2} - \vec{E}_{T1} = \vec{0}, \quad E_\theta(r = a) = 0 \ , \ E_\theta(r = B) = 0$$

$$-E_\theta = \frac{\omega}{k}B_r \quad \Rightarrow B_r = 0 \qquad \text{(IV-64)}$$

So : the magnetic field coordinates are:

$$\vec{B}\begin{pmatrix} 0 \\ B_\theta = \frac{k}{\omega}\frac{A\prime}{r} \\ 0 \end{pmatrix}$$

Maxwell-Ampère equation:

$$\overrightarrow{rot}\vec{B} = \mu_0\varepsilon\frac{\partial \vec{E}}{\partial t}$$

$$\overrightarrow{rot}\vec{B} = \mu_0\varepsilon\frac{\partial\vec{E}}{\partial t} \quad \left\{\begin{array}{l} \frac{1}{r}\frac{\partial B_z}{\partial\theta} - \frac{\partial B_\theta}{\partial z} = \mu_0\varepsilon\frac{\partial E_r}{\partial t} \\ \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \mu_0\varepsilon\frac{\partial E_\theta}{\partial t} \\ \frac{1}{r}\left(\frac{\partial rB_\theta}{\partial r} - \frac{\partial B_r}{\partial\theta}\right) = \mu_0\varepsilon\frac{\partial E_z}{\partial t} \end{array}\right\} \quad \text{(IV-65)}$$

$$\overrightarrow{rot}\vec{B} = \mu_0\varepsilon\frac{\partial\vec{E}}{\partial t} \quad \Rightarrow \quad \left\{\begin{array}{l} -ikB_\theta = -i\omega\mu_0\varepsilon E_r \\ \frac{\partial rB_\theta}{\partial r} = 0 \end{array}\right. \quad \Rightarrow B_\theta = \frac{\omega}{k}\mu_0\varepsilon E_r \quad \text{(IV-66)}$$

IV-6.3 Dispersion relations:

We have :

$$\left\{\begin{array}{l} B_r = \frac{k}{\omega}\frac{A'}{r} \\ E_r = \frac{A'}{r} \\ B_\theta = \frac{\omega}{k}\mu_0\varepsilon E_r \end{array}\right. \quad \Rightarrow \quad k^2 = \omega^2\mu_0\varepsilon \quad \text{(IV-67)}$$

TEM mode is dominant.

IV-6.4 Cut-off frequency for TEM mode:

The cut-off frequency of a coaxial line is written as:

$$f_c = \frac{ck_c}{2\pi\sqrt{\varepsilon_r}} \quad \text{(IV-68)}$$

Where :

k_c ; Cut-off wave vector.

C : speed of light $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$

ε_r : Relative Permittivity.

IV-6.5 Wavelength of a coaxial line:

The wavelength in a waveguide is given by the following relationship:

$$\lambda_g = \frac{2\pi}{k_z} = \frac{1}{\omega\sqrt{\varepsilon\mu_0}} \quad \text{(IV-69)}$$

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